Chapter 1

Applications of Geometric Algebra in Physics and Links with Engineering

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1.1 Introduction

While the early applications of geometric algebra (GA) were confined to physics, there has been significant progress over recent years in applying geometric algebra to areas of engineering and computer science. The beauty of using the same language for these applications is that both engineers and physicists should be able to understand the work done in each others fields. It is the aim of this paper to give brief outlines of the use of GA in the areas of relativity, quantum mechanics and gravitation – all using tools with which anyone working with GA should be familiar. Taking one particular area, multiparticle quantum mechanics, it is shown that the same mathematics may have some interesting applications in the fields of computer vision and robotics.

In this contribution, we review some of the physical applications in which a geometric algebra formulation is particularly helpful. This includes electromagnetism, quantum mechanics and gravitational theory. Then we show how some of these same techniques are of use and interest in engineering. More generally, we show how the availability of a unified mathematical language, able to span both disciplines, is an advantage in allowing professionals from each area to increase their understanding of previously inaccessible material, and make contributions outside their usual areas of expertise. As a case study involving new material, we examine a generalization to 4-d space, of the new conformal representation of 3-d Euclidean space being developed by Hestenes and collaborators (see e.g. chapter **??**). This conformal representation is already finding application in robotics [2] and may also be important in interpolation of rigid body motion [3]. We show how the 4-d version has unexpected links with the mathematics of sophisticated objects called 'twistors', and perhaps even more surprisingly, with multiparticle quantum mechanics. These links then suggest a novel method for carrying out such interpolation, allowing consideration of *velocities* as well as positions in 3-d.

In order to start the discussion of applications of geometric algebra in physics, it is necessary to introduce the *spacetime algebra* or STA – the geometric algebra of relativistic 4-dimensional spacetime. This may seem overly complicated to someone who wishes to see examples of links between physics and engineering expressed in geometric algebra. It is true that relativity theory impinges hardly at all on most engineering practice and applications. However, we shall see below that setting up the STA at the start is useful in areas as diverse as computer vision, quantum computing and (as mentioned before) interpolation of rigid body motion. Also, it is what will allow us to consider applications in physics such as electromagnetism and gravitational theory. Thus this contribution begins with an introduction to the STA and shows briefly how a concept called the *projective split* allows an easy articulation between four dimensions and the concepts of ordinary 3-dimensional geometric algebra.

1.2 The Spacetime Algebra

The *spacetime algebra* or STA is the geometric algebra of Minkowski spacetime. We introduce an orthonormal frame of vectors $\{\gamma_\mu\}$, $\mu = 0 \ldots 3$, such that

$$\gamma_\mu \cdot \gamma_\mu = \eta_{\mu\nu} = (+ - - -) \tag{1.1}$$

The STA has the basis

1	$\{\gamma_\mu\}$	$\{\gamma_\mu \wedge \gamma_\nu\}$	$\{i\gamma_\mu\}$	$i \equiv \gamma_0\gamma_1\gamma_2\gamma_3$
1 scalar	4 vectors	6 bivectors	4 trivectors	1 pseudoscalar

The pseudoscalar i *anti*-commutes with vectors.

At this point we note that generally, when working with single particle algebras, the standard has become to use I for the pseudoscalar; however, here, we will use i to denote the pseudoscalar to avoid later confusion when we discuss the multiparticle STA.

1.2.1 The spacetime split, special relativity and electromagnetism

In special relativity (SR) we deal with a 4-dimensional space; the three dimensions of ordinary Euclidean space, and time. Suppose we have a stationary observer with whom we can associate coordinates of space and time; this observer will observe events from his *spacetime* position. Now suppose that we have another observer travelling at a velocity v – he too will observe events from his continuously changing spacetime position. Relative vectors for an observer moving with velocity v are modelled as bivectors, so $a \wedge v$ gives the vector a seen in the v frame. Usually we take $v = \gamma_0$ and define

$$\sigma_k \equiv \gamma_k \gamma_0 \qquad k = 1, 2, 3 \tag{1.2}$$

The even subalgebra of the STA is then the algebra of relative space, spanned by

$$1,\ \{\sigma_k\},\ \{i\sigma_k\},\ i \tag{1.3}$$

The distinction between relative vectors and relative bivectors is frame-dependent and the process of moving between a vector a in the 4-d STA and its representation $\boldsymbol{a}$ in the relative space is known as the *spacetime split*;

$$\boldsymbol{a} = a \wedge \gamma_0 \tag{1.4}$$

In practical geometric problems occuring in computer vision and computer graphics it is common to move up from our 3-d Euclidean space to work in a 4-d projective space, where non-linear transformations become linear and where intersections of lines, planes etc., are easy to compute. This extra dimension is analogous to the γ_0 in the STA (although in projective geometry one can have either a $(+,+,+,+)$ or a $(+,-,-,-)$ signature) and moving between projective space and Euclidean space can similarly be carried out using the *projective split*, given by

$$\boldsymbol{a} = \frac{a \wedge \gamma_0}{a \cdot \gamma_0} \tag{1.5}$$

Here we are again relating the vectors in relative space (3-d) with bivectors in the higher (4-d) space. Alternatively, we can define the vector in 3-space, $\boldsymbol{a}$ as $a^j \gamma_j$, $j = 1, 2, 3$ and the associated vector in the higher space, a, by

$$a = a_0(a^j \gamma_j + \gamma_0), \quad j = 1, 2, 3 \tag{1.6}$$

so that we have

$$\boldsymbol{a} = \frac{(a \wedge \gamma_0)\gamma_0}{a \cdot \gamma_0} \tag{1.7}$$

Both of the above interpretations have been used in the literature in discussions of projective and conformal geometry, [5, 6].

Returning to the STA, one can conventionally derive a coordinate transformation between the frames of two observers, and to move between these two frames one applies a matrix transformation known as a **Lorentz Boost**. Geometric algebra provides us with a beautifully simple way of dealing with special relativistic transformations using the simple formula for rotations that we will discuss below, namely $a' = Ra\tilde{R}$ ([7, 8]).

We have seen in other contributions in this volume that in any geometric algebra, rotations are achieved by quantities called *rotors*. A rotor R can be written as

$$R = \pm \exp(-B/2) \tag{1.8}$$

where B is a bivector representing the plane in which the rotation takes place. It is then easy to show that a rotation of a vector a to a vector a' is achieved by the equation

$$a' = Ra\tilde{R}$$

In 3-d R is made up of scalar and bivector parts while in 4-d it has scalar, bivector and pseudoscalar parts; in each case it has a double-sided action. We should stress here that a rotor is simply part of the algebra and need not have special operator status. We will see in the following sections that rotors are crucial quantities in much of physics, in particular, we will see that simple rotations in the STA will allow us to understand most of special relativity and will play an important role in quantum mechanics.

The Lorentz boost turns out to be simply a rotor R which takes the time axis to a different position in 4-d ; $R\gamma_0\tilde{R}$. So, in an elegant coordinate-free way we are able to give the transformations of SR an intuitive geometric meaning. All the usual results of SR follow very quickly from this starting point.

Moving now to electromagnetism, the electromagnetic field strength is given by the bivector

$$F \equiv \tfrac{1}{2}F^{\mu\nu}\gamma_\mu\wedge\gamma_\nu \tag{1.9}$$

where the Greek indices μ and ν run over $0,1,2,3$. In the γ_0 frame this decomposes into bivectors of the form $\gamma_i\gamma_0$ and $\gamma_i\gamma_j$ $(i,j,=1,2,3, i\neq j)$, so that we can write

$$F = \boldsymbol{E} + i\boldsymbol{B} \tag{1.10}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields and are given by $\boldsymbol{E} = E^k\sigma_k = \frac{1}{2}(F - \gamma_0 F\gamma_0)$ and $\boldsymbol{B} = B^k\sigma_k = \frac{1}{2}(F + \gamma_0 F\gamma_0)$. Here, sandwiching between γ_0 flips the sign of the $\gamma_i\gamma_0$ bivectors but leaves the $\gamma_i\gamma_j$ bivectors unaltered. This form of F explains the usefulness of complex numbers in electromagnetism. Now, let us define the 4-d gradient operator as

$$\nabla \equiv \gamma^\mu \frac{\partial}{\partial x^\mu} \tag{1.11}$$

It is not hard to show that the full Maxwell equations can then be written simply as

$$\nabla F = J \tag{1.12}$$

where J is the source current. The above formulation of electromagnetism is also being used in several engineering applications, e.g. surface scattering of EM waves from objects such as ships, antennae design etc.

As an example of the simplifications that this approach can afford, consider what the electric and magnetic fields look like under a Lorentz boost. The conventional complicated formulae for the transformation are now replaced by the result

$$\boldsymbol{E}' + i\boldsymbol{B}' = R(\boldsymbol{E} + i\boldsymbol{B})\tilde{R}$$

where dashes denote transformed quantities and R is the rotation in the STA representing the boost.

To illustrate this explicitly, and to make the link with the standard formulae, we consider a boost with velocity parameter u (so the actual velocity is $\tanh u$) in the x direction, where the original field is $\boldsymbol{E} = E\sigma_y$, i.e. an electric field in the y direction only with no magnetic field component. We have $R = e^{\frac{u}{2}\sigma_x}$ and so

$$\begin{aligned} \boldsymbol{E}' + i\boldsymbol{B}' &= Ee^{\frac{u}{2}\sigma_x}\sigma_y e^{-\frac{u}{2}\sigma_x} \\ &= Ee^{u\sigma_x}\sigma_y = E(\cosh u + \sigma_x \sinh u)\sigma_y \\ &= E(\cosh u\sigma_y + i\sigma_z \sinh u) \end{aligned} \tag{1.13}$$

We can clearly see a $\boldsymbol{B}$ field is induced in the z-direction, with amplitude $E\sinh u$. Note the electromagnetic invariants arise immediately via the relation

$$F'^2 = RF\tilde{R}RF\tilde{R} = RF^2\tilde{R} = F^2 \tag{1.14}$$

since F^2 contains only scalar and pseudoscalar parts. Specifically we have

$$\begin{aligned} \text{Scalar part of } F^2 &= \boldsymbol{E}^2 - \boldsymbol{B}^2 \\ \text{Pseudoscalar part of } F^2 &= 2i\boldsymbol{E}\cdot\boldsymbol{B} \end{aligned} \tag{1.15}$$

and so $\boldsymbol{E}^2 - \boldsymbol{B}^2$ and $\boldsymbol{E}\cdot\boldsymbol{B}$ are invariant under any Lorentz transformation (as may be checked for the example above).

The complicated tensor formula for the electromagnetic stress energy tensor becomes extremely simple in the STA. We find the flow of energy/momentum through a hypersurface normal to the vector n is given by

$$T(n) = \frac{1}{2}Fn\tilde{F} \tag{1.16}$$

That is, we just rotate n by F!

This then easily leads to the standard Heaviside and Poynting formulae for the energy density and momentum flow in a given frame (e.g. the γ_0 frame). (For further examples and details see e.g. [8, 9, 10] and [11].) The STA really does seem to capture the essence of electromagnetism in a very compact and useful formalism!

1.3 Quantum mechanics

In non-relativistic quantum mechanics there are important quantities known as *Pauli spinors* – using these spinors we are able to write down the *Pauli equation* which governs the behaviour of a quantum mechanical state in some external field. The equation involves quantities called *spin operators* which are conventionally seen as completely different entities to the *states*. Using the 3-d geometric algebra we are able to write down the equivalent to the Pauli equation where the operators and states are all real-space multivectors – indeed the spinors become proportional to rotors of the type we have discussed earlier. The algebra of the $\{\sigma_i\}$ is isomorphic to the algebra of Pauli spin matrices.

To see how this works in a simple context, we consider the case of an electron in a magnetic field. A conventional quantum Pauli spinor $|\psi\rangle$, which is normally written as a two component complex column 'vector', is put into 1-1 correspondence with a GA spinor ψ (an even element of the geometric algebra of 3-d space) via:

$$|\psi\rangle = \begin{pmatrix} a^0 + ja^3 \\ -a^2 + ja^1 \end{pmatrix} \leftrightarrow \psi = \; a^0 + a^k i\sigma_k \tag{1.17}$$

(Note the symbol j is used for the unit scalar imaginary of quantum mechanics). We are interested in how the electron spin behaves, and will ignore any spatial variation. It is then easy to show that the GA form of the Pauli equation for this setup is

$$\frac{d\psi}{dt} = \tfrac{1}{2}\gamma i \boldsymbol{B}\psi \tag{1.18}$$

Here $\boldsymbol{B}$ is the magnetic field as described in the previous section and γ is the 'gyromagnetic ratio'. ($\gamma \approx e/m$ for an electron, where e and m are the electon charge and mass.) Any Pauli spinor can be decomposed as $\psi = \rho^{\frac{1}{2}}R$, where ρ is a scalar and R is a rotor. Substituting this form into (1.18), multiplying by $\tilde{\psi}$ and denoting time derivatives by an overdot, we obtain

$$\tfrac{1}{2}\dot{\rho} + \rho\dot{R}\tilde{R} = \tfrac{1}{2}\rho\gamma i\boldsymbol{B} \tag{1.19}$$

It is straightforward to show that $R\tilde{R} = 1$ implies $\dot{R}\tilde{R}$ is a bivector. The right hand side of (1.19) is also a bivector, so we deduce $\dot{\rho} = 0$. The scale thus drops out of the problem and the dynamics reduces to the rotor equation

$$\dot{R} = \tfrac{1}{2}\gamma i \boldsymbol{B} R \tag{1.20}$$

The conventional approach is unable to work with this single rotor equation, but instead has to work with two coupled complex equations, one for each of the components of the quantum state. Although the underlying physics is the same, the rotor form is often significantly easier to solve (e.g., for a constant field $\boldsymbol{B} = B_0\sigma_3$ along the z-axis, we can immediately intergrate to find $\psi(t) = \exp(\gamma B_0 t i\sigma_3/2)\psi_0$) and makes the analogue with the corresponding classical system much more transparent.

Relativistic quantum mechanics is conventionally described by the Dirac algebra, where the *Dirac equation* again tells us about the state of the particle in an external field. Here we use the 4-d spacetime geometric algebra with the algebra of the $\{\gamma_\mu\}$ isomorphic to that of the Dirac matrices. Again the *wavefunction* in conventional quantum mechanics becomes an instruction to rotate a basis set of axes and align them in certain directions – analogous to the theory of rigid body mechanics! We see therefore that there is a significant shift in interpretation; in GA, the states and operators no longer live in different spaces but are instead simply multivector elements of the geometric algebra.

Thus, with the STA, we can eliminate matrices and complex numbers from the Dirac theory. Suppose we start with the standard Dirac matrices:

$$\hat{\gamma}^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \hat{\gamma}^i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ -\hat{\sigma}_i & 0 \end{pmatrix} \tag{1.21}$$

where the $\{\hat{\sigma}_i\}$ are the usual Pauli spin matrices and I is the 2×2 identity matrix. A Dirac column spinor $|\psi\rangle$ maps onto an element of the 8-d even subalgebra (a spinor) of the STA via the following:

$$|\psi\rangle = \begin{pmatrix} a^0 + ja^3 \\ -a^2 + ja^1 \\ b^0 + jb^3 \\ -b^2 + jb^1 \end{pmatrix} \leftrightarrow \psi = \begin{matrix} a^0 + a^k i\sigma_k \\ +(b^0 + b^k i\sigma_k)\sigma_3 \end{matrix} \tag{1.22}$$

Dirac matrix operations are now replaced by:

$$\hat{\gamma}^\mu |\psi\rangle \leftrightarrow \gamma^\mu \psi \gamma_0 \qquad j|\psi\rangle \leftrightarrow \psi i\sigma_3 \tag{1.23}$$

This enables us to write the Dirac equation as

$$\nabla \psi i\sigma_3 - eA\psi = m\psi\gamma_0 \tag{1.24}$$

where $\nabla = \gamma^\mu \partial_\mu$ is the gradient operator defined in the previous section and A is the 4-potential of the external electromagnetic field. Note this equation — often referred to as the Hestenes form of the Dirac equation —

is independent of choice of matrix representation and is therefore the best form in which to expose the geometric content of Dirac theory.

In the conventional approach it is usual to define an additional operator $\hat{\gamma}_5 = -j\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$ — in our approach this is replaced by right multiplication by σ_3. Two of the main observables of Dirac theory (J and s, the so-called bilinear covariants) become:

$$J^\mu = \langle\bar{\psi}\hat{\gamma}^\mu\psi\rangle \quad \leftrightarrow \quad \langle\tilde{\psi}\gamma^\mu\psi\gamma_0\rangle = \gamma^\mu \cdot J$$
$$s^\mu = \langle\bar{\psi}\hat{\gamma}_5\hat{\gamma}^\mu\psi\rangle \quad \leftrightarrow \quad \langle\tilde{\psi}\gamma^\mu\psi\gamma_3\rangle = \gamma^\mu \cdot s$$

where $\bar{\psi}$ is the *Dirac adjoint.* The key quantities are the STA vectors J and s:

$$J = \psi\gamma_0\tilde{\psi} \qquad s = \psi\gamma_3\tilde{\psi} \tag{1.25}$$

(Note that there exist a set of identities called the Fierz identities which, in the above formulation, reduce to simple vector manipulations.)
It is now possible to decompose the spinor ψ in a Lorentz invariant manner;

$$\psi\tilde{\psi} = \rho e^{i\beta} \quad = \quad \text{scalar} + \text{pseudoscalar} \tag{1.26}$$

Using this decomposition we can write ψ as follows

$$\psi = (\rho e^{i\beta})^{1/2} R \tag{1.27}$$

where R is a spacetime rotor. The observables now become

$$J = \rho R\gamma_0\tilde{R} \qquad s = \rho R\gamma_3\tilde{R} \tag{1.28}$$

so the spinor reduces to an instruction to rotate the $\{\gamma_\mu\}$ frame onto the frame of observables. The STA framework for quantum mechanics has been applied in tunnelling theory [12] where it is capable of plotting streamlines representing the path of a particle inside a barrier. It is then easy to calculate *tunnelling times*, the time a particle spends within a barrier, – something which is much harder to do in conventional quantum mechanics where the concepts of imaginary time or momentum preclude straightforward calculations. Applications in electron scattering [13] have reformulated much of conventional theory allowing spin sums to be done straightforwardly and revealing rotor-structure at the heart of the formulation. For further details of applications to quantum theory, see [14]. The above once again illustrates that using geometric algebra one is able to deal with complex subjects such as relativistic quantum mechanics using those same tools used in current engineering applications of geometric algebra.

1.4 Gravity as a gauge theory

A gauge theory occurs if we stipulate that global symmetries must also be local symmetries – electromagnetism is a gauge theory where the symmetries are called *phase rotations*. Making these local, i.e. able to change arbitrarily from one spacetime position to the next, implies the introduction of *forces*. In geometric algebra, gravity can also be regarded as a gauge theory. If we require that the physics at all points of spacetime is invariant under arbitrary local displacements and rotations (recall that a 4-d rotation is a Lorentz boost), the *gauge field* that results is the gravitational field. Thus, the aim is to produce a gauge theory of gravity employing fields in a 'flat' background spacetime (defined by the STA); we then have no need for the complex notions of curved spacetime that are associated with Einstein's theory of general relativity. How can we construct such a theory without imposing some form of *absolute* Newtonian space? We start by ensuring that the following criteria are satisfied:

1. The physical content of a field equation must be unchanged under arbitrary local field displacements.
2. The physical content of a field equation must be unchanged under arbitrary local rotations of the fields.

In looking at how the resulting gauge theory differs from past gauge-theoretic approaches to gravity, we note the following points:

1. It is different from Poincaré gauge theory, which retains the ideas of a curved spacetime background.
2. There is no need to restrict to infinitesimal transformations; within GA we can work with finite rotations.
3. The need for principle 2 only emerges fully from a theory based on the Dirac equation.

To see mathematically what the symmetry constraints impose we first consider a relation of the type

$$a(x) = b(x) \tag{1.29}$$

which equates spacetime vectors at the same point. Now we introduce new fields

$$a'(x) \equiv a(x') \qquad b'(x) \equiv b(x') \tag{1.30}$$

where $x' = f(x)$ is some arbitrary (nonlinear) mapping between position vectors. The equation

$$a'(x) = b'(x) \tag{1.31}$$

has exactly the same content as the original equation, since the value of x is irrelevant provided it covers all of spacetime. This is true for *arbitrary* displacements.

In order to satisfy our previous conditions we require that this holds for all physical equations.

Next consider a relation of the type $a(x) = \nabla\phi(x)$. If we replace $\phi(x)$ with $\phi'(x) = \phi(x')$ we must now consider ∇ acting on the new scalar field $\nabla\phi(f(x)))$; by using the definition of the vector derivative it can be shown that [15]

$$\nabla\phi'(x) = \bar{\mathsf{f}}[\nabla_{x'}\phi(x')] \tag{1.32}$$

where

$$\mathsf{f}(a) = a\cdot\nabla f(x) \tag{1.33}$$

Here, $\mathsf{f}(a) = \mathsf{f}(a, x)$ is a linear function of its vector argument, and a nonlinear function of position, $\bar{\mathsf{f}}(a)$ is its adjoint. The appearance of this function means that the equation does not have the required transformation property.

We repair this by replacing ∇ with a new derivative $\bar{\mathsf{h}}(\nabla)$, where $\mathsf{h}(a)$ is a linear function of a and has arbitrary position dependence; we call $\mathsf{h}(a) = \mathsf{h}(a, x)$ the *position gauge field*. The adjoint function is written $\bar{\mathsf{h}}(a)$. Under a local displacement, this is defined to transform as

$$\bar{\mathsf{h}}(a, x) \mapsto \bar{\mathsf{h}}'(a, x) = \bar{\mathsf{h}}[\bar{\mathsf{f}}^{-1}(a), x'] \tag{1.34}$$

This law ensures that the equation

$$a(x) = \bar{\mathsf{h}}[\nabla\phi(x)] \tag{1.35}$$

is now *covariant*, in the required manner, i.e. under a change of position the equation takes the same form but is evaluated at that new position.

Recovering General Relativity

Using the linear function h it is now possible to recover classical general relativity (GR). To do this we first introduce a set of local coordinates $x^\mu = x^\mu(x)$, with coordinate frames

$$e_\mu = \partial_\mu x \qquad e^\mu = \nabla x^\mu \tag{1.36}$$

we can then recover a metric as follows:

$$g_{\mu\nu} = \mathsf{h}^{-1}(e_\mu)\cdot\mathsf{h}^{-1}(e_\nu) \tag{1.37}$$

This metric is then treated as a field in a flat background spacetime.

Rotations

As we have indicated in previous sections, rotations are often the key to the simplifications provided by GA. In this application it is again true that rotations are key to the novelty of this new approach, and also the key to *torsion*. Let us return to the equation $a(x) = b(x)$. Note that the physical content of this equation is unchanged if we replace a and b by

$$a'(x) = Ra(x)\tilde{R} \qquad b'(x) = Rb(x)\tilde{R} \tag{1.38}$$

since $a = b \implies a' = b'$. The physics is unchanged, provided the absolute direction of the vector in the STA does not enter (the second of our two principles). Again, this argument holds for *arbitrary*, *local* rotations.
To ensure that relations of the type

$$a = \bar{\mathsf{h}}(\nabla\phi) \tag{1.39}$$

remain unchanged, we are led to the transformation law

$$\bar{\mathsf{h}}(a) \mapsto \bar{\mathsf{h}}'(a) = R\bar{\mathsf{h}}(a)\tilde{R} \tag{1.40}$$

for $\bar{\mathsf{h}}$ under local rotations.

What does general relativity have to say about this transformation? — surprisingly, *nothing!*
The metric $g_{\mu\nu}$ is unchanged by this transformation, as are the components of covariant quantities:

$$F_{\mu\nu} = F\cdot[\mathsf{h}^{-1}(e_\mu)\wedge\mathsf{h}^{-1}(e_\nu)] \tag{1.41}$$

Both F and h rotate to leave the components unchanged.
(Most of) classical general relativity can be formulated in the STA without mentioning the rotation gauge. But do we also need to consider the Poincaré group? In fact, it is already fully encompassed by allowing arbitrary displacements.

This then leads us to ask the question of whether we have to address the rotation group at all? The answer to this question is *Yes!*; it is indeed unavoidable in the Dirac theory. We can see this by recalling the fact that observables such as $J = \psi\gamma_0\tilde{\psi}$ imply the spinor transformation law

$$\psi \mapsto \psi' = R\psi \tag{1.42}$$

Since this cannot be hidden, we are forced to introduce a new gauge field to make the Dirac theory invariant under local rotations.

Now let us look at the directional derivatives of $\nabla(R\psi)$:

$$\begin{aligned} a\cdot\nabla(R\psi) &= a\cdot\nabla R\psi + Ra\cdot\nabla\psi \\ &= R[\tilde{R}a\cdot\nabla R\,\psi + a\cdot\nabla\psi] \end{aligned}$$

Note that the quantity $\tilde{R}a\cdot\nabla R$ is a *bivector*. We now define the spinor covariant derivative as

$$D_a\psi \equiv a\cdot\nabla\psi + \tfrac{1}{2}\Omega(a)\psi \tag{1.43}$$

$\Omega(a)$ is a bivector-valued linear function of a, with nonlinear position dependence which has the transformation law

$$\Omega(a) \mapsto \Omega'(a) = R\Omega(a)\tilde{R} - 2a\cdot\nabla R\,\tilde{R} \tag{1.44}$$

We are now able to write down the minimally-coupled Dirac equation:

$$\bar{\mathsf{h}}(\partial_a)D_a\psi i\sigma_3 = m\psi\gamma_0 \tag{1.45}$$

The $\{\partial_a, a\}$ construction is a frame-free way of writing a contraction (see [15] for further details).

Observables

We see that it is now possible to differentiate covariant vectors:

$$a\cdot\nabla J = a\cdot\nabla\psi\gamma_0\tilde{\psi} + \psi\gamma_0 a\cdot\nabla\tilde{\psi} \tag{1.46}$$

which suggests that we define the derivative

$$\begin{aligned}\mathcal{D}_a J &= (D_a\psi)\gamma_0\tilde{\psi} + \psi\gamma_0(D_a\psi)\tilde{} \\ &= a\cdot\nabla J + \Omega(a)\times J\end{aligned}$$

This is the covariant derivative for multivectors, where $A\times B = \frac{1}{2}(AB - BA)$ represents the Hestenes commutator product [25].

From $\Omega(a)$ we define

$$\omega(a) = \Omega\mathsf{h}(a) \tag{1.47}$$

which is covariant under local displacements, and only sees the rotation group. When the rotation gauge is fixed, the quantities in $\omega(a)$ become physical observables (measurable). Classical general relativity has no analogue of these.

Note here that the full covariant derivative is

$$\mathcal{D} = \bar{\mathsf{h}}(\partial_a)\mathcal{D}_a. \tag{1.48}$$

The rest of the theory then proceeds by defining the following field strength tensor

$$[D_a, D_b]\psi = R(a\wedge b)\psi \tag{1.49}$$

By a double contraction we can get the Ricci scalar

$$\mathcal{R} = [\bar{\mathsf{h}}(\partial_b)\wedge\bar{\mathsf{h}}(\partial_a)]\cdot R(a\wedge b) \tag{1.50}$$

This can then be used in an action principle requiring stationarity of

$$\int d^4x \det\mathsf{h}^{-1}(\frac{1}{2}\mathcal{R} - \kappa\mathcal{L}_m) \tag{1.51}$$

where the matter Lagrangian is $\mathcal{L}_m$ and with $\Omega(a)$ and $\bar{\mathsf{h}}(a)$ as the dynamical variables ($\kappa = 8\pi G$ is the gravitational coupling constant).

The result is a theory which locally reproduces the equations of the ECKS (Einstein, Cartan, Kibble, Sciama) extension of GR with the following notable differences:

- it sits in a topologically trivial flat spacetime
- has all the advantages of flat-space STA still available for calculations
- finite gauge rotations and displacements are allowed
- the torsion type is uniquely picked out ($\partial_{\Omega(a)}\mathcal{L}_m = \mathcal{S}(a) =$ torsion tensor. This must be of the Dirac type, i.e. $\partial_a \cdot \mathcal{S}(a) = 0$ for minimal coupling)
- Physical observables and gauge covariant quantities of the theory are clearly picked out.

1.4.1 Some Applications

In this section we briefly outline some of the applications of this gauge theory of gravity (GTG).

1. *Covariant and gauge-invariant calculation of cosmic microwave background (CMB) anisotropies.*
   The GTG approach provides a completely unified scheme for scalar, vector and tensor quantities. It has been applied very successfully to the gauge-invariant calculation of CMB anisotropies [16] and to the development of perturbations, where it recovers the covariant approach of Ellis and coworkers [17].
2. *Topological applications.*
   Despite sitting in a topologically trivial flat spacetime, the GTG can in fact be applied to some situations which would conventionally be thought of as involving topology. It is found that entities like *cosmic strings* are allowed and can be treated (similar to the Aharanov-Bohm effect in electromagnetism), but that *wormholes, kinks, Kruskal-Szekeres and all forms of double cover* are ruled out under this theory.
3. *Cosmic Stings*
   A new *spinning* cosmic string solution [18] has been found which corrected an earlier GR-based attempt.
4. *Singularities*
   The availability of integral theorems (Gauss etc.) means that we

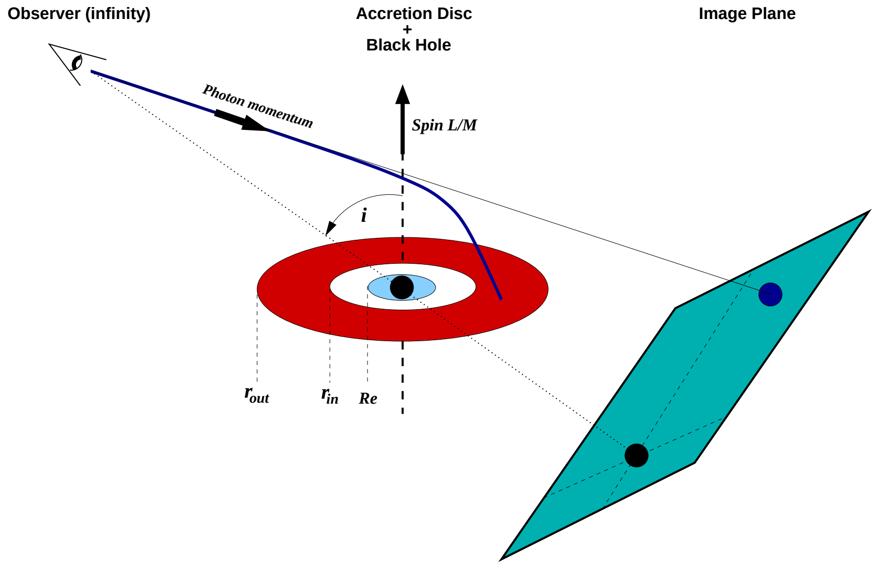

FIGURE 1.1. Setup for computing spectral lineshapes using the GTG approach .

can study the structure of singularities in new ways. For example, the singularity at the centre of the Kerr solution is revealed to be a ring of matter rotating at light-like velocity, but with a ring of *pure tension* stretched across it [19]. Such conclusions are gauge invariant.

5. Spectral lineshapes
A recent project [20] has concentrated on calculating spectral lineshapes from iron-line fluoresence in accretion discs around black holes in active galactic nuclei (AGN). Here the GTG provides an efficient calculational tool and gives a clear approach to the physical (gauge invariant) predictions, see figure 1.1. Results so far, for a particular active galactic nucleus, show that if a is the specific angular momentum of the black hole line and M is its mass, then $a/M > 0.9$ at 90 percent confidence, giving some of the first quantitative evidence for a *spinning* black hole. In this approach the 2nd order GR geodesic equations are replaced by first order equations for a rotor which describes the photon momentum. Integrating the rotor equations in such a setup has links with the procedures required when dealing with buckling beams and deforming elastic fibres (see below).

6. *Black holes*
In the GTG approach, black holes have a memory of the direction of time in which they were formed encoded in them. This means that

the first order (in derivatives) nature of the GTG results in time-reversal properties which are slightly different than those predicted in GR based on metric (second order) theory. A full discussion of this may be found in [15].

7. *Spinning Black Holes*
The GTG has produced a new and very simple form of the Kerr solution for spinning black holes [21]. This is called the *Newtonian Kerr* and takes the form

$$\bar{\mathsf{h}}(a) = a - a\cdot\hat{e}_u \sqrt{\frac{2M \sinh u}{L \cosh^2 u}} V \tag{1.52}$$

where we work in oblate spheroidal coordinates (t, u, ϕ, v), and the velocity vector V is given by

$$V = \frac{\cosh u \gamma_0 + \cos v \hat{\phi}}{\sqrt{\cosh^2 u - \cos^2 v}} \tag{1.53}$$

This provides a *global* solution which is not much more complicated than the Schwarzschild solution for stationary black holes.

1.4.2 Summary

This section has given an outline of how GA can be used to formulate a gauge theory of gravity and in the process reduces the tensor manipulations of general relativity to nothing more than linear algebra. The same tools are used throughout. Indeed it may be possible to use linear functions, which act in the same way as the h functions, to model elasticity. The concept of a frame of reference that varies in either space or time (or both) is also at the heart of much work that tries to understand deforming bodies. A very simple example is provided by a beam of uniform cross-section subject to some loading along its length. We can describe this deformation by splitting up the beam into very small segments and attaching a frame to the centre of mass of each segment. As the beam deforms and is subjected to torsional forces, we can describe its position at a given time by a series of translations and rotations specifying the positions and orientations of each element, see figure 1.2.

Current work [22] has focussed on rewriting conventional buckling equations in terms of GA which has the advantage of allowing us to deal with finite rotations and to interpolate resulting rotor fields. However, the methods outlined in this section present us with the possibility of employing more sophisticated techniques for such problems and for more general problems involving the deformation of long elastic fibres under given boundary conditions.

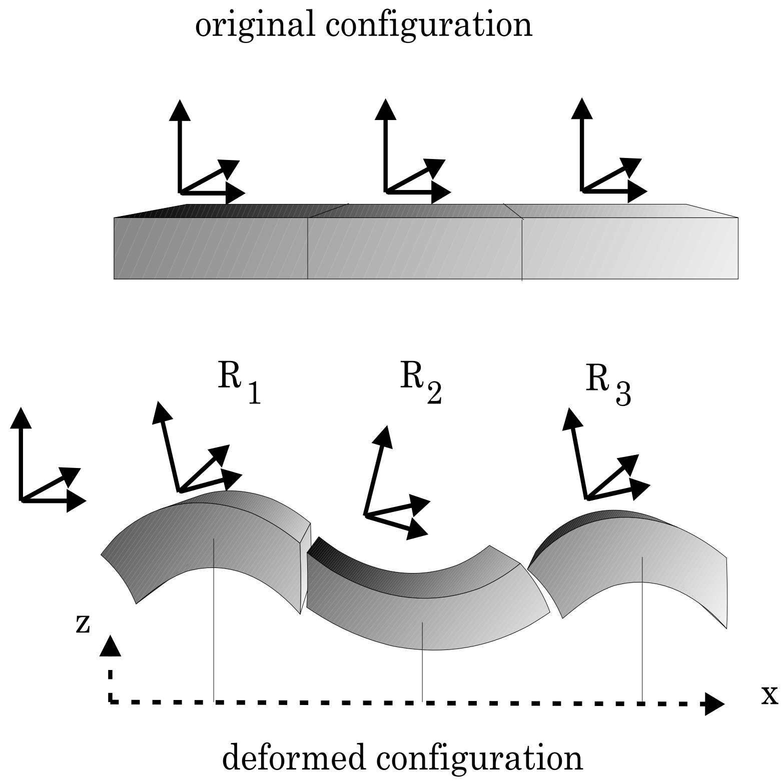

FIGURE 1.2. Model of a beam split into very small segments – the deformation is described by the positon and orientation of each segment

1.5 A New Representation of 6-d Conformal Space

A useful new application of geometric algebra to Euclidean geometry has been given by Hestenes *et al.* [**?**]. This uses a 5-d space to provide a conformal model of Euclidean geometry. Specifically two null vectors, e and e^* are adjoined to Euclidean space, which anticommute with the three basis vectors of Euclidean space and satisfy

$$e \cdot e^* = 1. \tag{1.54}$$

Two key results are

1. If x and y are the 5-d vectors representing 3-d points $\boldsymbol{x}$ and $\boldsymbol{y}$, then the inner product in 5-d gives a measure of the distance between the points in 3-d:

$$x \cdot y = -\frac{1}{2} (\boldsymbol{x} - \boldsymbol{y})^2 . \tag{1.55}$$

2. Secondly, both translations and rotations in 3-d are representable by *versor* multiplication in 5-d. We can write

$$x' = DxD^{-1}, \tag{1.56}$$

where $D = T_{\boldsymbol{a}}R$, R represents a rotation about a direction in 3-d and $T_{\boldsymbol{a}} = 1 + \frac{1}{2}\boldsymbol{a}e$ is a translation in the direction $\boldsymbol{a}$. Note $\boldsymbol{a}e$ is a null bivector in the 5-d space, and so $T_{\boldsymbol{a}}$ can be written in the exponential form $\exp(\frac{1}{2}\boldsymbol{a}e)$.

This new conformal model of Euclidean geometry appears to be rich in applications to computer vision and robotics (see chapter **??**, where the authors use the conformal model for the algebra of incidence and for estimating Euclidean motion). One possible application is to the problem of the joint interpolation of rotational and translational motion of robot arms (see e.g. [3, 4]), where the ability to write the motion in versor form could be of great benefit. Here, we consider a similar model but applied to relativistic rather than Euclidean geometry. This can be achieved in two ways. Firstly, one could use a 6-d space in which two extra null vectors satisfying $e \cdot e^* = 1$ have been added to a 4-d Lorentzian space. This is the obvious generalization of Hestenes' method to one dimension up, and should work very well as something to apply to relativistic problems (e.g. it may allow the problem of motion interpolation to be extended to include interpolation of *velocities* as well as positions – this is currently being investigated). However, as a novel method, one may instead use the 2-particle space of the 'multiparticle STA' (see [14, 11]), which is in fact 8-dimensional, and it is this we consider in detail here. The reason for wishing to stress this new method, is that it sheds wholly unexpected light on the links between such disparate concepts as multiparticle quantum mechanics, relativity, twistors, 2-spinors and the Hestenes conformal representation. Many of these are things which 'engineers' might never have expected to find out about, or to be related to things they wish to know, but here we show how they are in fact intimately related. In particular, we show that the re-expression of twistor theory in multiparticle GA, shows that the main results of the Hestenes conformal representation method are already-known aspects of twistor theory! The links between both of these and multiparticle quantum mechanics appear to be wholly new. (There is even an exciting hint in the work that it will allow a new and concrete expression of the particle physics concept of *supersymmetry*.) We give here just the bare outline of the method – a more detailed exposition is in preparation.

1.5.1 The multiparticle STA

To get started on this topic we need to understand aspects of the multiparticle theory within geometric algebra. The MSTA (Multiparticle SpaceTime Algebra) approach is capable of encoding multiparticle wavefunctions, and describing the correlations between them. The presentation here is hopefully complementary to the presentation given in the context of quantum computing by Havel (see chapter **??**) and parallels that given in Chapter 11 of the Banff Lectures [11] by Lasenby, Gull and Doran and in the review paper by Doran *et al.* 'Spacetime Algebra and Electron Physics' [14].

The n-particle STA is created simply by taking n sets of basis vectors $\{\gamma_\mu^i\}$, where the superscript labels the particle space, and imposing the geometric algebra relations

$$\begin{aligned} \gamma_\mu^i\gamma_\nu^j + \gamma_\nu^i\gamma_\mu^j &= 0, && i \neq j \\ \gamma_\mu^i\gamma_\nu^j + \gamma_\nu^i\gamma_\mu^j &= 2\eta_{\mu\nu} && i = j. \end{aligned} \tag{1.57}$$

These relations are summarised in the single formula

$$\gamma_\mu^i \cdot \gamma_\nu^j = \delta^{ij}\eta_{\mu\nu}. \tag{1.58}$$

The fact that the basis vectors from distinct particle spaces anticommute means that we have constructed a basis for the geometric algebra of a $4n$-dimensional configuration space. (Note the extra dimensions serve simply to label the properties of each individual particle, and should not be thought of as existing in anything other than a mathematical sense.)

Throughout, Roman superscripts are employed to label the particle space in which the object appears. So, for example, ψ^1 and ψ^2 refer to two copies of the same 1-particle object ψ, and not to separate, independent objects. Separate objects are given distinct symbols while the absence of superscripts denotes that all objects have been collapsed into a single copy of the STA.

1.5.2 2-Particle Pauli States and the Quantum Correlator

As an introduction to the properties of the multiparticle STA, we first consider the 2-particle Pauli algebra and the spin states of pairs of spin-1/2 particles. As in the single-particle case, the 2-particle Pauli algebra is just a subset of the full 2-particle STA. A set of basis vectors is defined by

$$\sigma_i^1 = \gamma_i^1\gamma_0^1 \tag{1.59}$$

$$\sigma_i^2 = \gamma_i^2\gamma_0^2 \tag{1.60}$$

which satisfy

$$\sigma_i^1\sigma_j^2 = \gamma_i^1\gamma_0^1\gamma_j^2\gamma_0^2 = \gamma_i^1\gamma_j^2\gamma_0^2\gamma_0^1 = \gamma_j^2\gamma_0^2\gamma_i^1\gamma_0^1 = \sigma_j^2\sigma_i^1. \tag{1.61}$$

So, in constructing multiparticle Pauli states, the basis vectors from different particle spaces commute rather than anticommute. Using the elements $\{1, i\sigma_k^1, i\sigma_k^2, i\sigma_j^1\, i\sigma_k^2\}$ as a basis, we can construct 2-particle states. Here we have introduced the abbreviation

$$i\sigma_i^1 \equiv i^1\sigma_i^1 \tag{1.62}$$

since, in most expressions, it is obvious which particle label should be attached to the i. In cases where there is potential for confusion, the particle label is put back on the i. The basis set $\{1, i\sigma_k^1, i\sigma_k^2, i\sigma_j^1\, i\sigma_k^2\}$ spans a 16-dimensional space, which is twice the dimension of the direct product space of two 2-component complex spinors. For example, the outer-product space of two spin-1/2 states can be built from complex superpositions of the set

$$\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}, \tag{1.63}$$

which forms a 4-dimensional complex space (8 real dimensions). Here the $(1,0)^T$ and $(0,1)^T$ symbols refer to the spin up and spin down states of conventional quantum mechanics, often written as $|\uparrow\rangle$ and $|\downarrow\rangle$ respectively. The dimensionality has doubled because we have not yet taken the complex structure of the spinors into account. While the role of j is played in the two single-particle spaces by right multiplication by $i\sigma_3^1$ and $i\sigma_3^2$ respectively, standard quantum mechanics does not distinguish between these operations. A projection operator must therefore be included to ensure that right multiplication by $i\sigma_3^1$ or $i\sigma_3^2$ reduces to the same operation. If a 2-particle spin state is represented by the multivector ψ, then ψ must satisfy

$$\psi i\sigma_3^1 = \psi i\sigma_3^2 \tag{1.64}$$

from which we find that

$$\begin{aligned} & \psi = -\psi i\sigma_3^1\, i\sigma_3^2 \\ \implies & \psi = \psi \tfrac{1}{2}(1 - i\sigma_3^1\, i\sigma_3^2). \end{aligned} \tag{1.65}$$

On defining

$$E = \tfrac{1}{2}(1 - i\sigma_3^1\, i\sigma_3^2), \tag{1.66}$$

we find that

$$E^2 = E \tag{1.67}$$

so right multiplication by E is a projection operation. (The relation $E^2 = E$ means that E is technically referred to as an 'idempotent' element.) It follows that the 2-particle state ψ must contain a factor of E on its right-hand side. We can further define

$$J = Ei\sigma_3^1 = Ei\sigma_3^2 = \tfrac{1}{2}(i\sigma_3^1 + i\sigma_3^2) \tag{1.68}$$

so that

$$J^2 = -E. \tag{1.69}$$

Right-sided multiplication by J takes on the role of j for multiparticle states.

The STA representation of a direct-product 2-particle Pauli spinor is now given by $\psi^1\phi^2 E$, where ψ^1 and ϕ^2 are spinors (even multivectors) in their own spaces. A complete basis for 2-particle spin states is provided by

$$\begin{aligned} \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} &\leftrightarrow E \\ \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} &\leftrightarrow -i\sigma_2^1 E \\ \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} &\leftrightarrow -i\sigma_2^2 E \\ \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} &\leftrightarrow i\sigma_2^1\, i\sigma_2^2 E. \end{aligned} \tag{1.70}$$

This procedure extends simply to higher multiplicities. All that is required is to find the 'quantum correlator' E_n satisfying

$$E_n i\sigma_3^j = E_n i\sigma_3^k = J_n \qquad \text{for all } j,\, k. \tag{1.71}$$

E_n can be constructed by picking out the $j = 1$ space, say, and correlating all the other spaces to this, so that

$$E_n = \prod_{j=2}^{n} \tfrac{1}{2}(1 - i\sigma_3^1\, i\sigma_3^j). \tag{1.72}$$

The value of E_n is independent of which of the n spaces is singled out and correlated to. The complex structure is defined by

$$J_n = E_n i\sigma_3^j, \tag{1.73}$$

where $i\sigma_3^j$ can be chosen from any of the n spaces. To illustrate this consider the case of $n = 3$, where

$$E_3 = \tfrac{1}{4}(1 - i\sigma_3^1\, i\sigma_3^2)(1 - i\sigma_3^1\, i\sigma_3^3) \tag{1.74}$$

$$= \tfrac{1}{4}(1 - i\sigma_3^1\, i\sigma_3^2 - i\sigma_3^1\, i\sigma_3^3 - i\sigma_3^2\, i\sigma_3^3) \tag{1.75}$$

and

$$J_3 = \tfrac{1}{4}(i\sigma_3^1 + i\sigma_3^2 + i\sigma_3^3 - i\sigma_3^1\, i\sigma_3^2\, i\sigma_3^3). \tag{1.76}$$

Both E_3 and J_3 are symmetric under permutations of their indices.

The above was framed for non-relativistic Pauli spinors, but in fact, the whole discussion also applies to Dirac spinors, since these are represented by even elements and multiplication by j is still right-sided multiplication by $i\sigma_3$. A significant feature of this approach is that all the operations defined for the single-particle STA extend naturally to the multiparticle algebra. The reversion operation, for example, still has precisely the same definition — it simply reverses the order of vectors in any given multivector. The spinor inner product also generalises immediately, to

$$(\psi, \phi)_S = \langle E_n \rangle^{-1} [\langle \tilde{\psi}\phi \rangle - \langle \tilde{\psi}\phi J_n \rangle i\sigma_3], \tag{1.77}$$

where the right-hand side is projected onto a single copy of the STA. The factor of $\langle E_n \rangle^{-1}$ is included so that the state '1' always has unit norm, which matches with the inner product used in the matrix formulation.

1.5.3 A 6-d representation in the MSTA

Much more could be said about the properties and applications of the MSTA, but here we wish to use it in a novel linking-together of quantum mechanics, twistors and conformal geometry.

Let ϕ be a (single particle) Dirac spinor, and $r = t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3$ be the position vector in 4-d space.

Consider the operator $\hat{r}$, mapping Dirac spinors to Dirac spinors, given by

$$\phi \mapsto \hat{r}(\phi) \equiv r\phi i\gamma_3 \tfrac{1}{2}(1 + \sigma_3). \tag{1.78}$$

The operator $(1 + \hat{r})$ has the remarkable property of leaving the inner product between Dirac spinors invariant. Specifically, we have

$$\langle \tilde{\psi}' \phi' \rangle_S = \langle \tilde{\psi} \phi \rangle_S, \tag{1.79}$$

where

$$\psi' = (1 + \hat{r})\psi \quad \text{and} \quad \phi' = (1 + \hat{r})\phi. \tag{1.80}$$

(The subscript S applied in this single-particle case just means the scalar and $i\sigma_3$ parts only are taken.) This relation is true for any Dirac spinors ψ and ϕ. We note further $\hat{r}^2 = 0$, so we can write $(1 + \hat{r})$ in the 'rotor' form $e^{\hat{r}}$.

Now consider the following two-particle quantum state:

$$\epsilon = \left(i\sigma_2^1 - i\sigma_2^2\right) \tfrac{1}{2}\left(1 - \sigma_3^1\right) \tfrac{1}{2}\left(1 - \sigma_3^2\right) \tfrac{1}{2}\left(1 - i\sigma_3^1 i\sigma_3^2\right), \tag{1.81}$$

This is a relativistic generalisation of the non-relativistic Pauli singlet state (see Doran et al [14]). Specifically it can be shown that it obeys

$$R^1 R^2 \epsilon = R^1 \tilde{R}^1 \epsilon = \epsilon \tag{1.82}$$

for any (Lorentz) rotor R, and is therefore relativististically invariant. We now use this to construct our first '6-d' point as follows:

$$\psi = e^{\hat{r}^1} e^{\hat{r}^2} \epsilon. \tag{1.83}$$

ψ here is a 2-particle wavefunction which provides a representation of the 4-d point r. We shall see shortly in what way it connects with 6 dimensions.

Firstly, however, note that this ψ has vanishing norm viewed as a 2-particle wavefunction:

$$\langle \tilde{\psi}\psi \rangle_S = 0. \tag{1.84}$$

More generally, let

$$\phi = e^{\hat{s}^1} e^{\hat{s}^2} \epsilon, \tag{1.85}$$

correspond to some different 4-d position s. Then we find

$$\langle \tilde{\phi}\psi \rangle_S = -\frac{1}{4}(r-s)^2. \tag{1.86}$$

Just as with the 'horosphere' construction used by Hestenes, we see we have found a way of turning differences into products, except here it is taking place in a relativistic context.

The way the quantum state links with 6 dimensions is as follows. The state space for relativistic spinors describing two particles is 16 complex dimensional (as effectively the outer product of two Dirac spinors) and splits into a 10-d space symmetric under particle interchange (i.e. swapping of the 1 and 2 labels) and a 6-d space anti-symmetric under interchange. This 6-d space is 'complex' (i.e. with 12 real degrees of freedom), but we can define a 'real' subspace of it via taking the following as being the general point:

$$\psi_P = (V+W)\epsilon' + P^1 \epsilon\, i\gamma_3^1 + P^2 \epsilon\, i\gamma_3^2 + (V-W)\epsilon. \tag{1.87}$$

Here

$$\epsilon' = -\gamma_0^1\gamma_0^2\epsilon\gamma_0^1\gamma_0^2 = \left(i\sigma_2^1 - i\sigma_2^2\right)\tfrac{1}{2}\left(1+\sigma_3^1\right)\tfrac{1}{2}\left(1+\sigma_3^2\right)\tfrac{1}{2}\left(1-i\sigma_3^1 i\sigma_3^2\right), \tag{1.88}$$

and

$$P = T\gamma_0 + X\gamma_1 + Y\gamma_2 + Z\gamma_3. \tag{1.89}$$

V, W, T, X, Y and Z are the coordinates of a 6-d real space with metric

$$ds^2 = dT^2 + dV^2 - dW^2 - dX^2 - dY^2 - dZ^2. \tag{1.90}$$

The extra dimensions V and W allow the formation of the combinations $V+W$ and $V-W$, which correspond to null directions in the 6-d space. (These directions are the equivalent of the e and e^* introduced by Hestenes in the 5-d case.)

The representation of 4-d points proceeds via working with points on the 'null cone' in 6-d. For these points we relate the 6-d space to ordinary 4-d Lorentz space projectively via

$$t = \frac{T}{V-W}, \quad x = \frac{X}{V-W}, \quad y = \frac{Y}{V-W}, \quad z = \frac{Z}{V-W}. \tag{1.91}$$

The way this relates to our previous construction is as follows:

$$\psi_P = (V-W)\, e^{\hat{r}^1} e^{\hat{r}^2} \epsilon, \tag{1.92}$$

i.e. it is simply a scaled version of the state generated by the rotor construction. We can see this by taking the length of ψ_P, via the norm of the quantum state:

$$\langle \tilde{\psi}_P \psi_P \rangle = \tfrac{1}{2}(V^2 - W^2 + T^2 - X^2 - Y^2 - Z^2). \tag{1.93}$$

This being null implies

$$T^2 - X^2 - Y^2 - Z^2 = -(V^2 - W^2), \tag{1.94}$$

i.e.

$$t^2 - x^2 - y^2 - z^2 = -\left(\frac{V+W}{V-W}\right). \tag{1.95}$$

ψ_P is thus just

$$(V - W)\left(-|r|^2 \epsilon' + r^1 \epsilon\, i\gamma_3^1 + r^2 \epsilon\, i\gamma_3^2 + \epsilon\right), \tag{1.96}$$

which is $(V - W)\, e^{\hat{r}^1} e^{\hat{r}^2} \epsilon$ as claimed.

1.5.4 Link with twistors

The above has been framed as a mixture of 2-particle relativistic quantum mechanics (written in the MSTA) and conformal geometry. It also links directly with *twistor* theory (see e.g. Penrose & Rindler, Vol. 2 [23]). Twistors were introduced by Penrose as objects describing the geometry of spacetime at a 'pre-metric' level (partially in an attempt to allow an alternative route to quantum gravity). Instead of points and a metric, the idea is that twistors can represent incidence relations between null rays. Spacetime points and their metric relations then emerge as a secondary concept, corresponding to the points of intersection of null lines. As discussed in Lasenby, Doran & Gull [24], in geometric algebra twistors are translated as Dirac spinors with a particular position dependence. Specifically, a twistor, which is written in 2-spinor notation as

$$\mathsf{Z}^\alpha = (\omega^A, \pi_{A'}) \tag{1.97}$$

is translated as the Dirac spinor Z given by

$$Z = \phi - r\phi i\gamma_3 \tfrac{1}{2}(1 + \sigma_3), \tag{1.98}$$

where r is the 4-d position vector and ϕ is the (constant) Dirac spinor

$$\phi = \omega_0 \tfrac{1}{2}(1 + \sigma_3) - \pi i\sigma_2 \tfrac{1}{2}(1 - \sigma_3), \tag{1.99}$$

with ω_0 and π the geometric algebra Pauli spinors corresponding to the Penrose & Rindler 2-spinors ω_0^A and $\pi_{A'}$. (ω_0^A is ω^A evaluated at the origin.)

What we can observe now is that Z is none other than $e^{-\hat{r}}\phi$. This links twistors with the previous section. In twistor theory, given two twistors Z and X satisfying certain conditions, we can find a spacetime point corresponding to their intersection via forming the skew $\begin{bmatrix}2\\0\end{bmatrix}$ twistor

$$\mathsf{R}^{\alpha\beta} = \mathsf{Z}^{\alpha}\mathsf{X}^{\beta} - \mathsf{X}^{\alpha}\mathsf{Z}^{\beta} \tag{1.100}$$

(see Penrose & Rindler, Vol. 2, [23], p. 65 and p. 305). Without going into the details, it turns out that what we have described in the previous section corresponds precisely to this construction, but instantiated in a concrete fashion in the MSTA. In particular, the twistor relation

$$\mathsf{S}^{\alpha\beta}\mathsf{R}_{\alpha\beta} = -(s^a - r^a)(s_a - r_a) \tag{1.101}$$

(Equation 6.2.30 of Penrose & Rindler, Vol. II) corresponds precisely to both our Equation 1.86 and the Hestenes 'horosphere' relation Equation 1.55. The latter is already prefigured therefore in twistor geometry.

It might be wondered why, if corresponding constructions exist in twistor theory, it is useful to have a version in geometric algebra. The advantages of the latter are twofold. Firstly, there is the economy of using a single algebraic system for all areas as different as quantum mechanics, conformal geometry, screw theory etc. Secondly, we can use the geometric algebra to do things which are not easily possible within twistor theory, but which extend its results in a very neat fashion. For example, in the next section we show how the full special conformal group of Lorentzian spacetime can be realized via very simple transformations in our two particle space. The corresponding operations would be much harder to display explicitly in twistor theory.

As a final remark in this area, we note that twistor theory encourages one to think about a *complexified* version of Lorentzian spacetime. The same occurs in our present constructions via the fact that the 2-particle anti-symmetric space is actually 12-dimensional, allowing us to have a complex version of the 6-d conformal space. In order to understand some areas of practical computer vision, we apparently require a complex projective space; this is the case particularly for camera calibration using the concepts of the *absolute conic* and *absolute quadric*. A complex version of our 6-d conformal space may turn out to be very useful in allowing us to find a natural home for such entities in geometric algebra. This area is currently being explored.

1.5.5 The special conformal group

We now look briefly at how rotations, dilations, inversions, translations and special conformal motions in Lorentzian spacetime can be represented via simple transformations in our two particle space. This parallels the equivalent analysis in the 5-d case given by Hestenes for motions in Euclidean space, except that here they emerge in a (perhaps surprising) fashion as operations within relativistic quantum mechanics. In 3-d the importance of such motions is that they preserve the *angles* between vectors, and thus are next in generality as regards rigid body motion if we wish to go beyond the strictly Euclidean transformations of translation and rotation. In 4-d, they are of great interest in physics from the point of view of conformally invariant theories, such as electromagnetism and massless fields, and may be of interest in engineering for the description of rigid body motion where velocities and not just positions are specified, and also in projective spaces. We now describe in each case the required operation, and indicate why it works.

Translations:

Here we just need to note that the operators $e^{\hat{r}}$ for different r's are all mutually commutative. Thus if we have a point r in 4-d that we wish to move to $r+s$, where s is another 4-d position vector, we just need to carry out the transformation

$$\psi_P \mapsto \psi'_P = e^{(\hat{s}^1 + \hat{s}^2)}\psi_P. \tag{1.102}$$

Rotations

These are easily accomplished. Given a Lorentz rotor R, we rotate in the 2-particle space via

$$\psi_P \mapsto \psi'_P = R^1 R^2 \psi_P. \tag{1.103}$$

This works since e.g. the r^1 term in the expansion for ψ_P responds like

$$R^1 R^2 r^1 \epsilon\, i\gamma_3 = R^1 r^1 R^2 \epsilon\, i\gamma_3 = R^1 r^1 \tilde{R}^1 \epsilon\, i\gamma_3. \tag{1.104}$$

Inversions

The aim here, in the 4-d space, is to have $r \mapsto r/|r|^2$. Since the coefficient of ϵ' in the ψ_P expansion is $-|r|^2$, the way to achieve this in the 2-particle space would be to swap the roles of ϵ and ϵ'. We can achieve this by multiplying on the right by $i\sigma_2^1 i\sigma_2^2$, since this swaps both ideals. At the same time one finds

$$r^1 \epsilon\, i\gamma_3^1 i\sigma_2^1 i\sigma_2^2 = -r^2 \epsilon\, i\gamma_3^2, \tag{1.105}$$

and *vice-versa*. Thus the required operation is

$$\psi_P \mapsto \psi'_P = \psi_P i\sigma_2^1 i\sigma_2^2. \tag{1.106}$$

Dilations

Here in 4-d space we want r to transform to $e^{\alpha} r$, where α is a scalar. In the 2-particle space we need a rotor operation which can accomplish this. Like inversion, it is clear that we need to swap the roles of the $\frac{1}{2}(1+\sigma_3)$ and $\frac{1}{2}(1-\sigma_3)$ ideals, only this time it needs to happen in a gradual fashion. It is easy to show that the required operation is

$$\psi_P \mapsto \psi'_P = \psi_P e^{\alpha/2(\sigma_3^1 + \sigma_3^2)}. \tag{1.107}$$

Special conformal motions

These motions are in fact composites of inversions and translations, so in a sense we have already done these. However, the resulting expression for the operation in the 2-particle space is quite neat, so we give the results explicitly. In 4-d space we want to achieve the motion

$$r \mapsto r\frac{1}{1+sr}, \tag{1.108}$$

where s is a constant vector. This can be generated via inverting r, translating by s and then inverting again (see Hestenes & Sobczyk [25], p. 218). In our case, the combination of two inversions amounts to changing the ideal used in $e^{\hat{r}}$ to its opposite, plus a change of sign for the vector. Thus if we define the new operator $\check{r}$ via

$$\phi \mapsto \check{r}(\phi) \equiv r\phi i\gamma_3 \tfrac{1}{2}(1-\sigma_3), \tag{1.109}$$

we see that the overall operation we want is

$$\psi_P \mapsto \psi'_P = e^{-(\check{s}^1 + \check{s}^2)}\psi_P. \tag{1.110}$$

1.5.6 6-d space operations

Although above we have confined ourselves to setting up the basic correspondence between conformal operations and 'quantum' operations in the 2-particle space, it is of interest to relate these operations directly to the operations that would be carried out in a 6-d space generalising the 'horosphere' construction. The simplest version of such a space uses the representation discussed at the end of Hestenes & Sobczyk [25]. At the risk of causing great confusion, we shall stick with the original Hestenes & Sobczyk notation, which has e and $\bar{e}$, satisfying

$$e^2 = -\bar{e}^2 = 1, \tag{1.111}$$

as the new vectors which would be added to make up a (1,3) space with vectors r say, up to a (2,4) conformal space. The null vectors formed from e and $\bar{e}$ are defined by

$$n = e + \bar{e}, \quad \bar{n} = e - \bar{e}. \tag{1.112}$$

The crucial representation formula, relating (in this case) 4-d vectors r to their 6-d equivalents $F(r)$ is

$$F(r) = -(r-e)e(r-e) + (r-e)^2\bar{e}, \tag{1.113}$$

([25], eqn 3.14). Re-expressing this in terms of the null vectors, one finds

$$F(r) = r^2 n + 2r - \bar{n}. \tag{1.114}$$

We should compare this equation (1.114), with our quantum representation (1.96). It is clear that how they work is that (up to signs) the relativistic singlet state ϵ takes on the role of the null vector n, its version using the opposite ideals, ϵ', takes on the role of $\bar{n}$ and the middle term $r^1\epsilon\, i\gamma_3^1 + r^2\epsilon\, i\gamma_3^2$ is an expanded version (appropriate to the 2-particle space) of the vector $2r$.

It is now very interesting to compare some of the actions of the conformal group in the two approaches. Taking inversion as an example, this operation is not discussed in Hestenes & Sobczyk, but it is easy to see that we invert a 4-d point, $r \mapsto r/|r|^2$, via reflection in the unit vector e. Explicitly, we carry out

$$F(r) \mapsto eF(r)e. \tag{1.115}$$

This swaps the roles of n and $\bar{n}$. In the 2-particle case, we know inversion is accomplished by right multiplication by $i\sigma_2^1 i\sigma_2^2$, since this swaps the quantities ϵ and ϵ'. Thus the quantum operation of swapping the spin states (up $\longleftrightarrow$ down) of the 2-particles (which is what the $i\sigma_2^1 i\sigma_2^2$ multiplication achieves), parallels the operation of *reflection* in the 6-d space. This hints at a deep geometrical connection between the two spaces, which will be investigated further elsewhere.

1.6 Summary and Conclusions

In this contribution we have seen that geometric algebra is able to span an enormous range of physics and mathematical physics. From the rest of this volume it is clear that GA is useful in many areas of engineering also. Thus GA stands ready to be adopted as a useful and efficient tool by scientists and engineers in a wide variety of fields, with consequent benefit for mutual comprehensibility. Even areas considered as difficult as general relativity have been shown to be understandable within GA using just simple tools of linear function theory. The links between the new conformal representation of Euclidean geometry, twistors and multiparticle quantum theory have been shown to be both fascinating and unexpected. Much more work is possible along this direction, including the possible role of *complex* projective and conformal geometry, and of relativistic spaces in allowing representation of velocity as well as position transformations.

References

[1] D. Hestenes. 2000. Old wine in new bottles: a new algebraic framework for computational geometry. In chapter **??**, this volume. Birkhauser.

[2] L. Dorst. 2000. Objects in contact: boundary collisions as geometric wave propagation. In chapter **??**, this volume. Birkhauser.

[3] K.R. Etzel and M. McCarthy. 1999. Interpolation of spatial displacements using the Clifford algebra of E^4. *J.Mech.Design.* 121, p.39.

[4] E. Bayro–Corrochano, G. Sommer and K. Daniilidis. 2000. Motor algebra for 3D kinematics. The case of the hand–eye calibration. To appear in *Int. Journal of Mathematical Imaging and Vision.*

[5] J. Lasenby and E.D. Bayro-Corrochano. 1999. Analysis and computation of projective invariants from multiple views in the geometric algebra framework. *International Journal of Pattern Recognition and Artificial Intelligence*, 13(8) p.1105. (Special Issue on Invariants for Pattern Recognition and Classification).

[6] L. Dorst. 2000. Honing geometric algebra for its use in the computer sciences. In G. Sommer, editor, *Geometric Computing with Clifford Algebras.* Springer. (In Press).

[7] D. Hestenes. 1966. *Space-Time Algebra.* Gordon and Breach, New York.

[8] S.F. Gull, A.N. Lasenby, and C.J.L. Doran. 1993. Imaginary numbers are not real — the geometric algebra of spacetime. *Found. Phys.*, 23(9) p.1175.

[9] A.N. Lasenby, C.J.L. Doran, and S.F. Gull. 1993. A multivector derivative approach to Lagrangian field theory. *Found. Phys.*, 23(10) p.1295.

[10] S.F. Gull, A.N. Lasenby, and C.J.L. Doran. 1993. Electron paths, tunnelling and diffraction in the spacetime algebra. *Found. Phys.*, 23(10) p.1329.

[11] C.J.L. Doran, A.N. Lasenby, S.F. Gull, and J. Lasenby. 1996. Lectures in geometric algebra. In W.E. Baylis, editor, *Clifford (Geometric) Algebras*, pp. 65–236. Birkhauser, Boston.

[12] A.D. Challinor, A.N. Lasenby, S.S. Somaroo, C.J.L. Doran, and S.F. Gull. 1997. Tunnelling times of electrons. *Phys. Lett. A*, 227, p.143.

[13] A.M. Lewis, A.N. Lasenby, and C.J.L. Doran. 2000. Electron scattering in the spacetime algebra. In R. Ablamowicz and B. Fauser, editors, *Proceedings of the 5th International Conference on Applications of Clifford Algebras, Mexico, 1999.* (In press.).

[14] C.J.L Doran, A.N. Lasenby, S.F. Gull, S.S. Somaroo, and A.D. Challinor. 1996. Spacetime algebra and electron physics. *Adv. Imag. & Elect. Phys.*, pp. 95–271.

[15] A.N. Lasenby, C.J.L. Doran, and S.F. Gull. 1998. Gravity, gauge theories and geometric algebra. *Phil. Trans. R. Soc. Lond. A*, 356, pp.487–582.

[16] A.D. Challinor and A.N. Lasenby. 1999. Cosmic microwave background anisotropies in the cold dark matter model: a covariant and gauge-invariant approach. *Astrophys.J.*, 513, p.1.

[17] G.F.R. Ellis and M. Bruni.1989. *Phys.Rev.D.* 40, p.1804.

[18] C.J.L. Doran, A.N. Lasenby, and S.F. Gull. 1996. Physics of rotating cylindrical strings. *Phys. Rev. D*, 54(10), p.6021.

[19] A.N. Lasenby, C.J.L. Doran, Y. Dabrowski, and A.D. Challinor. 1997. Rotating astrophysical systems and a gauge theory approach to gravity. In N. Sánchez and A. Zichichi, editors, *Current Topics in Astrofundamental Physics, Erice 1996*, p.380. World Scientific, Singapore.

[20] Y. Dabrowski, A. Fabian, K. Iwasawa, A. Lasenby, and C. Reynolds. 1997. The profile and equivalent width of the X-ray iron emission line from a disk around a Kerr black hole. *Mon. Not. R. Astron. Soc.*, 288, L11.

[21] C. Doran. 2000. New form of the Kerr solution. To appear in *Phys. Rev. D.*

[22] J. Clements. 1999, Beam buckling using geometric algebra. M.Eng. project report, Cambridge University Engineering Department.

[23] R. Penrose and W. Rindler. 1986. *Spinors and space-time, Volume II: spinor and twistor methods in space-time geometry.* Cambridge University Press.

[24] A.N. Lasenby, C.J.L. Doran, and S.F. Gull. 1993. 2-spinors, twistors and supersymmetry in the spacetime algebra. In Z. Oziewicz, B. Jancewicz, and A. Borowiec, editors, *Spinors, Twistors, Clifford Algebras and Quantum Deformations*, p.233. Kluwer Academic, Dordrecht.

[25] D. Hestenes and G. Sobczyk. 1984. *Clifford Algebra to Geometric Calculus.* Reidel, Dordrecht.