Multilinear Representations of Rotation Groups within Geometric Algebra

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Abstract

It is shown that higher-weighted representations of rotation groups can be constructed using multilinear functions in geometric algebra. Methods for obtaining the irreducible representations are found, and applied to the spatial rotation group, $SO(3)$, and the proper Lorentz group, $SO^+(1,3)$. It is also shown that the representations can be generalised to non-linear functions, with applications to relativistic wave equations describing higher-spin particles, such as the Rarita-Schwinger equations. The internal spin degrees of freedom and the external spacetime degrees of freedom are handled within the same mathematical structure.

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1 Introduction

The vector and spinor representations of rotation groups are well known. In this paper we show that it is possible to extend these to higher-weighted representations by using multilinear functions of vector variables. Our representations are constructed using the language of geometric algebra, an introduction to which is given in the following subsection.

First we show how to construct the representations of an arbitrary rotation group $SO^+(p,q)$ and demonstrate the methods for finding their irreducible components. The most important method is monogenic decomposition [1], which can be refined by considering the symmetry between the arguments of the function. A monogenic function belongs to the null space of the vector derivative (the Dirac operator). Analysis of the monogenic properties of functions forms an important part of Clifford analysis.

We use these methods to find the irreducible representations of the rotation group of Euclidean 3-space $SO(3)$ and the proper Lorentz group $SO^+(1,3)$. In both cases we show how to find all the integral- and half-integral-spin representations of the groups. These are useful in constructing wave equations describing higher-spin particles.

In the last section we examine the Dirac equation and its generalisations to higher-spin wave equations, such as the Rarita-Schwinger equations. We show that the multilinear representations can be extended to encompass non-linear functions, and these can be used to construct the wavefunctions. This provides a unified method for describing the internal and external degrees of freedom of a higher-spin particle. In using geometric algebra we are able to overcome one of the deficiencies in the matrix formulation of linear algebra,

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that it is not easily able to relate linear and non-linear methods. Polynomial projection [1] enables us to immediately construct the non-linear functions which form the basis of the spacetime part of the wavefunction from the multilinear functions derived earlier for the spin part of the wavefunction.

1.1 Geometric Algebra

Geometric algebra is based upon Clifford algebra with emphasis on the geometric interpretation of elements of the algebra rather than their matrix representations. Here we present a brief introduction to geometric algebra and the conventions that are used in this paper. More comprehensive introductions can be found in Hestenes and Sobczyk [2] and Gull, Lasenby and Doran [3].

Geometric algebra is a graded algebra in which the fundamental product is the geometric product. Vectors, which have grade 1, will be written in lower case Roman letters (a,b, *etc.*). The geometric product of vectors ab can be split into its symmetric and anti-symmetric parts

$$ab = a\cdot b + a\wedge b \tag{1}$$

where

$$\begin{aligned} a\cdot b &\equiv \tfrac{1}{2}(ab+ba) \\ a\wedge b &\equiv \tfrac{1}{2}(ab-ba) \end{aligned} \tag{2}$$

The inner product $a\cdot b$ is scalar-valued (grade 0). The outer product $a\wedge b$ represents a directed plane segment produced by sweeping a along b. This new element is called a bivector and has grade 2. By introducing more vectors, higher-grade elements can be constructed, trivectors (grade-3) $a\wedge b\wedge c$ represent volumes, and so on, up to the dimension of the space under consideration. A grade-r element of the algebra is called an r-vector and a general element of the geometric algebra, possibly consisting of many grades, is called a multivector. If we multiply an r-vector A_r by an s-vector B_s the geometric product can be expanded as follows

$$A_rB_s = \langle A_rB_s\rangle_{r+s} + \langle A_rB_s\rangle_{r+s-2} + \ldots + \langle A_rB_s\rangle_{|r-s|}\,, \tag{3}$$

where $\langle C\rangle_t$ denotes projecting out the grade-t elements of C. The notation for inner and outer products is retained for the lowest and highest grade elements in this expansion,

$$\begin{aligned} A_r\wedge B_s &\equiv \langle A_rB_s\rangle_{r+s}\,, \\ A_r\cdot B_s &\equiv \langle A_rB_s\rangle_{|r-s|}\,. \end{aligned} \tag{4}$$

We use the convention that inner and outer products take precedence over geometric products in expressions.

In addition we define the commutator product, which is equal to half the commutator bracket

$$A\times B = \tfrac{1}{2}(AB-BA) = \tfrac{1}{2}[A,B]. \tag{5}$$

Reversion is the operation by which the order of vectors in a geometric product is reversed, it is defined by

$$\widetilde{AB} = \tilde{B}\tilde{A}, \tag{6}$$

where a vector reverses to give itself, $\tilde{a} = a$.

Although geometric algebra can be constructed in an entirely basis-free form, in many applications it is useful to introduce a set of basis vectors. The geometric algebra with signature (p, q) which will be denoted as $\mathcal{G}_{p,q}$ has $n = p + q$ orthonormal basis vectors, $e_i, i = 1, \ldots, n$ which obey

$$e_i \cdot e_j = \begin{cases} 1 & i = j = 1, \ldots, p \\ -1 & i = j = p+1, \ldots, p+q \\ 0 & i \neq j \end{cases} \tag{7}$$

The grade-r space in this algebra is denoted $\mathcal{G}^r_{p,q}$.

The highest-grade element that can be formed from this basis has grade n and is called the pseudoscalar. It is defined by

$$I \equiv e_1 \wedge e_2 \wedge \ldots \wedge e_n = e_1 e_2 \ldots e_n. \tag{8}$$

All grade-n elements must be a scalar multiple of the pseudoscalar.

This paper makes use of some results of geometric calculus, which is based on the vector derivative. If we write a vector in terms of its components, $a = a^i e_i$, then the vector derivative with respect to a is defined as

$$\partial_a \equiv e^i \frac{\partial}{\partial a^i} \tag{9}$$

where the reciprocal vectors are defined by

$$e^i \equiv (-1)^{i-1} (e_1 \wedge \ldots \wedge \check{e}_i \wedge \ldots \wedge e_n) I^{-1} \tag{10}$$

and the check on $\check{e}_i$ denotes that it is omitted from the product. The vector derivative, besides being an operator, also inherits the properties of a vector from a. The vector derivative with respect to position x is given the symbol ∇,

$$\nabla \equiv \partial_x. \tag{11}$$

This paper uses the properties of multilinear functions which are functions in geometric algebra that are linear in all of their arguments. A function

$$M(a_1, \ldots, a_k) \tag{12}$$

which takes values in any part of the geometric algebra and is linear in its k vector arguments will be called k-linear.

There are two geometric algebras, the Pauli algebra and the spacetime algebra, which for reasons of convention have their own notation.

1.2 The Pauli Algebra

The Pauli algebra $\mathcal{G}_{3,0}$ is the geometric algebra of three dimensional Euclidean space. The three basis vectors are σ_1, σ_2 and σ_3, which satisfy

$$\sigma_j \cdot \sigma_k = \delta_{jk}. \tag{13}$$

The pseudoscalar of the algebra is

$$i \equiv \sigma_1 \sigma_2 \sigma_3. \tag{14}$$

The algebra has eight basis elements

1	$\{\sigma_k\}$	$\{i\sigma_k\}$	i
1 scalar	3 vectors	3 bivectors	1 pseudoscalar
grade 0	grade 1	grade 2	grade 3

(15)

1.3 The Spacetime Algebra

The spacetime algebra (STA) is the geometric algebra of Minkowski spacetime, $\mathcal{G}_{1,3}$. The basis vectors are γ_0, γ_1, γ_2 and γ_3, which satisfy

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \mathrm{diag}(+ - - -). \tag{16}$$

We also define the bivectors,

$$\sigma_k = \gamma_k \gamma_0 \quad (k = 1, 2, 3), \tag{17}$$

and the pseudoscalar,

$$i = \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3. \tag{18}$$

These definitions are compatible with those for the Pauli algebra given above, because the Pauli algebra is a sub-algebra of the STA. The STA has 16 basis elements,

1	$\{\gamma_\mu\}$	$\{\sigma_k, i\sigma_k\}$	$\{i\gamma_\mu\}$	i
1 scalar	4 vectors	6 bivectors	4 trivectors	1 pseudoscalar
grade 0	grade 1	grade 2	grade 3	grade 4

(19)

Vectors in the STA have a projective relation to those in the Pauli algebra,

$$a\gamma_0 = a_0 + \mathbf{a} \tag{20}$$

where $a_0 = a \cdot \gamma_0$ and $\mathbf{a} = a \wedge \gamma_0$. The bivector $\mathbf{a}$ in the STA corresponds to the vector in the Pauli algebra relative to the frame defined by the unit timelike vector γ_0.

2 Representations of Rotation Groups

In this section we shall develop the formalism we need to describe higher-weighted representations of the rotation group $SO^+(p, q)$ and find its irreducible representations.

2.1 The Representations

Any rotation belonging to the group $SO^+(p, q)$ leaves the inner product of two vectors in the geometric algebra $\mathcal{G}_{p,q}$ unchanged. Such a rotation is described by a rotor R, an even-graded element of the algebra which satisfies

$$R\tilde{R} = 1. \tag{21}$$

Rotors are used to define the two types of representation of the the rotation group. The one-sided representations, which will be denoted as $\hat{\mathrm{U}}_R$, in which the rotor acts one-sidedly on a multivector ψ,

$$\hat{\mathrm{U}}_R[\psi] = R\psi, \tag{22}$$

and the two-sided representations, which will be denoted as $\hat{\mathrm{V}}_R$, in which the rotor R acts two-sidedly on a multivector M,

$$\hat{\mathrm{V}}_R[M] = RM\tilde{R}. \tag{23}$$

The group $Spin^+(p,q)$ of which the one-sided representations are members is a double covering of the group $SO^+(p,q)$. This follows from the fact that in the two-sided representations, the rotation represented by a rotor R could equally well be represented by $-R$,

$$\hat{\mathrm{V}}_{(-R)} = \hat{\mathrm{V}}_R,$$

but this does not hold for the one-sided representations.

We extend the carrier spaces of these two types of representations to multilinear functions of vector variables by defining their actions on k-linear functions to be

$$\hat{\mathrm{U}}_R[\psi(a_1,\ldots,a_k)] = R\psi(\tilde{R}a_1R,\ldots,\tilde{R}a_kR), \tag{24}$$

$$\hat{\mathrm{V}}_R[M(a_1,\ldots,a_k)] = RM(\tilde{R}a_1R,\ldots,\tilde{R}a_kR)\tilde{R}. \tag{25}$$

Most rotors can be written in the form

$$R = \exp\left(\tfrac{1}{2}C\right), \tag{26}$$

where C is a bivector. If C is a simple bivector then it defines the plane in which the rotation occurs. The algebra of the bivectors $\mathcal{G}^2_{p,q}$, which is closed under the commutator product (5), is equivalent to the Lie algebra $so(p,q)$ of the rotation group [4].

By writing the operator $\hat{\mathrm{U}}_R$ in a form similar to (26) we can define the linear operator $\hat{\mathrm{L}}_C$ which corresponds to C,

$$\hat{\mathrm{U}}_R \equiv \exp\left(\tfrac{1}{2}\hat{\mathrm{L}}_C\right). \tag{27}$$

Take for example a linear function $\psi(a)$ which transforms under $\hat{\mathrm{U}}_R$. In order to find the action of $\hat{\mathrm{L}}_C$ on $\psi(a)$ we write $C = \alpha B$, where α is a scalar and use (26) to substitute for R,

$$\hat{\mathrm{U}}_R[\psi(a)] = \exp\left(\tfrac{1}{2}\alpha\hat{\mathrm{L}}_B\right)\psi(a) = e^{\frac{1}{2}\alpha B}\psi(e^{-\frac{1}{2}\alpha B}ae^{\frac{1}{2}\alpha B}). \tag{28}$$

Differentiating with respect to α and setting $\alpha = 0$, we find that

$$\tfrac{1}{2}\hat{\mathrm{L}}_B\psi(a) = \tfrac{1}{2}B\psi(a) + \psi(a\cdot B). \tag{29}$$

The factors of $\frac{1}{2}$ in this equation arise naturally from the definition of the rotor (26). The set of operators $\hat{\mathrm{L}}_B$ form a representation of the Lie algebra which obeys the commutation relation

$$[\hat{\mathrm{L}}_B, \hat{\mathrm{L}}_C] = \hat{\mathrm{L}}_{[B,C]}, \tag{30}$$

or writing the same expression using the commutator product,

$$\hat{\mathrm{L}}_B \times \hat{\mathrm{L}}_C = \hat{\mathrm{L}}_{B\times C}. \tag{31}$$

The geometric algebra $\mathcal{G}_{p,q}$ has $\frac{1}{2}n(n-1)$ linearly-independent bivectors, thus $SO(p,q)$ has $\frac{1}{2}n(n-1)$ generators.

Extending $\hat{\mathrm{L}}_B$ to act on functions with k arguments is straightforward

$$\tfrac{1}{2}\hat{\mathrm{L}}_B[\psi(\ldots)] = \tfrac{1}{2}B\psi(\ldots) + \sum_{i=1}^{k}\psi(\ldots,a_i\cdot B,\ldots), \tag{32}$$

each argument makes a contribution of one term to the expression. A similar procedure gives the action of $\hat{\mathrm{L}}_B$ on functions which transform two-sidedly,

$$\tfrac{1}{2}\hat{\mathrm{L}}_B[M(\ldots)] = B\times M(\ldots) + \sum_{i=1}^{k}M(\ldots,a_i\cdot B,\ldots). \tag{33}$$

2.2 Invariant Subspaces

We must find the irreducible representations of the rotation group. To this end we now seek the invariant subspaces of the representations.

The simplest kind of invariant subspace is one in which elements have a definite symmetry among their arguments. An operator $\hat{\mathrm{T}}_{ij}$ can be defined which exchanges the ith and jth arguments of a function,

$$\hat{\mathrm{T}}_{ij}\psi(\ldots,a_i,\ldots,a_j,\ldots) = \psi(\ldots,a_j,\ldots,a_i,\ldots), \tag{34}$$

which is used to construct operators which symmetrise or antisymmetrise between the ith and jth arguments of a function.

$$\begin{aligned} \hat{\mathrm{S}}_{ij} &\equiv \tfrac{1}{2}(1+\hat{\mathrm{T}}_{ij}), \\ \hat{\mathrm{A}}_{ij} &\equiv \tfrac{1}{2}(1-\hat{\mathrm{T}}_{ij}). \end{aligned} \tag{35}$$

These operators commute with the action of the group, so if a function has symmetric or antisymmetric arguments, the action of the group does not change this property. Combinations of $\hat{\mathrm{S}}_{ij}$ and $\hat{\mathrm{A}}_{ij}$ can be constructed to represent all distinct symmetries between the arguments of a k-linear function. A general product of these operators will be denoted as $\hat{\mathrm{Y}}$. It can be shown that there is a one-to-one correspondence between these functions and the order-k Young tableaux.

A two-sided representation preserves the grade of an object which it transforms. This property can be shown by considering the action of $\hat{\mathrm{V}}_R$ on an r-vector $A_r = a_1\wedge a_2\wedge\ldots\wedge a_r$ where $a_1, a_2, \ldots, a_r$ are vectors. Since $R\tilde{R} = 1$,

$$\hat{\mathrm{V}}_R[A_r] = RA_r\tilde{R} = (Ra_1\tilde{R})\wedge(Ra_2\tilde{R})\wedge\ldots\wedge(Ra_r\tilde{R}) \tag{36}$$

which is also be an r-vector, so $\hat{\mathrm{V}}_R$ preserves the grades of all r-vectors, formally,

$$\hat{\mathrm{V}}_R[\mathcal{G}^r_{p,q}] = \mathcal{G}^r_{p,q}. \tag{37}$$

A one-sided representation does not preserve the grade of an object, however, since rotors are even-graded elements of the algebra, this representation separates the space of multivectors into even and odd grades, formally,

$$\begin{aligned} \hat{\mathrm{U}}_R[\mathcal{G}^+_{p,q}] &= \mathcal{G}^+_{p,q}, \\ \hat{\mathrm{U}}_R[\mathcal{G}^-_{p,q}] &= \mathcal{G}^-_{p,q}. \end{aligned} \tag{38}$$

We shall call an one-sided irreducible representation a spinor representation. A spinor representation can also accommodate an ideal structure. A spinor or spinor-valued function ψ can be multiplied on the right by an idempotent element of the algebra $E = E^2$ to project it into a right ideal of the algebra,

$$\psi = \psi E. \tag{39}$$

The ideal structure is preserved by the action of $\hat{\mathrm{U}}_R$ since the rotor acts only on the left.

Another property that is not changed by the action of the group is that of monogenicity. If a function $\mathcal{M}(a_1,\ldots,a_k)$ is monogenic, it has the property

$$\partial_{a_j}\mathcal{M}(a_1,\ldots,a_k) = 0 \tag{40}$$

for all $j = 1, 2, \ldots, k$. This property is unchanged by the action of a rotor. Take, for example, a function which transforms under $\hat{\mathrm{U}}_R$

$$\partial_{a_j} \hat{\mathrm{U}}_R[\mathcal{M}(a_1, \ldots, a_k)] = \partial_{a_j} R\mathcal{M}(a_1', \ldots, a_k') \tag{41}$$

where $a_j' = \tilde{R} a_j R$. Using the chain rule to rearrange the transformed function,

$$\begin{aligned} \partial_{a_j} R\mathcal{M}(a_1', \ldots, a_k') &= \partial_{a_j} (\tilde{R} a_j R) \cdot \partial_{a_j'} R\mathcal{M}(a_1', \ldots, a_k') \\ &= R\partial_{a_j'} \mathcal{M}(a_1', \ldots, a_k'). \end{aligned} \tag{42}$$

Thus, under the action of $\hat{\mathrm{U}}_R$, the monogenic properties of a function are not changed,

$$\hat{\mathrm{U}}_R[\partial_{a_j} \mathcal{M}(a_1, \ldots, a_k)] = R\partial_{a_j'} \mathcal{M}(a_1', \ldots, a_k'). \tag{43}$$

A similar argument applies to the double-sided transformation (25). From this, it can be shown that monogenicity is also preserved under the action of elements of the Lie algebra. We shall denote any monogenic function by the primary label $\mathcal{M}$.

The grade-invariance property of the two-sided transformation and monogenic invariance do not define mutually exclusive invariant subspaces and so cannot be used simultaneously to find the irreducible representations. We conclude that the irreducible representations of rotation groups consist of basis functions which have distinct symmetry among their arguments. For functions which transform one-sidedly, they correspond to particular terms in the monogenic decomposition and possibly have an ideal structure or, for functions which transform two-sidedly, they possess definite grade.

2.3 Monogenic decomposition

In the last section we proved that the monogenic properties of a function are not changed by the action of the rotation group or its Lie algebra. Arbitrary k-linear functions can be decomposed into monogenic and non-monogenic parts in a similar manner to Taylor decomposition of functions in one dimension.

An inner product can be defined between two k-linear functions, ξ and η,

$$\langle \xi | \eta \rangle = \langle \xi(\partial_{a_1}, \ldots, \partial_{a_k}) \tilde{\eta}(a_1, \ldots, a_k) \rangle \tag{44}$$

which is invariant under the action of both types of representation of the rotation group. A complex structure can always be added to this inner product. The functions are orthogonal if $\langle \xi | \eta \rangle = 0$. For simplicity it will be assumed that at least one of the functions is not null. A function $\psi(a_1, \ldots, a_k)$ can be decomposed into a monogenic part and a part orthogonal to it:

$$\psi(a_1, \ldots, a_k) = \mathcal{M}(a_1, \ldots, a_k) + M(a_1, \ldots, a_k) \tag{45}$$

where

$$\begin{aligned} M(a_1, \ldots, a_k) &= a_1 \psi_1(a_2, \ldots, a_k) + a_2 \psi_2(a_1, a_3, \ldots, a_k) + \ldots \\ &\qquad + a_k \psi_k(a_1, \ldots, a_{k-1}). \end{aligned} \tag{46}$$

The fact that each term in M is orthogonal to $\mathcal{M}$ follows easily from the definition of the inner product, since for any function $\xi(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k)$ which does not depend on a_j,

$$\begin{aligned} \langle a_j \xi(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k) | \mathcal{M}(a_1, \ldots, a_k) \rangle &= \\ \big\langle \xi(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k) | \partial_{a_j} \mathcal{M}(a_1, \ldots, a_k) \big\rangle &= 0 \end{aligned} \tag{47}$$

by the monogenic properties of $\mathcal{M}$. Recursive use of the partial decomposition produces the complete monogenic decomposition

$$\psi(a_1,\ldots,a_k)=\sum_{l=0}^{k}M_l(a_1,\ldots,a_k) \tag{48}$$

where

$$\begin{aligned} M_0(a_1,\ldots,a_k) &= \mathcal{M}(a_1,\ldots,a_k), \\ M_l(a_1,\ldots,a_k) &= \textstyle\sum_\pi a_{\pi(1)}\ldots a_{\pi(l)}\mathcal{M}_\pi(a_{\pi(l+1)},\ldots,a_{\pi(k)}), \end{aligned} \tag{49}$$

where the sum runs over all permutations π of the numbers $1,\ldots,k$. The decomposition is unique and the M_l are orthogonal

$$\langle M_j|M_l\rangle = 0 \quad \text{for} \quad j\neq l. \tag{50}$$

Under the action of $\hat{\mathrm{U}}_R$, the function M_l transforms as

$$\hat{\mathrm{U}}_R[M_l(a_1,\ldots,a_k)]=\sum_\pi a_{\pi(1)}\ldots,a_{\pi(l)}\hat{\mathrm{U}}_R[\mathcal{M}_\pi(a_{\pi(l+1)},\ldots,a_{\pi(k)})], \tag{51}$$

similarly for $\hat{\mathrm{V}}_R$. This shows that only the monogenic part of the function is transformed, the polynomial of vectors remaining unchanged.

We conclude that any arbitrary k-linear function can be constructed out of monogenic functions as in (48).

The symmetry between the arguments can be used to refine the monogenic decomposition,

$$\psi(a_1,\ldots,a_k)=\sum_{l=0}^{k}\sum_{\hat{\mathrm{Y}}}\hat{\mathrm{Y}}M_l(a_1,\ldots,a_k) \tag{52}$$

where M_l is defined as before (49) and the decomposition is summed over all possible symmetries among the arguments, $\hat{\mathrm{Y}}$. This is known as the Fischer decomposition [1].

3 Representations of SO(3)

In this section we will investigate the multilinear representations of the rotation group of Euclidean space, $SO(3)$.

3.1 Generators and Eigenstates

The Pauli algebra $\mathcal{G}_{3,0}$ has three linearly-independent bivectors, $i\sigma_1$, $i\sigma_2$ and $i\sigma_3$, so $SO(3)$ has three generators, for which the following notation is introduced.

$$\hat{\mathrm{J}}_k \equiv \tfrac{1}{2}\hat{\mathrm{L}}_{i\sigma_k} \tag{53}$$

for $k=1,2,3$. Since $i\sigma_k^2=-1$ these are compact generators. The $\{\hat{\mathrm{J}}_k\}$ obey the commutation relation

$$[\hat{\mathrm{J}}_k,\hat{\mathrm{J}}_l]=-\epsilon_{klm}\hat{\mathrm{J}}_m. \tag{54}$$

This relation differs from the usual convention because the generators are related to the bivectors of the Pauli algebra, rather than the vectors as in more conventional treatments.

The representations will be classified as eigenstates of the generators. To do this we need to introduce a complex structure, but rather than using the scalar imaginary we shall use an operation within the real geometric algebra to play the same role. The geometric operation that we choose will be denoted by $\hat{\jmath}$.

The spin of a particle is described by two quantum numbers which give the total spin and the spin in a particular plane. The 'total spin' operator is the Casimir operator of the group,

$$\hat{\mathbf{J}}^2 \equiv \hat{\mathrm{J}}_1{}^2 + \hat{\mathrm{J}}_2{}^2 + \hat{\mathrm{J}}_3{}^2. \tag{55}$$

Denoting this as the square of a vector is somewhat misleading, but we shall retain this notation for the sake of convention. One of the generators must be chosen to complete our commuting set of operators. The conventional choice is $\hat{\mathrm{J}}_3$ which measures the spin in the $i\sigma_3$ plane. The states are described in terms of the eigenstates of these two operators, the eigenvalues of which are s and m respectively. Acting on an abstract state $|sm\rangle$,

$$\hat{\mathbf{J}}^2 \,|sm\rangle = -s(s+1)\,|sm\rangle\,, \tag{56}$$

$$\hat{\mathrm{J}}_3 \,|sm\rangle = \hat{\jmath} m \,|sm\rangle\,. \tag{57}$$

The raising and lowering operators are the same as in conventional treatments,

$$\hat{\mathrm{J}}_\pm = \hat{\mathrm{J}}_1 \pm \hat{\jmath}\hat{\mathrm{J}}_2. \tag{58}$$

The spin-s irreducible representation will be referred to as $D(s)$. Products of the representations obey the familiar Clebsch-Gordon decomposition,

$$D(s_1) \otimes D(s_2) \equiv D(s_1+s_2) \oplus \ldots \oplus D(|s_1-s_2|). \tag{59}$$

3.2 The Spin-$\frac{1}{2}$ Representation

The spin-$\frac{1}{2}$ representation is the fundamental spinor representation which could take values in the even or odd sectors of the algebra. It is more convenient to let spinor ψ take values in the even sub-algebra $\mathcal{G}^+_{3,0}$, of which the basis elements are

$$\{1, i\sigma_1, i\sigma_2, i\sigma_3\}. \tag{60}$$

Translating the conventional representation into geometric algebra [5, 6] it is found that the spin-$\frac{1}{2}$ representation is complexified by right-multiplication by $i\sigma_3$

$$\hat{\jmath}\psi \equiv \psi i\sigma_3. \tag{61}$$

The even sub-algebra is closed under this operation.

Using the raising and lowering operators (58) the spin-up and spin-down basis states $\chi_\pm$ are found to be

$$\begin{aligned} \chi_+ &= 1, \\ \chi_- &= i\sigma_1. \end{aligned} \tag{62}$$

Coefficients of these states are defined over the 'complex' field, $\{1, \hat{\jmath}\}$ so a general state takes values over the whole even sub-algebra. The convention for the basis states used here differs slightly from those of Doran *et al.* [6] since we have adopted a different convention for the generators.

3.3 The Spin-1 Representation

The vector space of the Pauli algebra $\mathcal{G}^1_{3,0}$ has three basis vectors which could form a basis for the spin-1 representation $D(1)$. We cannot use the same complexification operation as the spin-$\frac{1}{2}$ representation here, but we note that the space of bivectors $\mathcal{G}^2_{3,0}$ could also form a suitable basis for $D(1)$. So to complexify this representation we employ an *ad hoc* construction: the pseudoscalar $i = \sigma_1\sigma_2\sigma_3$, which has a negative square, is used to complexify the representation,

$$\jmath A \equiv iA. \tag{63}$$

A consequence of this is that the $D(1)$ representation is vector- and bivector-valued and has 3 complex or 6 real degrees of freedom. The complexification operations we have defined for the spin-$\frac{1}{2}$ and the spin-1 representations at first sight appear to be incompatible, but later we shall show that they can be reconciled.

Using the raising and lowering operators the three basis states A_{sm} can be constructed.

$$\begin{aligned} A_{1,1} &= \tfrac{1}{\sqrt{2}}(\sigma_1 + i\sigma_2) \\ A_{1,0} &= -i\sigma_3 \\ A_{1,-1} &= \tfrac{1}{\sqrt{2}}(\sigma_1 - i\sigma_2) \end{aligned} \tag{64}$$

the coefficients of which take values over the 'complex' field $\{1, \jmath\}$.

3.4 Relating Spin-$\frac{1}{2}$ and Spin-1 Representations

Here we shall attempt to relate the spin-$\frac{1}{2}$ representations to the spin-1 representations and reconcile the differing complexification operators. This process will be very useful later in constructing higher-spin representations.

We can attempt to construct a spin-1 state from two spin-$\frac{1}{2}$ states, ψ_1 and ψ_2, in the form

$$M = \psi_1 \Gamma \tilde{\psi}_2, \tag{65}$$

where Γ is a fixed multivector to be determined. If the spinors are transformed under $\hat{\mathrm{U}}_R$ we find that,

$$\psi_1 \Gamma \tilde{\psi}_2 \xmapsto{R} \hat{\mathrm{U}}_R[\psi_1]\Gamma\hat{\mathrm{U}}_R[\tilde{\psi}_2] = R\psi_1\Gamma(R\psi_2)\tilde{} = R\psi_1\Gamma\tilde{\psi}_2\tilde{R} = \hat{\mathrm{V}}_R[\psi_1\Gamma\tilde{\psi}_2], \tag{66}$$

so M transforms like an integral-spin representation. Taking the product of the representations, M will transform as

$$D(\tfrac{1}{2}) \otimes D(\tfrac{1}{2}) = D(1) \oplus D(0). \tag{67}$$

Since Γ is a multivector, M will take values in all possible grades of the algebra. The scalar and pseudoscalar parts can be identified with the spin-0 representation, since these grades are invariant under rotation,

$$\phi = \left\langle \psi_1 \Gamma \tilde{\psi}_2 \right\rangle_{0,3} \tag{68}$$

and the vector and bivector parts are identified with the spin-1 representation as in the previous section,

$$A = \left\langle \psi_1 \Gamma \tilde{\psi}_2 \right\rangle_{1,2}. \tag{69}$$

Inserting the spin-$\frac{1}{2}$ basis states into (69), we find that Γ is

$$\Gamma \equiv (1+\sigma_3)\sigma_1. \tag{70}$$

The presence of the idempotent $\frac{1}{2}(1+\sigma_3)$ in Γ ensures that the differing complexification operators defined earlier are consistent.

3.5 Half-Integral-Spin Representations

To find a general half-integral-spin representation, we will be looking for spinor-valued functions which transform under $\hat{\mathrm{U}}_R$. The complexification operator for these functions is the same as that for the spinor representation.

Consider a spinor-valued linear function, $\psi(a)$, which transforms under $\hat{\mathrm{U}}_R$. The monogenic decomposition (48) can be used to analyse the transformation properties of the function,

$$\psi(a) = \mathcal{M}(a) + a\chi \tag{71}$$

where $\mathcal{M}(a)$ is the monogenic part of the function, and the second term is the residual component which is defined by

$$\chi = \tfrac{1}{3}\partial_a \psi(a) \tag{72}$$

which will take values in the odd-graded part of the algebra $\mathcal{G}_{3,0}^{-}$. This function will transform as the

$$D(\tfrac{1}{2}) \otimes D(1) = D(\tfrac{3}{2}) \oplus D(\tfrac{1}{2}) \tag{73}$$

representation. Under the action of $\hat{\mathrm{U}}_R$, $\psi(a)$ transforms to

$$R\psi(\tilde{R}aR) = R\mathcal{M}(\tilde{R}aR) + aR\chi. \tag{74}$$

It can be seen that the second term transforms with one rotor like the spin-$\frac{1}{2}$ representation, so the monogenic part of the function can be identified with the spin-$\frac{3}{2}$ representation.

To construct the higher-spin representations we will need some 'complex'-valued functions of a vector variable from which to construct the contributions of the arguments,

$$\begin{aligned}
\phi_1(a) &= \tfrac{1}{\sqrt{2}}(a\cdot\sigma_1 + a\cdot\sigma_2\hat{\jmath}),\\
\phi_0(a) &= -a\cdot\sigma_3\hat{\jmath},\\
\phi_{-1}(a) &= \tfrac{1}{\sqrt{2}}(a\cdot\sigma_1 - a\cdot\sigma_2\hat{\jmath}).
\end{aligned} \tag{75}$$

These are closely related to the spin-1 basis states. They have been written using $\hat{\jmath}$ since they will be useful in constructing both integral and half-integral-spin states.

The spin-$\frac{3}{2}$ multiplet can be constructed from these functions and the spin-$\frac{1}{2}$ basis states $\chi_\pm$ (62),

$$\begin{aligned}
\psi_{\frac{3}{2},\frac{3}{2}}(a) &= \chi_+\phi_1(a),\\
\psi_{\frac{3}{2},\frac{1}{2}}(a) &= \tfrac{1}{\sqrt{3}}(\chi_-\phi_1(a) + \sqrt{2}\chi_+\phi_0(a)),\\
\psi_{\frac{3}{2},-\frac{1}{2}}(a) &= \tfrac{1}{\sqrt{3}}(\chi_+\phi_{-1}(a) + \sqrt{2}\chi_-\phi_0(a)),\\
\psi_{\frac{3}{2},-\frac{3}{2}}(a) &= \chi_-\phi_{-1}(a).
\end{aligned} \tag{76}$$

These four basis states are monogenic functions. The associated spin-$\frac{1}{2}$ doublet consists of the residual functions from the monogenic decomposition,

$$\begin{aligned}
\psi_{\frac{1}{2},\frac{1}{2}}(a) &= \tfrac{1}{\sqrt{3}}(\sqrt{2}\chi_-\phi_1(a) - \chi_+\phi_0(a)) &&= \tfrac{1}{\sqrt{3}}ai,\\
\psi_{\frac{1}{2},-\frac{1}{2}}(a) &= \tfrac{1}{\sqrt{3}}(-\sqrt{2}\chi_+\phi_{-1}(a) + \chi_-\phi_0(a)) &&= -\tfrac{1}{\sqrt{3}}a\sigma_1.
\end{aligned} \tag{77}$$

These functions are orthonormal under the functional inner product (44).

Now consider a spinor-valued bilinear function, $\psi(a,b)$, which will transform under $\hat{\mathrm{U}}_R$ as

$$D(\tfrac{1}{2})\otimes D(1)\otimes D(1) = D(\tfrac{5}{2})\oplus 2D(\tfrac{3}{2})\oplus 2D(\tfrac{1}{2}) \tag{78}$$

The monogenic decomposition of $\psi(a,b)$ is

$$\psi(a,b) = \mathcal{M}(a,b) + a\mathcal{M}'(b) + b\mathcal{M}''(a) + ab\xi_1 + ba\xi_2 \tag{79}$$

The symmetry between the arguments need not be taken into account since there cannot be any antisymmetric monogenic functions in $\mathcal{G}_{3,0}$. If there were such a monogenic $\mathcal{M}_A(a,b)$, it could be written as a function of a bivector $\mathcal{M}_A(a\wedge b)$. By linearity,

$$\partial_a \mathcal{M}_A(a\wedge b) = \partial_a (a\wedge b)\cdot\partial_B \mathcal{M}_A(B) = 0, \tag{80}$$

where the chain rule has been applied. Expanding B over the bivector basis elements, this becomes

$$b\cdot i\sigma_1 \mathcal{M}_A(i\sigma_1) + b\cdot i\sigma_2 \mathcal{M}_A(i\sigma_2) + b\cdot i\sigma_3 \mathcal{M}_A(i\sigma_3) = 0, \tag{81}$$

which must hold for all values of b. It is easy to show that this implies that each of the components of $\mathcal{M}_A(a\wedge b)$ vanishes. It follows that in 3 dimensions a k-linear monogenic cannot be antisymmetric in any two of its arguments, it therefore must be totally symmetric.

Under the action of $\hat{\mathrm{U}}_R$, $\psi(a,b)$ transforms to

$$\begin{aligned} R\psi(\tilde{R}aR,\tilde{R}bR) &= R\mathcal{M}(\tilde{R}aR,\tilde{R}bR) + aR\mathcal{M}'(\tilde{R}bR) \\ &\quad + bR\mathcal{M}''(\tilde{R}aR) + abR\xi_1 + baR\xi_2 \end{aligned} \tag{82}$$

Again it can be seen that it is the monogenic part which behaves as the irreducible representation with the highest spin, in this case spin-$\frac{5}{2}$. There are also two spin-$\frac{3}{2}$ and two spin-$\frac{1}{2}$ representations, as predicted in (78). The $m=\frac{5}{2}$ state will be

$$\psi_{\frac{5}{2},\frac{5}{2}}(a,b) = \chi_+\phi_1(a)\phi_1(b), \tag{83}$$

and the remaining basis states of the $D(\frac{5}{2})$ representation can be constructed using the lowering operator.

The properties of a function which corresponds to a general half-integral-spin irreducible representation can now be deduced. Consider a spinor-valued k-linear function

$$\psi(a_1,\ldots,a_k) \tag{84}$$

which transforms under $\hat{\mathrm{U}}_R$ as

$$D(\tfrac{1}{2})\otimes \underbrace{D(1)\otimes\ldots\otimes D(1)}_{k \text{ times}} = D(k+\tfrac{1}{2})\oplus D(k-\tfrac{1}{2})\oplus\ldots\oplus D(\tfrac{1}{2}). \tag{85}$$

The monogenic part of this function

$$\mathcal{M}(a_1,\ldots,a_k), \tag{86}$$

which must be totally symmetric, will correspond to the $D(k+\frac{1}{2})$ irreducible representation. The $m=k+\frac{1}{2}$ basis state will be

$$\psi_{k+\frac{1}{2},k+\frac{1}{2}}(a_1,\ldots,a_k) = \chi_+\phi_1(a_1)\ldots\phi_1(a_k). \tag{87}$$

The rest of the spin-$(k+\frac{1}{2})$ basis states can be constructed from this using the lowering operator.

3.6 Integral-Spin Representations

In the same way as we constructed the spin-1 representation from the spin-$\frac{1}{2}$ representation, we can construct any integral-spin representation from the half-integral-spin representations deduced above. A linear function which transforms under $\hat{\mathrm{V}}_R$ can be constructed from a spinor-valued linear function $\psi_1(a)$ and a spinor ψ_2.

$$M(a) = \psi_1(a)\Gamma\tilde{\psi_2} \tag{88}$$

where $\Gamma = (1+\sigma_3)\sigma_1$ as before (70). $M(a)$ will be multivector-valued, and it will belong to the

$$D(\tfrac{1}{2}) \otimes D(1) \otimes D(\tfrac{1}{2}) \tag{89}$$

representation. If $\psi_1(a)$ is restricted to be monogenic $\mathcal{M}(a)$, then the function

$$M'(a) = \mathcal{M}(a)\Gamma\tilde{\psi_2} \tag{90}$$

belongs to the

$$D(\tfrac{3}{2}) \otimes D(\tfrac{1}{2}) = D(2) \oplus D(1) \tag{91}$$

representation. This can be restricted further by selecting the vector and bivector part

$$A(a) = \langle \mathcal{M}(a)\Gamma\tilde{\psi_2}\rangle_{1,2}. \tag{92}$$

which will transform as the $D(2)$ irreducible representation. The scalar and pseudoscalar part of $M'(a)$ will transform as D(1). Inserting the $D(\frac{3}{2})$ and $D(\frac{1}{2})$ basis states from (62) and (76) the basis states of the D(2) irreducible representation are

$$\begin{aligned}
A_{2,2}(a) &= A_1\phi_1(a), \\
A_{2,1}(a) &= \tfrac{1}{\sqrt{2}}(A_1\phi_0(a) + A_0\phi_1(a)), \\
A_{2,0}(a) &= \tfrac{1}{\sqrt{6}}(A_1\phi_{-1}(a) + A_{-1}\phi_1(a) + 2A_0\phi_0(a)), \\
A_{2,-1}(a) &= \tfrac{1}{\sqrt{2}}(A_{-1}\phi_0(a) + A_0\phi_{-1}(a)), \\
A_{2,-2}(a) &= A_{-1}\phi_{-1}(a).
\end{aligned} \tag{93}$$

where A_1, A_0 and A_{-1} are shorthand for the $m=1$, $m=0$ and $m=-1$ basis states of the $D(1)$ representation respectively.

This procedure can now be generalised to find the properties of a function which corresponds to any integral-spin irreducible representation. Consider a k-linear function which transforms under $\hat{\mathrm{V}}_R$, constructed from a k-linear spinor-valued function $\psi_1(a_1,\ldots,a_k)$ and a spinor ψ_2,

$$M(a_1,\ldots,a_k) = \psi_1(a_1,\ldots,a_k)\Gamma\tilde{\psi_2}. \tag{94}$$

It will transform as

$$D(\tfrac{1}{2}) \otimes \underbrace{D(1) \otimes \ldots \otimes D(1)}_{k \text{ times}} \otimes D(\tfrac{1}{2}) = D(k+1) \oplus D(k) \oplus \ldots \oplus D(0) \tag{95}$$

If $\psi_1(a_1,\ldots,a_k)$ is restricted to be a monogenic then the function

$$M'(a) = \mathcal{M}(a_1,\ldots,a_k)\Gamma\tilde{\psi_2} \tag{96}$$

will transform as the

$$D(k+\tfrac{1}{2}) \otimes D(\tfrac{1}{2}) = D(k+1) \oplus D(k) \tag{97}$$

representation. Again if this is restricted to the vector and bivector parts, the resulting function

$$A(a_1, \ldots, a_k) = \langle \mathcal{M}(a_1, \ldots, a_k)\Gamma\tilde{\psi}_2\rangle_{1,2} \tag{98}$$

corresponds to the $D(k+1)$ irreducible representation.

We have now identified the functions which correspond to all of the irreducible representations of $SO(3)$ and in the process discovered many monogenic functions in the Pauli algebra.

4 Representations of the Lorentz Group

We now turn our attention the the proper Lorentz group $SO^+(1,3)$. The representations we find in this section will be useful in describing relativistic wave equations.

4.1 Generators

The spacetime algebra has six linearly-independent bivectors, $\{\sigma_k, i\sigma_k\}$, therefore the Lorentz group has six generators, for which we introduce the following shorthand notation,

$$\hat{\mathrm{J}}_k = \tfrac{1}{2}\hat{\mathrm{L}}_{i\sigma_k}, \quad \hat{\mathrm{K}}_k = \tfrac{1}{2}\hat{\mathrm{L}}_{\sigma_k}, \tag{99}$$

where $k = 1, 2, 3$. These operators obey the commutation relations

$$[\hat{\mathrm{J}}_i, \hat{\mathrm{J}}_j] = -\epsilon_{ijk}\hat{\mathrm{J}}_k, \quad [\hat{\mathrm{K}}_i, \hat{\mathrm{K}}_j] = \epsilon_{ijk}\hat{\mathrm{J}}_k, \quad [\hat{\mathrm{J}}_i, \hat{\mathrm{K}}_j] = \epsilon_{ijk}\hat{\mathrm{K}}_k. \tag{100}$$

The $\{\hat{\mathrm{J}}_k\}$ form a closed sub-algebra, which is identical to the Lie algebra of $SO(3)$ in the previous section. Again these relations differ slightly from conventional treatments.

4.2 Classification of Irreducible Representations

We shall introduce complex structure to the Lorentz group within the real STA, using the operator $\hat{\jmath}$ to stand for the complexification operation and deducing its action on the representations. The complex structure can be used to define two sets of generators from the generators of the Lorentz group [7]

$$\hat{\mathrm{A}}_k = \tfrac{1}{2}(\hat{\mathrm{J}}_k + \hat{\jmath}\hat{\mathrm{K}}_k), \quad \hat{\mathrm{B}}_k = \tfrac{1}{2}(\hat{\mathrm{J}}_k - \hat{\jmath}\hat{\mathrm{K}}_k), \tag{101}$$

where $k = 1, 2, 3$. These operators have the following commutation relations

$$[\hat{\mathrm{A}}_k, \hat{\mathrm{A}}_l] = -\epsilon_{klm}\hat{\mathrm{A}}_m, \quad [\hat{\mathrm{B}}_k, \hat{\mathrm{B}}_l] = -\epsilon_{klm}\hat{\mathrm{B}}_m, \quad [\hat{\mathrm{A}}_k, \hat{\mathrm{B}}_l] = 0. \tag{102}$$

The operators form two uncoupled sets of generators which individually behave like the generators of $SO(3)$. The group generated by these uncoupled operators will therefore behave as a direct product of two copies of $SO(3)$. If s_a is the spin of the $\hat{\mathrm{A}}_k$ irreducible representation and s_b is the spin of the $\hat{\mathrm{B}}_k$ irreducible representation then their direct product space, which can be described by a pair of spin weights (s_a, s_b), will have $(2s_a + 1)(2s_b + 1)$ complex degrees of freedom. An eigenstate will be equivalent to the product of two $SO(3)$ eigenstates

$$|s_a, s_b, m_a, m_b\rangle \cong |s_a, m_a\rangle \otimes |s_b, m_b\rangle\,. \tag{103}$$

The operators

$$\begin{aligned}\hat{\mathbf{A}}^2 &= \hat{\mathrm{A}}_1{}^2 + \hat{\mathrm{A}}_2{}^2 + \hat{\mathrm{A}}_3{}^2, && (104)\\ \hat{\mathbf{B}}^2 &= \hat{\mathrm{B}}_1{}^2 + \hat{\mathrm{B}}_2{}^2 + \hat{\mathrm{B}}_3{}^2, && (105)\end{aligned}$$

are the Casimir operators of the group, where the 'vector' notation has again been used. These can be written in terms of the $\hat{\mathrm{J}}_k$ and $\hat{\mathrm{K}}_k$ operators by taking the sum and difference,

$$\begin{aligned}\hat{\mathbf{A}}^2 + \hat{\mathbf{B}}^2 &= \tfrac{1}{2}(\hat{\mathbf{J}}^2 - \hat{\mathbf{K}}^2), && (106)\\ \hat{\mathbf{A}}^2 - \hat{\mathbf{B}}^2 &= -\jmath\hat{\mathbf{J}}\cdot\hat{\mathbf{K}}. && (107)\end{aligned}$$

It will be found that a particular irreducible representation can be distinguished from other representations with the same $s = s_a + s_b$ by its grade or algebraic properties. We shall refer to all such representations as spin-s representations, for example, $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $(1,0) \oplus (0,1)$ are both spin-1 representations.

Because the representations have a direct product structure, the Lorentz group analogue of the Clebsh-Gordon decomposition is quite simple,

$$(s_{a_1}, s_{b_1}) \otimes (s_{a_2}, s_{b_2}) = \sum_{s_a, s_b} \oplus (s_a, s_b) \tag{108}$$

where the sum runs independently over the values

$$\begin{aligned}s_a &= s_{a_1} + s_{a_2}, s_{a_1} + s_{a_2} - 1, \ldots, |s_{a_1} - s_{a_2}|,\\ s_b &= s_{b_1} + s_{b_2}, s_{b_1} + s_{b_2} - 1, \ldots, |s_{b_1} - s_{b_2}|.\end{aligned} \tag{109}$$

Now we have identified the irreducible representations, we can classify their basis states. The basis states will be expressed as eigenstates of the operators $\hat{\mathrm{J}}_3$ and $\hat{\mathrm{K}}_3$ with eigenvalues m and n respectively, because these are related to simple bivector generators. These states will still be eigenstates of $\hat{\mathrm{A}}_3$ and $\hat{\mathrm{B}}_3$, the eigenvalues are related by

$$\begin{aligned}m &= m_a + m_b, && (110)\\ n &= m_a - m_b. && (111)\end{aligned}$$

The basis states are thus characterised by four spin quantum numbers, s_a, s_b, m and n. Operating on an eigenstate $|s_a s_b m n\rangle$ we find

$$\begin{aligned}\hat{\mathbf{A}}^2\,|s_a s_b m n\rangle &= -s_a(s_a+1)\,|s_a s_b m n\rangle && (112)\\ \hat{\mathbf{B}}^2\,|s_a s_b m n\rangle &= -s_b(s_b+1)\,|s_a s_b m n\rangle && (113)\\ \hat{\mathrm{J}}_3\,|s_a s_b m n\rangle &= \jmath m\,|s_a s_b m n\rangle && (114)\\ \hat{\mathrm{K}}_3\,|s_a s_b m n\rangle &= n\,|s_a s_b m n\rangle && (115)\end{aligned}$$

The raising and lowering operators

$$\begin{aligned}\hat{\mathrm{A}}_\pm &= \hat{\mathrm{A}}_1 \pm \jmath\hat{\mathrm{A}}_2\\ \hat{\mathrm{B}}_\pm &= \hat{\mathrm{B}}_1 \pm \jmath\hat{\mathrm{B}}_2\end{aligned} \tag{116}$$

which raise and lower the m_a and m_b quantum numbers respectively can be used to move between the basis states of the representation, which is show in terms of the m and n states in Figure 1.

$$\begin{tikzcd}
\left|s_a, s_b, m+1, n-1\right\rangle & & \left|s_a, s_b, m+1, n+1\right\rangle \\
& \left|s_a, s_b, m, n\right\rangle \arrow[ul, "\hat{\mathrm{B}}_+"'] \arrow[ur, "\hat{\mathrm{A}}_+"] \arrow[dl, "\hat{\mathrm{A}}_-"'] \arrow[dr, "\hat{\mathrm{B}}_-"] & \\
\left|s_a, s_b, m-1, n-1\right\rangle & & \left|s_a, s_b, m-1, n+1\right\rangle
\end{tikzcd}$$

Figure 1: Raising and Lowering Operators for Spin-s Irreducible Representation of the Lorentz Group.

An alternative, incomplete method of classifying the irreducible representations of the Lorentz group is to decompose them into representations of the subgroup $SO(3)$, as eigenstates of $\hat{\mathbf{J}}^2$. The (s_a, s_b) representation contains the

$$D(s_a) \otimes D(s_b) \quad = \quad D(s_a + s_b) \oplus \ldots \oplus D(|s_a - s_b|) \tag{117}$$

representations of $SO(3)$ and the $(s_a, s_b) \oplus (s_b, s_a)$ representation will contain two copies of each of these subgroup representations. The operator $\hat{\mathrm{J}}_3$ will be used to find the eigenstates of these subgroup representations. The alternative quantum numbers which corresponds to the eigenstates of $\hat{\mathbf{J}}^2$ and $\hat{\mathrm{J}}_3$ are j and m,

$$\begin{aligned}
(\hat{\mathbf{J}}^2 - \hat{\mathbf{K}}^2) \left|s_a s_b j m\right\rangle &= -2(s_a(s_a+1) + s_b(s_b+1)) \left|s_a s_b j m\right\rangle && (118) \\
\hat{\mathbf{J}}^2 \left|s_a s_b j m\right\rangle &= -j(j+1) \left|s_a s_b j m\right\rangle && (119) \\
\hat{\mathrm{J}}_3 \left|s_a s_b j m\right\rangle &= \hat{\jmath} m \left|s_a s_b j m\right\rangle && (120)
\end{aligned}$$

These are not a complete set of operators because $\hat{\mathbf{J}}^2$ does not commute with $\hat{\mathbf{J}} \cdot \hat{\mathbf{K}}$. Also $\hat{\mathrm{K}}_3$ commutes with $\hat{\mathrm{J}}_3$ but not with $\hat{\mathbf{J}}^2$, so it does not behave as a lowering or raising operator, but mixes the $\left|s_a s_b j m\right\rangle$ state with all the other possible j eigenstates leaving the eigenvalue s unchanged (Figure 2).

4.3 The Spin-$\frac{1}{2}$ Representation

The Lorentz group has one spin-$\frac{1}{2}$ irreducible representation, $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$. Translating the Dirac spinor into the STA we find it most convenient if it takes values in the even subalgebra [5, 6]. The complexification operation is identical to that of the non-relativistic spin-$\frac{1}{2}$ representation, multiplication on the right by $i\sigma_3$.

$$\hat{\jmath} \psi \equiv \psi i \sigma_3. \tag{121}$$

The basis states ψ_{mn} of this representation are,

$$\begin{array}{rclrcl}
\psi_{\frac{1}{2},\frac{1}{2}} &=& \frac{1}{2}(1+\sigma_3) & \psi_{\frac{1}{2},-\frac{1}{2}} &=& \frac{1}{2}(1-\sigma_3) \\
\psi_{-\frac{1}{2},\frac{1}{2}} &=& i\sigma_1 \frac{1}{2}(1-\sigma_3) & \psi_{-\frac{1}{2},-\frac{1}{2}} &=& i\sigma_1 \frac{1}{2}(1+\sigma_3)
\end{array} \tag{122}$$

$$
\begin{tikzcd}
 & {|s_a, s_b, j, m+1\rangle} & \\
{|s_a, s_b, j-1, m\rangle} & {|s_a, s_b, j, m\rangle} \arrow[l, "\hat{\mathrm{K}}_3"'] \arrow[r, "\hat{\mathrm{K}}_3"] \arrow[u, "\hat{\mathrm{J}}_+"'] \arrow[d, "\hat{\mathrm{J}}_-"] & {|s_a, s_b, j+1, m\rangle} \\
 & {|s_a, s_b, j, m-1\rangle} &
\end{tikzcd}
$$

Figure 2: Alternative Raising and Lowering Operators for Spin-s Irreducible Representation of the Lorentz Group.

These basis states have an ideal structure, therefore they correspond most closely to the Weyl representation.

Decomposing the $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$ irreducible representation in terms of the $SO(3)$ subgroup,

$$(\tfrac{1}{2},0) \oplus (0,\tfrac{1}{2}) \cong D(\tfrac{1}{2}) \oplus D(\tfrac{1}{2}), \tag{123}$$

there are two $D(\frac{1}{2})$ doublets which are transformed into each other by the action of $\hat{\mathrm{K}}_3$,

$$\begin{array}{rclcrcl} \chi_+ & = & 1, & \overset{\hat{\mathrm{K}}_3}{\longleftrightarrow} & \xi_+ & = & \sigma_3, \\ \chi_- & = & i\sigma_1, & & \xi_- & = & \sigma_2. \end{array} \tag{124}$$

The $\chi_\pm$ doublet is identical to the $D(\frac{1}{2})$ representation of $SO(3)$, the $\xi_\pm$ doublet is new, coming from the action of $\hat{\mathrm{K}}_3$ on the $\chi_\pm$ doublet.

4.4 The Spin-1 Representations

The Lorentz group has two spin-1 representations, $\left(\frac{1}{2},\frac{1}{2}\right)$ and $(1,0) \oplus (0,1)$. Like the integral-spin representations of $SO(3)$, we can construct the spin-1 representations from the spinor representation. We use the same form,

$$M = \psi_1 \Gamma \tilde{\psi_2}. \tag{125}$$

where Γ is an arbitrary multivector to be chosen later. Since the STA and the Pauli Algebra share the same pseudoscalar we can use the same construction as before to complexify these representations,

$$\hat{\jmath} M \equiv iM. \tag{126}$$

In general, M will transform under $\hat{\mathrm{V}}_R$ as the

$$\left[\left(\tfrac{1}{2},0\right) \oplus \left(0,\tfrac{1}{2}\right)\right] \otimes \left[\left(\tfrac{1}{2},0\right) \oplus \left(0,\tfrac{1}{2}\right)\right] = 2\,(0,0) \oplus 2\left(\tfrac{1}{2},\tfrac{1}{2}\right) \oplus \left[(1,0) \oplus (0,1)\right] \tag{127}$$

representation.

The $\left(\frac{1}{2},\frac{1}{2}\right)$ irreducible representation is obtained by choosing

$$\Gamma = \gamma_1 + i\gamma_2 \tag{128}$$

which restricts M to be vector- and trivector-valued. Inserting the $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$ basis states from (122) it is found that the basis states A_{mn} of the $\left(\frac{1}{2},\frac{1}{2}\right)$ representation are

$$\begin{array}{rclrcl} A_{0,1} &=& \gamma_0+\gamma_3, & A_{0,-1} &=& \gamma_0-\gamma_3, \\ A_{1,0} &=& \gamma_1+i\gamma_2, & A_{1,-1,0} &=& \gamma_1-i\gamma_2, \end{array} \tag{129}$$

which is identical to the null tetrad $(l,n,m,\bar{m})$ of Penrose and Rindler [8, 9].

If Γ is chosen to be

$$\Gamma = \sigma_1 + i\sigma_2 \tag{130}$$

then M will be even-valued. Further restricting (125) to the symmetric part

$$F = \tfrac{1}{2}(\psi_1\Gamma\tilde{\psi_2} + \psi_2\Gamma\tilde{\psi_1}) \tag{131}$$

then F will be bivector-valued, which corresponds to the $(1,0)\oplus(0,1)$ irreducible representation.

We have chosen the operation (126) to perform the complexification. Multiplication by the pseudoscalar, i, also generates the duality operation, so by this process we only obtain the self-dual set of basis states for F. Inserting the spin-$\frac{1}{2}$ basis states from (122) the basis states F_{mn} are found to be

$$\begin{array}{rcl} F_{1,1} &=& \sigma_1+i\sigma_2, \\ F_{0,0} &=& i\sigma_3, \\ F_{-1,-1} &=& \sigma_1-i\sigma_2. \end{array} \tag{132}$$

Instead, if we had chosen $\jmath M = -iM$ to be the complexification operation, which is consistent with choosing $\Gamma = \sigma_1 - i\sigma_2$, we would obtain the anti-self-dual set of basis states.

The corresponding antisymmetric part of (125)

$$\phi = \tfrac{1}{2}(\psi_1\Gamma\tilde{\psi_2} - \psi_2\Gamma\tilde{\psi_1}) \tag{133}$$

is scalar- and pseudoscalar-valued, which is invariant under the action of $\hat{\mathrm{V}}_R$. This corresponds to the scalar $(0,0)$ irreducible representation.

Above, identifying the grades of a multivector with the representations, we have only obtained half of the irreducible representations that should be contained in the decomposition of the complete multivector representation (127). This is a consequence of the minimalist route we have chosen, introducing complexification as a geometric operation.

4.5 Half-Integral-Spin Representations

We now search for half-integral-spin irreducible representations, which we expect to be multilinear spinor-valued functions. Consider a linear spinor-valued function $\psi(a)$ which transforms under $\hat{\mathrm{U}}_R$. Its monogenic decomposition is

$$\psi(a) = \mathcal{M}(a) + a\chi \tag{134}$$

where the first term is monogenic and the second is the residual term in which χ defined is by $\chi = \frac{1}{4}\partial_a\psi(a)$. The function $\psi(a)$ belongs to the

$$\left[\left(\tfrac{1}{2},0\right)\oplus\left(0,\tfrac{1}{2}\right)\right]\otimes\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \left[\left(1,\tfrac{1}{2}\right)\oplus\left(\tfrac{1}{2},1\right)\right]\oplus\left[\left(\tfrac{1}{2},0\right)\oplus\left(0,\tfrac{1}{2}\right)\right] \tag{135}$$

representation. The second term of the monogenic decomposition (134) transforms like the $\big(\frac{1}{2},0\big)\oplus\big(0,\frac{1}{2}\big)$ irreducible representation so the monogenic component is identified with the $\big(1,\frac{1}{2}\big)\oplus\big(\frac{1}{2},1\big)$ irreducible representation, a 'spin-$\frac{3}{2}$' representation:

$$
\begin{aligned}
\big(1,\tfrac{1}{2}\big)\oplus\big(\tfrac{1}{2},1\big) \quad &\leftrightarrow \quad \mathcal{M}(a),\\
\big(\tfrac{1}{2},0\big)\oplus\big(0,\tfrac{1}{2}\big) \quad &\leftrightarrow \quad \chi.
\end{aligned}
$$

The monogenic functions which form the basis states for the $\big(1,\frac{1}{2}\big)\oplus\big(\frac{1}{2},1\big)$ representation can be constructed from the spin-$\frac{1}{2}$ basis states and 'complex'-valued linear functions derived from the $\big(\frac{1}{2},\frac{1}{2}\big)$ representation. For example,

$$
\psi_{\frac{1}{2},\frac{3}{2}}(a) = \psi_{\frac{1}{2},\frac{1}{2}}\,\phi_{0,1}(a) \tag{136}
$$

is monogenic, where $\phi_{0,1}(a) = a\cdot(\gamma_0+\gamma_3)$.

Now consider a bilinear spinor-valued function $\psi(a,b)$ which transforms under $\hat{\mathrm{U}}_R$. Its monogenic decomposition is

$$
\psi(a,b) = \mathcal{M}(a,b) + a\mathcal{M}'(b) + b\mathcal{M}''(a) + ab\chi_1 + ba\chi_2, \tag{137}
$$

which can be further refined by taking into account the symmetry between the arguments

$$
\begin{aligned}
\psi(a,b) \quad = \quad & \mathcal{M}_S(a,b) + \mathcal{M}_A(a,b) + a\mathcal{M}'_S(b) + b\mathcal{M}'_S(a) \\
& + a\mathcal{M}'_A(b) - b\mathcal{M}'_A(a) + a\cdot b\chi_S + a\wedge b\chi_A,
\end{aligned} \tag{138}
$$

where the subscripts S and A denote the symmetric and antisymmetric parts respectively,

$$
\begin{aligned}
\mathcal{M}_S(a,b) \quad &= \quad \mathcal{M}_S(b,a),\\
\mathcal{M}_A(a,b) \quad &= \quad -\mathcal{M}_A(b,a).
\end{aligned} \tag{139}
$$

The function $\psi(a,b)$ belongs to the

$$
\begin{gathered}
\big[\big(\tfrac{1}{2},0\big)\oplus\big(0,\tfrac{1}{2}\big)\big]\otimes\big(\tfrac{1}{2},\tfrac{1}{2}\big)\otimes\big(\tfrac{1}{2},\tfrac{1}{2}\big) = \\
\big[\big(\tfrac{3}{2},1\big)\oplus\big(1,\tfrac{3}{2}\big)\big]\oplus\big[\big(\tfrac{3}{2},0\big)\oplus\big(0,\tfrac{3}{2}\big)\big]\oplus 2\,\big[\big(1,\tfrac{1}{2}\big)\oplus\big(\tfrac{1}{2},1\big)\big]\oplus 2\,\big[\big(\tfrac{1}{2},0\big)\oplus\big(0,\tfrac{1}{2}\big)\big]
\end{gathered} \tag{140}
$$

representation. The are two new terms in this decomposition compared to that of a linear spinor-valued function, $\big(\frac{3}{2},1\big)\oplus\big(1,\frac{3}{2}\big)$ and $\big(\frac{3}{2},0\big)\oplus\big(0,\frac{3}{2}\big)$. Counting the degrees of freedom, we make the following identifications:

$$
\begin{aligned}
\big(\tfrac{3}{2},1\big)\oplus\big(1,\tfrac{3}{2}\big) \quad &\leftrightarrow \quad \mathcal{M}_S(a,b),\\
\big(\tfrac{3}{2},0\big)\oplus\big(0,\tfrac{3}{2}\big) \quad &\leftrightarrow \quad \mathcal{M}_A(a,b),\\
\big(1,\tfrac{1}{2}\big)\oplus\big(\tfrac{1}{2},1\big) \quad &\leftrightarrow \quad \mathcal{M}'_S(a),\mathcal{M}'_A(a),\\
\big(\tfrac{1}{2},0\big)\oplus\big(0,\tfrac{1}{2}\big) \quad &\leftrightarrow \quad \chi_S,\chi_A.
\end{aligned}
$$

We can now proceed to find a general half-integral-spin representation. Consider a spinor-valued k-linear function

$$
\psi(a_1,\ldots,a_k) \tag{141}
$$

which will transform under the action of the group as

$$\left[\left(\tfrac{1}{2},0\right)\oplus\left(0,\tfrac{1}{2}\right)\right]\otimes\underbrace{\left(\tfrac{1}{2},\tfrac{1}{2}\right)\otimes\ldots\otimes\left(\tfrac{1}{2},\tfrac{1}{2}\right)}_{k\text{ times}}. \tag{142}$$

The highest-spin representation in the decomposition will be a symmetric monogenic function which will correspond to the $\left(\frac{k+1}{2},\frac{k}{2}\right)\oplus\left(\frac{k}{2},\frac{k+1}{2}\right)$ irreducible representation. This is confirmed by counting the degrees of freedom in this function. A symmetric spinor-valued function in 4 dimensions has $8\frac{(k+3)!}{3!k!}$ real degrees of freedom. A monogenic version of the same therefore has

$$8\left(\frac{(k+3)!}{3!k!}-\frac{(k+2)!}{3!(k-1)!}\right)=4(k+2)(k+1) \tag{143}$$

real degrees of freedom which is exactly the number contained in the $\left(\frac{k+1}{2},\frac{k}{2}\right)\oplus\left(\frac{k}{2},\frac{k+1}{2}\right)$ irreducible representation.

4.6 Integral-Spin Representations

Again the integral-spin representations can be constructed from half-integral-spin representations. Consider a multivector-valued linear function constructed in the form

$$T(a)=\psi_1(a)\Gamma\tilde{\psi}_2. \tag{144}$$

In general, $T(a)$ will belong to the

$$\left[\left(\tfrac{1}{2},0\right)\oplus\left(0,\tfrac{1}{2}\right)\right]\otimes\left(\tfrac{1}{2},\tfrac{1}{2}\right)\otimes\left[\left(\tfrac{1}{2},0\right)\oplus\left(0,\tfrac{1}{2}\right)\right] \tag{145}$$

representation. If $\psi_1(a)$ is restricted to be monogenic, then the function will belong to the

$$\left[\left(1,\tfrac{1}{2}\right)\oplus\left(\tfrac{1}{2},1\right)\right]\otimes\left[\left(\tfrac{1}{2},0\right)\oplus\left(0,\tfrac{1}{2}\right)\right] \tag{146}$$

representation which contains the $(1,1)$ and $\left(\frac{3}{2},\frac{1}{2}\right)\oplus\left(\frac{1}{2},\frac{3}{2}\right)$ irreducible representations as the highest-spin irreducible representations. Using $\Gamma=\gamma_1+i\gamma_2$ projects out the vector and trivector part which corresponds to the $(1,1)$ representation,

$$A(a)=\psi_1(a)(\gamma_1+i\gamma_2)\tilde{\psi}_2, \tag{147}$$

and choosing $\Gamma=\sigma_1+i\sigma_2$ and projecting out the bivector part corresponds to the $\left(\frac{3}{2},\frac{1}{2}\right)\oplus\left(\frac{1}{2},\frac{3}{2}\right)$ representation,

$$F(a)=\langle\psi_1(a)(\sigma_1+i\sigma_2)\tilde{\psi}_2\rangle_2. \tag{148}$$

We can now generalise to to an arbitrary integral-spin representation

$$T(a_1,\ldots,a_k)=\psi_1(a_1,\ldots,a_k)\Gamma\tilde{\psi}_2 \tag{149}$$

which belongs to the

$$\left[\left(\tfrac{1}{2},0\right)\oplus\left(0,\tfrac{1}{2}\right)\right]\otimes\underbrace{\left(\tfrac{1}{2},\tfrac{1}{2}\right)\otimes\ldots\otimes\left(\tfrac{1}{2},\tfrac{1}{2}\right)}_{k\text{ times}}\otimes\left[\left(\tfrac{1}{2},0\right)\oplus\left(0,\tfrac{1}{2}\right)\right] \tag{150}$$

representation. If we select the monogenic, totally-symmetric part of the function, we can restrict it to the

$$\left[\left(\tfrac{k+1}{2},\tfrac{k}{2}\right)\oplus\left(\tfrac{k}{2},\tfrac{k+1}{2}\right)\right]\otimes\left[\left(\tfrac{1}{2},0\right)\oplus\left(0,\tfrac{1}{2}\right)\right] \tag{151}$$

representation which contains the $\left(\frac{k+1}{2}, \frac{k+1}{2}\right)$ and $\left(\frac{k+2}{2}, \frac{k}{2}\right) \oplus \left(\frac{k}{2}, \frac{k+2}{2}\right)$ irreducible representations as the highest-spin irreducible representations, which again correspond to the vector-trivector and bivector parts respectively. These are both 'spin-$(k+1)$' representations.

We have now found the method for identifying the irreducible representations of the Lorentz group with multilinear functions.

5 Wave Equations

So far we have restricted our attention to describing higher-weighted representations of rotation groups with multilinear functions. The wavefunction describing a quantum-mechanical particle depends in an arbitrary non-linear fashion on the position vector x. The method described in this paper can be generalised to include non-linear arguments, and thus handle these 'external' spatial or spacetime degrees of freedom within the same mathematical structure as the 'internal' spin degrees of freedom.

Consider a function which transforms under $\hat{\mathrm{U}}_R$ which has linear arguments $a_1, \ldots, a_k$ and non-linear arguments $x_1, \ldots, x_l$, we shall separate the linear and non-linear arguments with a semi-colon,

$$\psi(a_1, \ldots, a_k; x_1, \ldots, x_l). \tag{152}$$

The non-linear arguments transform under rotation in the same manner as the linear ones,

$$\hat{\mathrm{U}}_R[\psi(a_1, \ldots, a_k; x_1, \ldots, x_l)] = R\psi(\tilde{R}a_1R, \ldots, \tilde{R}a_kR; \tilde{R}x_1R, \ldots, \tilde{R}x_lR), \tag{153}$$

but the action of the Lie group is modified,

$$\begin{aligned} \tfrac{1}{2}\hat{\mathrm{L}}_B[\psi(\ldots;\ldots)] &= \tfrac{1}{2}B\psi(\ldots;\ldots) + \textstyle\sum_{i=1}^k \psi(\ldots, a_i\cdot B, \ldots;\ldots) \\ &\quad - \textstyle\sum_{i=1}^l B\cdot(x_i\wedge\partial_{x_i})\psi(\ldots;\ldots). \end{aligned} \tag{154}$$

Each non-linear argument x_i makes a contribution

$$-B\cdot(x_i\wedge\partial_{x_i}) = (x_i\cdot B)\cdot\partial_{x_i} \tag{155}$$

to $\frac{1}{2}\hat{\mathrm{L}}_B$. If x_i were a linear argument then this would reduce to the same form as for a linear argument, so this generalisation is consistent with our earlier definitions. This operator (155) could also be interpreted as finding the component of angular momentum, $-\jmath(x\wedge\nabla)$, in the plane defined by B.

A similar generalisation can be made for functions which transform under $\hat{\mathrm{V}}_R$, only the external transformation properties of the function differ from those transforming under $\hat{\mathrm{U}}_R$.

The Dirac equation written in the STA is

$$\nabla\psi(x)i\sigma_3 = m\psi(x)\gamma_0. \tag{156}$$

This has been investigated extensively elsewhere [10, 6]. The wavefunction $\psi(x)$ is a spinor function of position x. Some of the functions which form a basis for the Dirac wavefunction can be obtained from multilinear functions by polynomial projection [1]. The l-th order contribution can be made by setting all the arguments of a l-linear function equal to x,

$$\psi_l(x) = \prod_{i=1}^{l}(x\cdot\partial_{a_i})\psi(a_1, \ldots, a_l). \tag{157}$$

Only the symmetric part of the function will survive this procedure, the antisymmetric parts vanish. The complete wavefunction can be expressed as a sum over all orders of these functions.

Using the representations of the Lorentz group we found earlier, we can investigate higher-spin wave equations. The Rarita-Schwinger equations [11] describe half-integral spin particles. Translating these into geometric algebra, the equation describing a spin-$\frac{3}{2}$ particle becomes

$$\nabla\psi(a;x)i\sigma_3 = m\psi(a;x)\gamma_0, \tag{158}$$

with the additional condition

$$\partial_a\psi(a;x) = 0 \tag{159}$$

which ensures that the spin part of $\psi(a;x)$ belongs to the $\left(1,\frac{1}{2}\right)\oplus\left(\frac{1}{2},1\right)$ representation. In general, a spinor-valued k-linear wavefunction which is symmetric in its arguments represents a spin-$(k+\frac{1}{2})$ particle, it obeys

$$\nabla\psi_S(a_1,\ldots,a_k;x)i\sigma_3 = m\psi_S(a_1,\ldots,a_k;x)\gamma_0, \tag{160}$$

$$\partial_{a_1}\psi_S(a_1,\ldots,a_k;x) = 0 \tag{161}$$

where the additional condition restricts the spin part of the wavefunction to the $\left(\frac{k+1}{2},\frac{k}{2}\right)\oplus\left(\frac{k}{2},\frac{k+1}{2}\right)$ representation.

The basis functions of the k-linear wavefunction can also be obtained by polynomial projection. The l-th order contribution to the wavefunction is obtained by setting l arguments of a suitable $(k+l)$-linear function equal to x,

$$\psi_l(a_1,\ldots,a_k;x) = \prod_{i=1}^{l}(x\cdot\partial_{a_{(k+i)}})\psi(a_1,\ldots,a_{k+l}). \tag{162}$$

The symmetry among the first k arguments must be correct for the wavefunction that is being described. Again, the complete wavefunction will be a sum over all orders of these functions.

Different relativistic wave equations with spin corresponding to different representations of the Lorentz group can be explored by selecting the appropriate multilinear wavefunctions. This presents opportunities for further investigation.

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