Electron Scattering in the Spacetime Algebra

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Abstract

The Spacetime Algebra provides an elegant language for studying the Dirac equation. Cross section calculations can be performed in an intuitive way following a method suggested by Hestenes.[1] The S-matrix is replaced with an operator which rotates the initial states into the scattered states. We show how the method neatly handles spin dependence by allowing the scattering operator to become a function of the initial spin. When the operator is independent of spin we can provide manifestly spin-independent results. Spin basis states are not needed, and we do no spin sums, instead dealing with the spin orientation directly. We perform some example calculations for single electron scattering and briefly discuss more complicated cases in QED.

Introduction

Methods for calculating spinor cross sections are well known, however these often involve complicated abstract calculations with gamma matrices. In this paper we show how to calculate cross sections in a more transparent and intuitive way. Instead of using spin basis states, summing over spins and using spin projection operators, we instead incorporate the spin orientation directly. This greatly streamlines the calculation of spin dependent results, and makes it clear when results are independent of spin. We first consider single electron scattering, where our method is most naturally applied, and then briefly discuss multi-particle scattering.

Spacetime Algebra (STA) is the geometric (Clifford) algebra of Minkowski spacetime, first developed by Hestenes.[1–3] The formulation of Dirac theory within the algebra replaces the matrices of the conventional theory with multivectors. We introduce the STA form of the Dirac equation, and show how the theory can be developed within the STA. Using the STA formulation Hestenes[1] has demonstrated an elegant method for performing cross section calculations. We extend and clarify this work, handling spin-dependence in a natural way.

Throughout we make use of the Geometric Algebra. We present a brief summary of the STA below to clarify our notation and conventions. Full details of Geometric Algebra can be found elsewhere.[2, 4]

Spacetime Algebra

We shall use the four orthogonal basis vectors of spacetime γ_μ, where $\gamma_0^2 = 1$, and $\gamma_k^2 = -1$ for $k = 1, 2, 3$. The Geometric Algebra has an associative product and the basis vectors satisfy the Dirac algebra

$$\gamma_\mu \cdot \gamma_\nu \equiv \tfrac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \mathrm{diag}(+ - - -).$$

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The antisymmetric part of the product defines the *outer product*

$$\gamma_\mu \wedge \gamma_\nu \equiv \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu).$$

By repeated multiplication of the basis vectors we can build up the 16 basis elements of STA multivectors:

1	$\{\gamma_\mu\}$	$\{\boldsymbol{\sigma}_k, \boldsymbol{\Sigma}_k\}$	$\{I\gamma_\mu\}$	I
scalar	vectors	bivectors	pseudovectors	pseudoscalar

The bivectors $\boldsymbol{\sigma}_k \equiv \gamma_k\gamma_0$ are isomorphic to the basis vectors for Euclidean 3-space satisfying $\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k = \delta_{jk}$. Similarly the $\boldsymbol{\Sigma}_k = I\boldsymbol{\sigma}_k$ are isomorphic to the basis bivectors of Euclidean 3-space. We define the highest grade element

$$I \equiv \sigma_1\sigma_2\sigma_3 = \gamma_1\gamma_0\gamma_2\gamma_0\gamma_3\gamma_0 = \gamma_0\gamma_1\gamma_2\gamma_3$$

which satisfies $I^2 = -1$ and anticommutes with all vectors.

We usually take γ_0 to be the lab frame velocity vector. The $\boldsymbol{\sigma}_k$ then represent a frame in space relative to the γ_0 vector. We can do a space-time split of a vector into the γ_0 frame by multiplying by γ_0. Bold letters are now used for relative 3-vectors (spacetime bivectors). The split is

$$a\gamma_0 = a_0 + \boldsymbol{a}$$

where $a_0 = a\cdot\gamma_0$ and $\boldsymbol{a} = a\wedge\gamma_0$. We use natural units throughout, where $c = \epsilon_0 = \hbar = 1$, so, for example, the momentum p is split

$$p\gamma_0 = p\cdot\gamma_0 + p\wedge\gamma_0 = E + \boldsymbol{p}.$$

It is often useful to project out a particular grade from a multivector. Angled brackets $\langle A\rangle_r$ are used to denote the grade r projection of A. For the scalar part $r = 0$ and we write it just as $\langle A\rangle$. The inner and outer products for grade r and s multivectors are defined as follows

$$\begin{aligned} A_r \cdot B_s &\equiv \langle A_r B_s\rangle_{|r-s|} \\ A_r \wedge B_s &\equiv \langle A_r B_s\rangle_{r+s}. \end{aligned}$$

In the case where $r = 1$, so that $a \equiv A_r$ is a vector, we have the relation

$$aB_s = a\wedge B_s + a\cdot B_s.$$

The symmetry of both the inner and outer product alternate with increasing grade of B_s:

$$\begin{aligned} a\cdot B_s \equiv \langle aB_s\rangle_{s-1} &= \frac{1}{2}(aB_s - (-1)^s B_s a) \\ a\wedge B_s \equiv \langle aB_s\rangle_{s+1} &= \frac{1}{2}(aB_s + (-1)^s B_s a). \end{aligned}$$

We adopt the useful convention that when the operands of the inner and outer product are in bold type, and are therefore spatial vectors, the inner and outer products take their three-dimensional meaning.

Another important operation is that of *reversion*. We write the reverse of A as $\widetilde{A}$, which reverses all the vector products making up the multivector. It has the property that

$$(AB)\tilde{}\,= \widetilde{B}\,\widetilde{A}.$$

Lorentz transformations are spacetime rotations and can be performed by use of a *rotor*, which can be written $R = \pm\exp(B/2)$. Here B is a bivector in the plane of the rotation

and $|B|$ determines the amount of rotation. The rotation of a multivector M is then given by

$$M \to RM\widetilde{R}.$$

Finally we need the vector derivative $\nabla \equiv \gamma^\mu \partial_\mu$, the derivative with respect to position. Its definition implies that it has the spacetime split

$$\nabla\gamma_0 = \partial_t - \boldsymbol{\nabla}.$$

The Dirac Equation

The Hestenes STA form of the Dirac equation is entirely equivalent to the usual equation.[5] However the STA approach brings out the geometric structure, leading to more physically transparent calculations. Here we show that it is possible to arrive at the Dirac equation by quantizing a classical equation. This 'derivation' has the advantage that the observables are then clearly related to the classical parameters, and the geometric structure of the theory is brought out.

Our classical model will consist of a small spinning symmetric top with four velocity v. We can represent v as a boosted version of the lab frame time vector γ_0:

$$v = L\gamma_0\widetilde{L}$$

where L is a boosting rotor. In this way the velocity can be represented by the rotor L. Similarly we can use a spatial rotor U to encode the spin plane as a rotation of some fixed reference plane. We write the rest spin of the top as

$$\hat{S}^0 = U\boldsymbol{\Sigma}\widetilde{U}$$

where $\boldsymbol{\Sigma}$ is some arbitrary constant reference bivector orthogonal to γ_0 ($\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_3$ is often chosen). Since U is a spatial rotor it does not affect the γ_0 direction so the momentum can be written

$$p = mR\gamma_0\widetilde{R}$$

where $R = LU$. This equation for p squares to give $p^2 = m^2$ which gives the Klein-Gordon equation on quantization. However the rotor equation contains much more information than the scalar equation given by its square.

As well as encoding the rotation of $\boldsymbol{\Sigma}$ into the spin plane the spatial rotor U can also include an arbitrary unobservable rotation in the reference plane $\boldsymbol{\Sigma}$. We can boost up $\hat{S}^0$ to define the relativistic spin bivector

$$\hat{S} = L\hat{S}^0\widetilde{L} = R\boldsymbol{\Sigma}\widetilde{R}.$$

The rotor R now encodes everything about the four velocity and spin direction of the top, plus some arbitrary unobservable rotation in the spin plane.

In the quantum version we wish to have probability densities. In the rest frame of the top this corresponds to some probability density ρ of finding it at each point. We want this to the $v\cdot J$ component of a four vector probability current J, with the lab frame probability density given by $\gamma_0\cdot J$. We therefore define the four vector $J = \rho v$ which can be written

$$J = \rho R\gamma_0\widetilde{R}.$$

We now wrap up ρ and R into a single even multivector $\psi = \rho^{1/2}R$ so that

$$J = \psi\gamma_0\widetilde{\psi}$$

and the rest frame probability density is given by $\rho = \psi\widetilde{\psi}$. We now want to put the equation for the momentum in terms of ψ. Multiplying the equation on the right by R we have

$$\begin{aligned} pR &= mR\gamma_0 \\ \implies p\psi &= m\psi\gamma_0. \end{aligned}$$

This equation now contains all the ingredients for successful quantization. The usual procedure is to make the replacement $p_\mu \to \hat{\jmath}\nabla\mu$, so we get

$$\hat{\jmath}\nabla\psi = m\psi\gamma_0$$

as our form of the Dirac equation. For a plane wave $\psi(x) = \psi e^{-\hat{\jmath}p\cdot x}$ this just gives us back our classical equation, as expected.

There is a remaining ambiguity in what $\hat{\jmath}$ is. It could be a scalar imaginary, or could it be something more physical? Multiplication by $\hat{\jmath}$ should just affect the phase of the wave function, we don't want the spin or momentum to be affected. So for plane waves, writing $\psi' = \hat{\jmath}\psi$, we want

$$S' = \psi'\mathbf{\Sigma}\widetilde{\psi'} = \psi\mathbf{\Sigma}\widetilde{\psi} \quad \text{and} \quad J' = \psi'\gamma_0\widetilde{\psi'} = \psi\gamma_0\psi.$$

These can be satisfied if

$$\psi' \equiv \hat{\jmath}\psi = \hat{S}\psi \quad \text{or} \quad \psi' \equiv \hat{\jmath}\psi = \psi\mathbf{\Sigma},$$

and indeed for plane wave states these are equivalent since

$$\psi\mathbf{\Sigma} = 1/\rho\psi\mathbf{\Sigma}\widetilde{\psi}\psi = \hat{S}\psi.$$

So the 'complex' phase factors of the form $e^{\hat{\jmath}\alpha}$ just encodes rotations in the spin plane — the rotations that were unobservable in the classical case.

Dirac Theory

Having 'derived' the Dirac equation we now take that equation as given and see what it implies. For positive energy plane wave states all the classical results still hold. However we now have two sets of plane wave solutions,

$$\psi^{(+)} = u(p)e^{-\hat{\jmath}p\cdot x} \quad \text{and} \quad \psi^{(-)} = v(p)e^{\hat{\jmath}p\cdot x}$$

where

$$mu - pu\gamma_0 = 0 \quad \text{and} \quad mv + pv\gamma_0 = 0.$$

We therefore have positive and negative energy states, with energy projection operators given by

$$\Lambda_\pm(\psi) = \frac{1}{2m}(m\psi \pm p\psi\gamma_0).$$

Since $\psi\widetilde{\psi}$ reverses to itself it can only contain scalar and pseudoscalar parts and we define

$$\rho e^{I\beta} \equiv \psi\widetilde{\psi}$$

where β and ρ are scalars. So the general form for ψ now is now

$$\psi = \rho^{1/2}e^{I\beta/2}R.$$

In addition to encoding a rotation and a dilation, the spinor also contains a 'β-factor'. This determines the ratio of particle to anti-particle solutions since

$$\Lambda_{\pm}(I\psi) = I\Lambda_{\mp}(\psi).$$

The transformation properties of ψ are inherited from its component rotor, so we have

$$\psi_L(x) = R\psi(\widetilde{R}xR).$$

We call an element of the STA which transforms as a rotor a *spinor*.

The Dirac equation can be obtained from the Lagrangian

$$\mathcal{L} = \langle \jmath\nabla\psi\gamma_0\widetilde{\psi} - m\psi\widetilde{\psi}\rangle$$

by using the multivector form of the Euler-Lagrange equations.[6, 7] The Lagrangian is invariant under

$$\psi \to \psi e^{\jmath\theta}$$

corresponding to invariance under rotation in the spin plane. Using the multivector form of Noether's theorem,[6] we find the corresponding conserved probability current

$$J = \psi\gamma_0\widetilde{\psi},$$

in agreement with our classical definition. The Dirac equation ensures that $\nabla\cdot J = 0$.

Plane waves and basis states

Using the decomposition $R = LU$ of a rotor into a spatial rotation and a boost we can write a spinor ψ as

$$\psi = \rho^{1/2}e^{I\beta/2}LU.$$

Consider a positive energy spinor $u = \Lambda_+(u)$ and a negative energy spinor $v = \Lambda_-(v)$. If the particle is at rest we have

$$\gamma_0 u^0\gamma_0 = u^0 \quad \text{and} \quad \gamma_0 v^0\gamma_0 = -v^0$$

which implies that

$$u^0 = \rho_u^{1/2}U_u \quad \text{and} \quad v^0 = \rho_v^{1/2}IU_v.$$

We can find the more general form by performing a boost to momentum p. The boost transforms $m\gamma_0$ into the momentum p:

$$p = mL\gamma_0\widetilde{L} \quad \Longrightarrow \quad pL\gamma_0 - mL = 0$$

so that $\Lambda_-(L) = 0$. A solution is therefore of the form $L = \Lambda_+(X)$. Choosing X equal to a constant so that $\widetilde{L}L = 1$ we have

$$L = \frac{m + p\gamma_0}{\sqrt{2m(E+m)}} = \frac{E + m + \boldsymbol{p}}{\sqrt{2m(E+m)}}. \tag{1}$$

Normalizing so that $\rho_u = \rho_v = 2m$ and performing the boost we get

$$\begin{aligned} u(p) &= Lu^0 = \sqrt{E+m}\left(1 + \frac{\boldsymbol{p}}{E+m}\right)U_u \\ v(p) &= Lv^0 = I\sqrt{E+m}\left(1 + \frac{\boldsymbol{p}}{E+m}\right)U_v. \end{aligned}$$

In addition to the energy projection operators there are also the projection operators

$$\chi_\pm(\psi) = \tfrac{1}{2}(\psi \mp P\psi\mathbf{\Sigma})$$

where P is a bivector with $P^2 = -1$. For a state a ψ satisfying $\psi = \chi_\pm(\psi)$ we have

$$\psi = \mp P\psi\mathbf{\Sigma}.$$

Multiplying on the right by $\widetilde{\psi}$ this gives $\rho = \mp\rho P\hat{S}$ and so $\hat{S} = \pm P$. The projection operator therefore projects out parts corresponding the two spin orientations in the plane P. The spin projection operators commute with the energy projection operators

$$\Lambda_\pm(\psi) = \frac{1}{2m}(m\psi \pm p\psi\gamma_0)$$

since $P \cdot p = 0$. We can therefore split an arbitrary spinor (eight real components) into scalar and $\hat{\jmath}$ multiples of four basis states

$$\begin{aligned} u_1 &= \chi_+(\Lambda_+(u_1)) & u_2 &= \chi_-(\Lambda_+(u_2)) \\ v_1 &= \chi_-(\Lambda_-(v_1)) & v_2 &= \chi_+(\Lambda_-(v_2)) \end{aligned}$$

With the normalization convention that $u\widetilde{u} = 2m$ the four basis states obey the orthogonality relations

$$\begin{aligned} \langle \widetilde{u}_r u_s \rangle_S &= 2m\delta^r_s & \langle \widetilde{v}_r v_s \rangle_S &= -2m\delta^r_s \\ \langle \widetilde{u}_r v_s \rangle_S &= 0 & \langle \widetilde{v}_r u_s \rangle_S &= 0 \end{aligned}$$

where $\langle A \rangle_S$ represents the $\{1, \mathbf{\Sigma}\}$ projection of A:

$$\langle A \rangle_S \equiv \langle A \rangle - \langle A\mathbf{\Sigma} \rangle \mathbf{\Sigma}.$$

By writing ψ as a sum over basis states it's easy to see that

$$\sum_r u_r \langle \widetilde{u}_r \psi \rangle_S = p\psi\gamma_0 + m\psi \quad \text{and} \quad \sum_r v_r \langle \widetilde{v}_r \psi \rangle_S = p\psi\gamma_0 - m\psi.$$

So we see that the usual basis state results of Dirac theory can be formulated in the STA approach. However we shall now develop the scattering theory largely without resort to basis states.

Feynman propagators

The minimally coupled Dirac equation can be written

$$\hat{\jmath}\nabla\psi\gamma_0 - m\psi = eA\psi\gamma_0$$

where $e = -|e|$ is the electron charge. We need a Greens' function for this equation satisfying

$$\hat{\jmath}\nabla_x S_F(x - x')\psi(x')\gamma_0 - mS_F(x - x')\psi(x') = \delta^4(x - x')\psi(x')$$

so that an integral solution can be found from

$$\psi(x) = \psi_i(x) + e\int d^4x' S_F(x - x')A(x')\psi(x')\gamma_0 \tag{2}$$

where ψ_i satisfies the free-particle equation. Taking the Fourier transform we have

$$pS_F(p)\psi\gamma_0 - mS_F(p)\psi = \psi$$

where

$$S_F(x-x') = \int \frac{d^4p}{(2\pi)^4} S_F(p) e^{-\hat{\jmath}p\cdot(x-x')}.$$

Operating on both sides with the energy projection operator Λ_+ we can solve for the momentum space Feynman propagator:

$$\begin{aligned}(p^2-m^2)S_F(p)\psi &= p\psi\gamma_0 + m\psi \\ \Rightarrow S_F(p)\psi &= \frac{p\psi\gamma_0 + m\psi}{p^2 - m^2 + \hat{\jmath}\epsilon}. \end{aligned} \tag{3}$$

The $\hat{\jmath}\epsilon$ ensures that the contour integral is in the $\mathbf{\Sigma}$ plane and that it is causal—positive energy waves propagate into the future and negative energy waves into the past. Fourier transforming back and performing the integral over dE we get

$$S_F(x-x')\psi = -2m\hat{\jmath}\int \frac{d^3\boldsymbol{p}}{2E_p(2\pi)^3}[\theta(t-t')\Lambda_+(\psi)e^{-\hat{\jmath}p\cdot(x-x')} + \theta(t'-t)\Lambda_-(\psi)e^{\hat{\jmath}p\cdot(x-x')}] \tag{4}$$

where $E = +\sqrt{\boldsymbol{p}^2 + m^2}$.

The photon propagator is the Greens' function for Maxwell's equations. In the Lorentz gauge $\nabla\cdot A = 0$ we have $\nabla^2 A = J$, so the Greens' function must satisfy

$$\nabla_x^2 D_F(x-x') = \delta^4(x-x').$$

Taking the Fourier transform we can solve for the Feynman propagator

$$D_F(p) = \frac{-1}{p^2 + \hat{\jmath}\epsilon}.$$

Electron scattering

For scattering calculations we write the wavefunction as the sum of an incoming plane wave and a scattered beam, $\psi = \psi_i + \psi_{\text{diff}}$, where ψ_{diff} is the solution at asymptotically large times given by

$$\psi_{\text{diff}}(x) = -2m\hat{\jmath}e\int d^4x' \int \frac{d^3\boldsymbol{p}}{2E_p(2\pi)^3}\Lambda_+ \left[A(x')\psi(x')\gamma_0\right] e^{-\hat{\jmath}p\cdot(x-x')}.$$

This can be written as a sum over final states

$$\psi_{\text{diff}}(x) = \int \frac{d^3\boldsymbol{p}_f}{2E_f(2\pi)^3}\psi_f(x),$$

the final states being plane waves of the form

$$\psi_f(x) \equiv \psi_f e^{-\hat{\jmath}p_f\cdot x} \equiv -\hat{\jmath}e\int d^4x'[p_f A(x')\psi(x') + mA(x')\psi(x')\gamma_0]e^{-\hat{\jmath}p_f\cdot(x-x')}. \tag{5}$$

With this definition the number of scattered particles is given by

$$\int d^3\boldsymbol{x}\gamma_0\cdot J_{\text{diff}} = \int \frac{d^3\boldsymbol{p}_f}{2E_f(2\pi)^3}\left[\frac{\gamma_0\cdot J_f}{2E_f}\right] \equiv \int \frac{d^3\boldsymbol{p}_f}{2E_f(2\pi)^3}N_f$$

where we have defined the number density per Lorentz invariant phase space interval to be

$$N_f \equiv \frac{\gamma_0 \cdot J_f}{2E_f} = \frac{\gamma_0 \cdot (\psi_f \gamma_0 \widetilde{\psi}_f)}{2E_f} = \frac{\rho_f}{2m}.$$

The Born series perturbative solution is generated by iterating (2). In the first order Born approximation this amounts to simply replacing $\psi(x')$ by $\psi_i(x')$. For plane waves of particles we have

$$\psi(x) = \psi e^{-\hat{\jmath} p \cdot x} \quad \text{and} \quad m\psi\gamma_0 = p\psi$$

so the final states become

$$\begin{aligned} \psi_f &= -\hat{\jmath} e \int d^4x' [p_f A(x') + A(x') p_i] \psi_i e^{\hat{\jmath} q \cdot x'} \\ &= -\hat{\jmath} e [p_f A(q) + A(q) p_i] \psi_i \end{aligned}$$

where $q \equiv p_f - p_i$.

More generally we define

$$\psi_f = S_{fi} \psi_i$$

where S_{fi} is the *scattering operator* which rotates and dilates the initial states into the final states. Here the f and i indices label the initial and final momenta and the initial spin, so in general $S_{fi} = S_{fi}(p_f, p_i, \hat{S}_i)$. However S_{fi} does not depend on the final spin—instead the final spin is determined from the initial spin by a rotation encoded in S_{fi}. Since S_{fi} consists of a rotation and dilation it is convenient to decompose it as

$$S_{fi} = \rho_{fi}^{1/2} R_{fi}$$

where R_{fi} is a rotor. There is no $e^{I\beta}$ part since we have particles scattering to particles, not a mixture of particles and antiparticles. The cross section will be determined by the ρ_{fi} factor, as detailed in the next section. The rotor R_{fi} rotates states with momentum p_i into states with momentum p_f. It also relates the initial and final spins by

$$\hat{S}_f = R_{fi} \hat{S}_i \widetilde{R}_{fi}$$

so the rest spins are related by

$$\hat{S}_f^0 = \widetilde{L}_f \hat{S}_f L_f = \widetilde{L}_f R_{fi} \hat{S}_i \widetilde{R}_{fi} L_f = \widetilde{L}_f R_{fi} L_i \hat{S}_i^0 \widetilde{L}_i \widetilde{R}_{fi} L_f.$$

We therefore define the *rest spin scattering operator*

$$U_{fi} \equiv \widetilde{L}_f R_{fi} L_i$$

so that

$$\hat{S}_f^0 = U_{fi} \hat{S}_i^0 \widetilde{U}_{fi}.$$

The rest spin scattering operator and the cross section contain all the information about scattering of states with momentum p_i and spin $\hat{S}_i$ into states with momentum p_f.

The form of the external line Feynman propagator (4) ensures that S_{fi} is of the form

$$S_{fi} = -\hat{\jmath}(p_f M + M p_i) \tag{6}$$

where in the Born approximation example $M = eA(q)$. However in general M can have some $\hat{\jmath}$-dependence in which case we can write

$$S_{fi}\psi_i = -\hat{\jmath}(p_f[M_r + \hat{\jmath} M_j] + [M_r + \hat{\jmath} M_j] p_i)\psi_i$$

where M_j and M_r are independent of $\hat{\jmath}$. Using $\hat{\jmath}\psi_i = \psi_i \mathbf{\Sigma} = \hat{S}_i \psi_i$ and the fact that $\hat{S}_i$ and p_i commute this can be written

$$S_{fi} = -\hat{\jmath}(p_f M + M p_i)$$

where

$$M = M_r + M_j \hat{S}_i$$

now depends on the initial spin. We can thus replace dependence on the 'imaginary' $\hat{\jmath}$ with dependence on the spin bivector.

Using $mL^2 = p\gamma_0$ we can obtain U_{fi} from

$$U_{fi} \propto L_f \gamma_0 M L_i + \widetilde{L}_f M \gamma_0 \widetilde{L}_i.$$

Positron scattering and pair annihilation

Adapting the above results to positron scattering is straightforward. We just consider a negative energy plane wave coming in from the future and scattering into the past, so $\psi_i(x) = \psi_2 e^{\hat{\jmath} p_2 x}$ and

$$S_{fi}\psi_i = -\hat{\jmath}(-p_1 M \psi_i + M \psi_i \gamma_0)$$

where p_1 is the incoming positron momentum and p_2 is the outgoing momentum. This then gives

$$S_{fi} = \hat{\jmath}(p_1 M + M p_2),$$

amounting to the substitution $p_f \to -p_1, p_i \to -p_2$.

The other case to consider is when the incoming electron gets scattered into the past, corresponding to pair annihilation. In this case we have

$$S_{fi} = -\hat{\jmath}(-p_2 M + M p_1)$$

where p_1 and p_2 are the incoming momenta of the electron and positron respectively. In this case we can decompose S_{fi} as

$$S_{fi} = \rho_{fi}^{1/2} I R_{fi}$$

since S_{fi} must now contain a factor of I to map electrons into positrons. This also implies

$$S_{fi}\widetilde{S}_{fi} = -\rho_{fi}.$$

Cross sections

The scattering rate into the final states per unit volume per unit time is given by

$$W_{fi} = \frac{1}{VT} N_f = \frac{1}{VT} \frac{\gamma_0 \cdot J_f}{2E_f} = \frac{\rho_f}{2mVT}$$

where ρ_f is given simply by

$$\rho_f = |S_{fi}|^2 \rho_i = \rho_{fi} \rho_i.$$

Here we have defined

$$|S_{fi}|^2 \equiv |S_{fi}\widetilde{S}_{fi}| = \pm S_{fi}\widetilde{S}_{fi}$$

where plus sign corresponds to electron to electron and positron to positron scattering, the minus sign to electron-positron annihilation. The cross section is defined as

$$d\boldsymbol{\sigma} = \frac{W_{fi}}{\text{Target density} \times \text{Incident flux}}.$$

When S_{fi} is of the form

$$S_{fi} = -\hat{\jmath}(2\pi)^4\delta^4(P_f - P_i)T_{fi}$$

we have

$$|S_{fi}|^2 = VT(2\pi)^4\delta^4(P_f - P_i)|T_{fi}|^2.$$

Working in the J_i frame the target density is just ρ_i, so writing the incident flux as χ we have

$$d\boldsymbol{\sigma} = \frac{1}{2m\chi}(2\pi)^4\delta^4(P_f - P_i)|T_{fi}|^2.$$

Alternatively we may have elastic scattering and

$$S_{fi} = -\hat{\jmath}2\pi\delta(E_f - E_i)T_{fi}$$

in which case

$$|S_{fi}|^2 = 2\pi T\delta(E_f - E_i)|T_{fi}|^2.$$

A target density of $1/V$ and an incident flux of $|\boldsymbol{J}_i| = \rho_i|\boldsymbol{p}_i|/m$ then gives

$$d\boldsymbol{\sigma} = \frac{\pi}{|\boldsymbol{p}_i|}\delta(E_f - E_i)|T_{fi}|^2.$$

Above we have considered the total number of particles scattered. If we are interested in the final spin we can find it using the spin scattering operator. However we might also like to consider the cross section when we only observe particles with final spins in a certain plane $\hat{S}_o$ (where $\hat{S}_o{\cdot}p_f = 0$). This is particularly relevant in examples like electron-positron annihilation where ψ_f is actually an input state and we would like to calculate the cross section for arbitrary initial spins.

The spin projection operators into the $\hat{S}_o$ plane are

$$\chi_\pm(\psi) = \tfrac{1}{2}(\psi \mp \hat{S}_o\psi\boldsymbol{\Sigma})$$

and we are interested in scattering into

$$\chi_\pm(\psi_f) = \chi_\pm(S_{fi}\psi_i).$$

Now if S_{fi} is in the form (6) we have

$$\begin{aligned}
\chi_\pm(S_{fi}\psi_i) &= -\tfrac{1}{2}\left[(p_fM + Mp_i)\psi_i\boldsymbol{\Sigma} \pm \hat{S}_o(p_fM + Mp_i)\psi_i\right] \\
&= -\tfrac{1}{2}\left[(p_fM + Mp_i)\hat{S}_i \pm \hat{S}_o(p_fM + Mp_i)\right]\psi_i \\
&= -\tfrac{1}{2}\left[p_f(M\hat{S}_i \pm \hat{S}_oM) + (M\hat{S}_i \pm \hat{S}_oM)p_i\right]\psi_i.
\end{aligned}$$

Defining $\chi_\pm(\psi_f) = S^\pm_{fi}\psi_i$ the scattering rate will be proportional to $\rho^\pm_{fi}$ given by

$$|S^\pm_{fi}|^2 = \left\langle (m^2M + p_fMp_i)(\widetilde{M} \mp \hat{S}_i\widetilde{M}\hat{S}_o)\right\rangle. \tag{7}$$

If we sum over final spins the $\hat{S}_o$ term cancels out and we get the expected result for the total ρ_{fi}:

$$|S_{fi}|^2 = \langle (p_fM + Mp_i)(\widetilde{M}p_f + p_i\widetilde{M})\rangle = 2\langle m^2M\widetilde{M} + p_fMp_i\widetilde{M}\rangle \tag{8}$$

Coulomb scattering

As our first simple example we consider the first Born approximation in electron Coulomb scattering where we have an external field given by

$$A(x) = \frac{-Ze}{4\pi|\boldsymbol{x}|}\gamma_0.$$

Taking the Fourier transform we have

$$A(q) = -\frac{2\pi Ze}{\boldsymbol{q}^2}\delta(E_f - E_i)\gamma_0$$

and $M = eA(q)$ in the first Born approximation. Writing

$$S_{fi} = -\hat{\jmath}2\pi\delta(E_f - E_i)T_{fi}$$

and using energy conservation we have

$$T_{fi} = -\frac{Ze^2}{\boldsymbol{q}^2}(2E + \boldsymbol{q})$$

so that the formula for the cross section becomes

$$d\boldsymbol{\sigma} = \left(\frac{Ze^2}{\boldsymbol{q}^2}\right)^2 \frac{\pi}{|\boldsymbol{p}_i|}\delta(E_f - E_i)(4E^2 - \boldsymbol{q}^2)\frac{d^3\boldsymbol{p}_f}{2E_f(2\pi)^3}.$$

Using $d^3\boldsymbol{p}_f = |\boldsymbol{p}_f|E_f dE_f d\Omega_f$ we recover the Mott cross section

$$\left(\frac{d\sigma}{d\Omega_f}\right)_{\text{Mott}} = \frac{Z^2\alpha^2}{\boldsymbol{q}^4}(4E^2 - \boldsymbol{q}^2) = \frac{Z^2\alpha^2}{4\boldsymbol{p}^2\beta^2\sin^4(\theta/2)}\big(1 - \beta^2\sin^2(\theta/2)\big),$$

where

$$\boldsymbol{q}^2 = (\boldsymbol{p}_f - \boldsymbol{p}_i)^2 = 2\boldsymbol{p}^2(1 - \cos\theta) \quad \text{and} \quad \beta = |\boldsymbol{p}|/E.$$

The derivation is manifestly independent of initial spin, so the cross section is spin independent. Of course the final and initial spins will be related by the rest spin scattering operator U_{fi}, where

$$U_{fi} \propto L_f L_i + \widetilde{L}_f \widetilde{L}_i \propto (E + m)^2 + \boldsymbol{p}_f\boldsymbol{p}_i.$$

If U_{fi} rotates by an angle δ in the $\hat{B}$ plane ($\hat{B}^2 = -1$) it is given by

$$U_{fi} = e^{\delta\hat{B}/2} = \cos(\delta/2) + \hat{B}\sin(\delta/2).$$

So we see that the rotation is in the $\boldsymbol{p}_f \wedge \boldsymbol{p}_i$ plane and by an angle δ given by

$$\tan(\delta/2) = \frac{|\langle U_{fi}\rangle_2|}{\langle U_{fi}\rangle} = \frac{|\boldsymbol{p}_f \wedge \boldsymbol{p}_i|}{(E + m)^2 + \boldsymbol{p}_f\cdot\boldsymbol{p}_i} = \frac{\sin\theta}{(E + m)/(E - m) + \cos\theta}.$$

Similar derivations of these result using the STA approach have been given before.[1, 8]

Compton scattering

In this case there are two Feynman diagrams which give two terms of the form

$$M_{12} = e^2 \int d^4x' \int d^4x'' \int \frac{d^4p}{(2\pi)^4} A_1(x') \frac{pA_2(x'') + A_2(x'')p_i}{p^2 - m^2 + \hat{\jmath}\epsilon} e^{\hat{\jmath}x'\cdot(p_f - p)} e^{\hat{\jmath}x''\cdot(p - p_i)}$$

where

$$A(x) = \epsilon e^{\mp \hat{\jmath} k\cdot x}$$

is different at each vertex and $\epsilon^2 = -1$. Performing the integrations and summing the two contributions we have

$$M = e^2(2\pi)^4 \delta^4(p_f + k_2 - p_i - k_i) \left[\epsilon_f \frac{(p_i + k_i)\epsilon_i + \epsilon_i p_i}{2k_i\cdot p_i} + \epsilon_i \frac{(p_i - k_f)\epsilon_f + \epsilon_f p_i}{-2p_i\cdot k_f} \right].$$

Choosing $p_i\cdot\epsilon_i = p_i\cdot\epsilon_f = 0$ this is simply

$$M = e^2(2\pi)^4 \delta^4(p_f + k_f - p_i - k_i) \left(\frac{\epsilon_f h_i \epsilon_i}{2k_i\cdot p_i} + \frac{\epsilon_i k_f \epsilon_f}{2p_i\cdot k_f} \right).$$

Writing

$$S_{fi} = -\hat{\jmath}(2\pi)^4 \delta^4(p_f + k_f - p_i - k_i) T_{fi}$$

and using (8) we then have

$$|T_{fi}|^2 = e^4 \left\langle \frac{m^2 \epsilon_f k_i \epsilon_i \epsilon_f k_f \epsilon_i + p_f \epsilon_f k_i \epsilon_i p_i \epsilon_f k_f \epsilon_f}{k_i\cdot p_i k_f\cdot p_i} + \frac{p_f \epsilon_f k_i \epsilon_i p_i \epsilon_i k_i \epsilon_f}{2(k_i\cdot p_i)^2} + \frac{p_f \epsilon_i k_f \epsilon_f p_i \epsilon_f k_f \epsilon_i}{2(k_f\cdot p_i)^2} \right\rangle.$$

The identities we need to calculate are now the same as in the traditional approach, only now we know that the result is independent of initial spin since we haven't done a spin sum. Using momentum conservation we know

$$p_f + k_f = p_i + k_i \qquad k_f\cdot p_i = k_i\cdot p_f \qquad p_i\cdot k_i = p_f\cdot k_f.$$

Applying these the result becomes, after some work,

$$|T_{fi}|^2 = e^4 \left[4(\epsilon_i\cdot\epsilon_f)^2 - 2 + \frac{p_i\cdot k_f}{p_i\cdot k_i} + \frac{p_i\cdot k_i}{p_i\cdot k_f} \right].$$

To calculate the cross section we work in the frame where the electron is initially at rest ($p_i = m\gamma_0$). The incoming photon flux is $2k_i^0$ so we have

$$d\boldsymbol{\sigma} = (2\pi)^4 \delta^4(p_f + k_f - p_i - k_i) \frac{|T_{fi}|^2}{2m2k_i^0} \frac{d^3k_f}{2k_f^0(2\pi)^3} \frac{d^3p_f}{2E_f(2\pi)^3}.$$

Now

$$\int d^3p_f d^3k_f \delta^4(p_f + k_f - p_i - k_i) = (k_f^0)^2 \frac{E_f k_f^0}{m k_i^0} d\Omega,$$

where we have done the integral over the final electron's momentum since we are primarily interested in the scattering of the photon. In the lab frame the result is therefore

$$\begin{aligned} \frac{d\boldsymbol{\sigma}}{d\Omega} &= \left(\frac{k_f}{k_i}\right)^2 \frac{|T_{fi}|^2}{4m^2(4\pi)^2} \\ &= \frac{\alpha^2}{4m^2} \left(\frac{k_f}{k_i}\right)^2 \left[\frac{k_f}{k_i} + \frac{k_i}{k_f} + 4(\epsilon_f\cdot\epsilon_i)^2 - 2 \right] \end{aligned}$$

in agreement with the Klein-Nishina formula. Again, the difference is that this derivation applies regardless of the initial electron spin. Of course if we had used circularly polarized photons we would have introduced some $\hat{\jmath}$-dependence and the result would have become spin-dependent.

Pair annihilation

A process closely related to Compton scattering is electron-positron annihilation. We just have to take account of the fact that the out state is a positron so the final states are of the form

$$\psi_f(x) \equiv \psi_f e^{\hat{\jmath} p_f \cdot x} \equiv -\hat{\jmath} e \int d^4x' [-p_f A(x')\psi(x') + mA(x')\psi(x')\gamma_0] e^{\hat{\jmath} p_f \cdot (x-x')}.$$

Writing

$$S_{fi} = -\hat{\jmath}(-p_f M + M p_i)$$

we have two terms of the form

$$M_{12} = e^2 \int d^4x' \int d^4x'' \int \frac{d^4p}{(2\pi)^4} A_1(x') \frac{pA_2(x'') + A_2(x'')p_i}{p^2 - m^2 + \hat{\jmath}\epsilon} e^{-\hat{\jmath} x' \cdot (p_f + p)} e^{\hat{\jmath} x'' \cdot (p - p_i)}$$

where

$$A(x) = \epsilon e^{\hat{\jmath} k \cdot x}$$

is different at each vertex and $\epsilon^2 = -1$. As for Compton scattering we now choose $p_i \cdot \epsilon_1 = p_i \cdot \epsilon_2 = 0$ and sum the two contributions to get

$$M = e^2 (2\pi)^4 \delta^4(p_f + p_i - k_1 - k_2) \left(\frac{\epsilon_2 h_1 \epsilon_1}{2k_1 \cdot p_i} + \frac{\epsilon_1 k_2 \epsilon_2}{2p_i \cdot k_2} \right).$$

For general positron and electron spins we should use (7) to calculate the cross section. However if either ψ_f or ψ_i are unpolarized the spin dependence will cancel out and the average just introduces a factor of two into equation (8). In this case $|T_{fi}|^2$ is obtained from the Compton case by the substitution $p_f \to -p_f$ and $k_i \to -k_1$, and an overall sign change because $T_{fi}\widetilde{T}_{fi} < 0$ in this case:

$$|T_{fi}|^2 = -\frac{e^4}{2} \left[4(\epsilon_i \cdot \epsilon_f)^2 - 2 - \frac{p_i \cdot k_2}{p_i \cdot k_1} - \frac{p_i \cdot k_1}{p_i \cdot k_2} \right].$$

To get the cross section just divide by flux factors and perform the integral as usual.

Second order Coulomb scattering

Second order Coulomb scattering is interesting as it is spin-dependent and so provides a good testing ground for our calculation techniques. To avoid problems with divergent integrals the potential is replaced with the screened potential

$$A(x) = -\frac{e^{-\lambda|\boldsymbol{x}|} Ze}{4\pi|\boldsymbol{x}|} \gamma_0$$

and the Coulomb result found in the limit λ goes to zero.[9,10] For this potential the first order analysis above can be applied with M given by

$$eA(q) = -\frac{2\pi Ze^2}{\lambda^2 + \boldsymbol{q}^2} \delta(E_f - E_i)\gamma_0.$$

To iterate to second order (5) is used, with the substitution

$$\psi(x') = \psi_i e^{-\hat{\jmath} p_i \cdot x'} + e \int d^4x'' \int \frac{d^4k}{(2\pi)^4} \frac{kA(x'') + A(x'')p_i}{k^2 - m^2 + \hat{\jmath}\epsilon} \psi_i e^{\hat{\jmath} x'' \cdot (k - p_i)} e^{-\hat{\jmath} k \cdot x'}$$

giving the extra contribution to M

$$M' = e^2 \int d^4x' \int d^4x'' \int \frac{d^4k}{(2\pi)^4} A(x') \frac{kA(x'') + A(x'')p_i}{k^2 - m^2 + \hat{\jmath}\epsilon} e^{\hat{\jmath}x'\cdot(p_f - k)} e^{\hat{\jmath}x''\cdot(k - p_i)}.$$

Carrying out the x' and x'' integrations and using one of the resultant δ-functions we have

$$M = 2\pi\delta(E_f - E_i) M_T$$

where the extra contribution to M_T is

$$M_T' = e^2 \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{a_0(p_f - k) a_0(k - p_i)}{k^2 - m^2 + \hat{\jmath}\epsilon} \gamma_0 [k\gamma_0 + \gamma_0 p_i]$$

and

$$a_0(p) = \int d^3\boldsymbol{x} e^{-\boldsymbol{p}\cdot\boldsymbol{x}} \gamma_0 \cdot A(\boldsymbol{x}) = \frac{-Ze}{\lambda^2 + \boldsymbol{p}^2}.$$

Using

$$k^2 - m^2 = \boldsymbol{p}_i^2 - \boldsymbol{k}^2$$

and the integrals

$$I_1 + \tfrac{1}{2}(\boldsymbol{p}_i + \boldsymbol{p}_f) I_2 \;=\; \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{1 + \boldsymbol{k}}{[(\boldsymbol{p}_f - \boldsymbol{k})^2 + \lambda^2][(\boldsymbol{p}_i - \boldsymbol{k})^2 + \lambda^2](\boldsymbol{p}_i^2 - \boldsymbol{k}^2 + \hat{\jmath}\epsilon)}$$

we have

$$M_T' = Z^2 e^4 \left[\gamma_0 \tfrac{1}{2}(\boldsymbol{p}_i + \boldsymbol{p}_f) I_2 + (p_i + \gamma_0 E) I_1\right].$$

In the limit $\lambda \to 0$ our total M_T to second order is therefore

$$M_T = \frac{-Ze^2}{\boldsymbol{q}^2}\gamma_0 + Z^2 e^4 \left[(E\gamma_0 - \tfrac{1}{2}[p_f + p_i]) I_2 + (p_i + \gamma_0 E) I_1\right]$$

where the integrals are[10]

$$\begin{aligned} I_1 &= \frac{-\hat{\jmath}}{16\pi|\boldsymbol{p}|^3 \sin^2(\theta/2)} \ln \frac{2|\boldsymbol{p}| \sin(\theta/2)}{\lambda} \\ I_2 &= \frac{1}{16\pi|\boldsymbol{p}|^3 \cos^2(\theta/2)} \left\{ \frac{\pi[\sin(\theta/2) - 1]}{2\sin^2(\theta/2)} - \hat{\jmath} \ln \frac{\lambda}{2|\boldsymbol{p}|} \right\} + \frac{I_1}{\cos^2(\theta/2)}. \end{aligned}$$

We see that M has some $\hat{\jmath}$ dependence, so writing $I_1 = (A + C)\hat{\jmath}$ and $I_2 = B + C\hat{\jmath}$ where A, B, and C are scalars, and replacing the $\hat{\jmath}$-dependence with $\hat{S}_i$-dependence, this becomes

$$M_T = \gamma_0 \left[-\frac{Ze^2}{\boldsymbol{q}^2} + EZ^2 e^4 \left\{ B + (2C + A)\hat{S}_i \right\} \right] + Z^2 e^4 \left[p_i(A\hat{S}_i - B) - \tfrac{1}{2} q (B + C\hat{S}_i) \right].$$

The term proportional to q does not contribute to T_{fi}. Using

$$p_f p_i + m^2 = E(2E + \boldsymbol{q}) - \boldsymbol{p}^2 - \boldsymbol{p}_f \boldsymbol{p}_i$$

we have

$$T_{fi} = (2E + \boldsymbol{q}) \left[-\frac{Ze^2}{\boldsymbol{q}^2} + 2EZ^2 e^4 (A + C)\hat{S}_i \right] + Z^2 e^4 (\boldsymbol{p}^2 + \boldsymbol{p}_f \boldsymbol{p}_i)(B - A\hat{S}_i).$$

Keeping terms up to α^3 the cross section is governed by

$$|T_{fi}|^2 \;=\; (4E^2 - \boldsymbol{q}^2) \frac{Z^2 e^4}{\boldsymbol{q}^4} - \frac{4Z^3 e^6}{\boldsymbol{q}^2} \left[EB(\boldsymbol{p}^2 + \boldsymbol{p}_f \cdot \boldsymbol{p}_i) + mA(\boldsymbol{p}_i \wedge \boldsymbol{p}_f) \cdot \hat{S}_i^0 \right]$$

where $\hat{S}_i^0$ is the initial rest spin. As expected the divergent parts of the integrals have cancelled out, and we are only left with the finite terms B and

$$A = \frac{\ln \sin(\theta/2)}{16\pi|\boldsymbol{p}|^3 \cos^2(\theta/2)}.$$

The cross section for unpolarized scattering is found by averaging over the initial spin. This gives the spin-independent part of the cross section since the spin dependent part averages to zero. The α^3 contribution is therefore

$$\begin{aligned} \frac{d\sigma}{d\Omega_f}' &= -\frac{4Z^3e^6BE}{4(2\pi)^2\boldsymbol{q}^2}(\boldsymbol{p}^2 + \boldsymbol{p}_f\cdot\boldsymbol{p}_i) \\ &= \frac{\pi\alpha^3 Z^3 E[1-\sin(\theta/2)]}{4|\boldsymbol{p}|^3\sin^3(\theta/2)}. \end{aligned}$$

Hence the unpolarized cross section, including the second Born approximation but ignoring radiative corrections, is

$$\frac{d\sigma}{d\Omega_f} = \left(\frac{d\sigma}{d\Omega_f}\right)_{\text{Mott}} \left\{1 + Z\alpha\pi\frac{\beta\sin(\theta/2)[1-\sin(\theta/2)]}{1-\beta\sin^2(\theta/2)}\right\}$$

in agreement with the result obtained by Dalitz[9] using the conventional matrices and spin-sums approach.

Spin dependence and double scattering

As an example of handling spin dependence we can work out the asymmetry parameter for double scattering from a Coulomb potential. The idea is that since the second order correction to Coulomb scattering is spin dependent the scattered beam will be partially polarized even with an unpolarized incident beam. The scattered beam can then impinge on a second target, which leads to an observable asymmetry in the scattered intensity. The asymmetry was first worked out by Mott.[11,12]

The first thing we need to know is the spin after the first scattering. This is given by

$$\hat{S}_f = R_{fi}\hat{S}_i\widetilde{R}_{fi}$$

so we have

$$\hat{S}_f \propto T_{fi}\hat{S}_i\widetilde{T}_{fi} = \frac{Z^2e^4}{\boldsymbol{q}^4}(2E+\boldsymbol{q})\hat{S}_i(2E-\boldsymbol{q}) - \frac{2Z^3e^6A}{\boldsymbol{q}^2}\left\langle(\boldsymbol{p}^2+\boldsymbol{p}_f\boldsymbol{p}_i)(2E-\boldsymbol{q})\right\rangle_2$$

where we have only kept the lowest order terms in the spin dependent and spin-independent parts. We now define S^0 to be the polarization in the plane $\hat{S}^0$. This is just a bivector in the plane of $\hat{S}^0$ with modulus equal to the polarization of the beam. Since the incoming beam is taken to be unpolarized the resultant polarization plane will be given by the spin-independent part of $\hat{S}_f$ deboosted to rest. To get the polarization we then just divide by the magnitude of the spin-dependent part:

$$\begin{aligned} S_f^0 &= -\frac{2Ze^2\boldsymbol{q}^2A}{(4E^2-\boldsymbol{q}^2)}\widetilde{L}_f\left\langle(\boldsymbol{p}^2+\boldsymbol{p}_f\boldsymbol{p}_i)(2E-\boldsymbol{q})\right\rangle_2 L_f \\ &= \frac{2Ze^2\boldsymbol{q}^2A}{(4E^2-\boldsymbol{q}^2)}2m\boldsymbol{p}_i\wedge\boldsymbol{p}_f. \end{aligned}$$

The spin-dependent part of the cross section for the second scattering is then given by

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega_f}\right)_{\rm spin} &= -\frac{4Z^3e^6mA_2}{\boldsymbol{q}_2^2(2\pi)^2}(\boldsymbol{p}_f\wedge\boldsymbol{p}_2)\cdot S_f^0 \\ &= -\frac{64(2\pi)^2Z^4\alpha^4\boldsymbol{q}_1^2m^2A_1A_2}{\boldsymbol{q}_2^2(4E^2-\boldsymbol{q}_1^2)}(\boldsymbol{p}_f\wedge\boldsymbol{p}_2)\cdot(\boldsymbol{p}_i\wedge\boldsymbol{p}_f)\end{aligned}$$

where the 1 and 2 subscripts refer to the first are second scattering respectively (e.g. $\boldsymbol{q}_2 = \boldsymbol{p}_2 - \boldsymbol{p}_f$). We see that the asymmetry will depend on the cosine of the angle ϕ between the $\boldsymbol{p}_f\wedge\boldsymbol{p}_2$ and $\boldsymbol{p}_i\wedge\boldsymbol{p}_f$ planes. The asymmetry parameter δ is defined so that the final intensity depends on ϕ through the factor

$$1 + \delta\cos\phi.$$

In the case where $\boldsymbol{p}_i\cdot\boldsymbol{p}_f = \boldsymbol{p}_f\cdot\boldsymbol{p}_2 = 0$ ($\boldsymbol{p}_i\cdot\boldsymbol{p}_2 = -\boldsymbol{p}^2\cos\phi$) we find that the first non-zero contribution to the asymmetry factor is

$$\begin{aligned}\delta &= \frac{64(2\pi)^2Z^4\alpha^4m^2A^2}{(4E^2-\boldsymbol{q}^2)}\boldsymbol{p}^4\frac{\boldsymbol{q}^4}{Z^2\alpha^2(4E^2-\boldsymbol{q}^2)} \\ &= Z^2\alpha^2(\ln 2)^2\frac{\beta^2(1-\beta^2)}{(2-\beta^2)^2}\end{aligned}$$

in agreement with the answer quoted by Dalitz.[9] It is of course only the first approximation, and for large Z nuclei higher order corrections will be far from negligible.

The partial spin-sum approach

The above formalism seems to work well for single particle scattering. Here we show how we can adapt a more traditional approach in more complicated cases, demonstrating the flexibility of the STA formalism. The scattering operator approach could equally well be used in the more complicated case, as we show below.

We use the two basis states u_r to write (5) as

$$\begin{aligned}\psi_f &= -\hat{\jmath}e\int d^4x'\sum_r u_r(p_f)\langle\widetilde{u}_r(p_f)A(x')\psi_i\gamma_0\rangle_S e^{\hat{\jmath}p_f\cdot x'} \\ &= \sum_r u_r(p_f)S_{fi}^r\end{aligned}$$

where S_{fi}^r is the traditional S-matrix. The total number density per Lorentz invariant phase space interval is then

$$N_f = \sum_r |S_{fi}^r|^2.$$

As an example we consider electron-muon scattering in which A is given by

$$A(x) = \int d^4x' D_F(x-x')J(x')$$

and $J(x')$ is the 'complex' conserved current given by

$$J_a = e\langle\widetilde{u}_s\gamma_a\psi_2\gamma_0\rangle_S.$$

Defining T^{rs} as usual we have

$$T^{rs} = -\frac{e^2}{q^2}\langle\widetilde{u}_r\gamma_a\psi_1\gamma_0\rangle_S\langle\widetilde{u}_s\gamma^a\psi_2\gamma_0\rangle_S.$$

where $q = p_1' - p_1 = p_2 - p_2'$ and dashed variables correspond to the final states. Summing over r and s

$$\begin{aligned}
|T|^2 &= \frac{e^4}{q^4}\langle\gamma_0\widetilde{\psi}_1\gamma^b(p_1'\gamma^a + \gamma^a p_1)\psi_1\rangle_S\langle\gamma_0\widetilde{\psi}_2\gamma_b(p_2'\gamma_a + \gamma_a p_2)\psi_2\rangle_S \\
&= \frac{2e^4\rho_1\rho_2}{m_1 m_2 q^4}\Big[\langle\gamma^b(p_1'\gamma^a + \gamma^a p_1)p_1\rangle\langle\gamma_b(p_2'\gamma_a + \gamma_a p_2)p_2\rangle \\
&\qquad\qquad - \langle\gamma^b(p_1'\gamma^a + \gamma^a p_1)p_1\hat{S}_1\rangle\langle\gamma_b(p_2'\gamma_a + \gamma_a p_2)p_2\hat{S}_2\rangle\Big] \\
&= \frac{2e^4\rho_1\rho_2}{m_1 m_2 q^4}\Big[p_1'\cdot p_2' p_1\cdot p_2 + p_2'\cdot p_1 p_2\cdot p_1' - m_1^2 p_2'\cdot p_2 + m_2^2 p_1'\cdot p_1 + 2m_2^2 m_1^2 \\
&\qquad\qquad - [q\cdot(\hat{S}_1\wedge p_1)]\cdot[q\cdot(\hat{S}_2\wedge p_2)]\Big].
\end{aligned}$$

This approach differs from the normal one in that we have only done one spin sum over the final spins. We can therefore explicitly retain information about the initial spins, and calculations that are spin-independent will be manifestly so. Spin averaging simply amounts to removing spin dependent terms in the cross-section.

The same result could be obtained using the scattering operator approach using

$$M = eD_F\gamma^a J_a$$

and summing over the final spin of the other particle. One ends up with exactly the same equation. However the scattering operator approach may be better for calculating spin effects. If we are interested in the spin dependence of a particular fermion line the scattering operator approach works well once we have summed over the spins of the other particles. For example we can calculate the final spin and polarization in the same way as we did for Coulomb scattering. In this approach we still have to perform a spin sum, but only over the spins of the other particles. We could of course introduce spin projection operators to single out particular spins of the other particles if necessary.

Conclusions

We have seen how Hestenes' STA formulation of Dirac theory provides a useful and elegant method of performing cross section calculations. Spin is handled in a simple manner, and the logic of calculating cross sections is simplified considerably. We don't perform unnecessary spin sums and spin dependence is manifest in the spin bivector dependence of the scattering operator. It's a simple matter to calculate spin precessions, polarizations and spin dependent results, and the results are automatically expressed in terms of physical spin bivectors and the other scattering parameters. We can perform unpolarized calculations simply by averaging over spins.

In the multiparticle case things are more complicated. We don't have a neat method for performing arbitrary spin dependent calculations, and still have to resort to spin sums over terms involving complex conserved currents. However we can still write down a scattering operator for any given fermion line, retaining the benefits of the scattering operator for calculations involving the spin of the particle.

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