

# New Approach To Quantum Field Theory For Arbitrary Observers In Electromagnetic Backgrounds

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A reformulation of fermionic QFT in electromagnetic backgrounds is presented which uses methods analogous to those of conventional multiparticle quantum mechanics. Emphasis is placed on the (Schrödinger picture) states of the system, described in terms of Slater determinants of Dirac states, and not on the field operator  $\hat{\psi}(x)$  (which is superfluous in this approach). The vacuum state 'at time  $\tau$ ' is defined as the Slater determinant of a basis for the span of the negative spectrum of the 'first quantized' Hamiltonian  $\hat{H}(\tau)$ , thus providing a concrete realisation of the Dirac Sea. The general S-matrix element of the theory is derived in terms of time-dependent Bogoliubov coefficients, demonstrating that the S-matrix follows directly from the definition of inner product between Slater determinants. The process of 'Hermitian extension', inherited directly from conventional multiparticle quantum mechanics, allows second quantized operators to be defined without appealing to a complete set of orthonormal modes, and provides an extremely straightforward derivation of the general expectation value of the theory. The concept of 'radar time', advocated by Bondi in his work on k-calculus, is used to generalise the particle interpretation to an arbitrarily moving observer. A definition of particle results, which depends *only* on the observer's motion and the background present, not on any choice of coordinates or gauge, or of the particle detector. We relate this approach to conventional methods by comparing and contrasting various derivations. Our particle definition can be viewed as a generalisation to arbitrary observers of Gibbons' approach [1].

*Key Words:* particle creation, fermion, observer, Slater determinant, radar time.

## 1. INTRODUCTION

We present here an initial-value formulation of fermionic QFT in an electromagnetic background. This formulation can be seen as the natural relativistic generalisation of non-relativistic multiparticle quantum mechanics. We emphasise the states of the system, described in terms of Slater de-

terminants of Dirac states, rather than the field operator. The vacuum is defined as the Slater determinant of a basis for the span of the negative spectrum of the ‘first quantized’ Hamiltonian, providing a concrete manifestation of the Dirac Sea. Simple derivations of the general S-Matrix element and expectation values in the theory then follow. Moreover, this approach suggests a consistent particle interpretation at all times, without requiring any ‘asymptotic niceness conditions’ on the ‘in’ and ‘out’ states. By using the concept of ‘radar time’ (originally made popular by Bondi in his work on k-calculus [2]) we generalise this particle interpretation to an arbitrarily moving observer, providing a definition of particle which depends *only* on the observer’s motion and on the background.

Perhaps the most surprising aspect of this description of relativistic multiparticle systems is that it is essentially new. Even in 1932, when the ‘Dirac Sea’ concept was first introduced [3, 4], the notion of using Slater determinants to describe the ‘filling of energy levels’ was well-known [5]. Indeed, a formulation of relativistic fermionic systems close to that presented here (although not in a classical background, not explicitly using Slater determinants, and not considering observer dependence) was presented in 1934 by Furry and Oppenheimer [6] (it is described from a more modern standpoint in the introduction to Weinberg’s book [7]). However, the view at the time was that the Dirac Sea was little more than a mathematical trick, and should be stable and trivial to describe. Even Dirac [8] stated that “The vacuum is quite a trivial thing physically, and we should expect it to correspond to a trivial solution of the Schrödinger equation”. The Furry-Oppenheimer theory was rapidly replaced by the Canonical approach, which treats the ‘holes’ in the Dirac Sea (rather than its constituents) on the same footing as the electrons, while defining the vacuum only implicitly, as ‘containing no electrons or holes’.

A more recent formulation of fermionic QFT in classical backgrounds, which also actively incorporates the Dirac Sea concept, has been given by Keifer et al. [9, 10, 11], Floreanini et al. [12] and others. This is based on the ‘functional Schrödinger equation’, where the state of the system is described by a functional of Grassmann fields, much as was proposed by Berezin [13]. This formulation has met with considerable success in applications [10, 11, 14], although questions regarding particle interpretation, the choice of boundary conditions, and the foliation of spacetime remain unresolved.

The formulation of QFT presented in the present paper resolves these problems. We support the claim [15] that, when considering QFT in electromagnetic or gravitational backgrounds (so that vacuum effects are important) it is clearer and faster to work directly with a concrete representation of the Dirac sea. Since this approach is the natural relativistic generalisation of methods already familiar in conventional multiparticle

quantum mechanics, we also believe that it will often provide a favourable alternative to the canonical approach.

Sections 2 and 3 set out the basis of our approach, as described in the rest frame of an inertial observer. The State space  $\mathcal{F}_\wedge(\mathcal{H})$  is presented in Section 2 in terms of Slater determinants of spinor-valued functions. In Section 3 we describe the time-dependent particle interpretation of state space. We show how the general S-Matrix element of the theory follows directly from the definition of inner product between Slater determinants, and we calculate the general expectation value using the theory. ‘Radar time’ is introduced in Section 4, and is used to generalise the particle interpretation to arbitrarily moving observers. In Section 5 we show how our formalism is related to more conventional methods, using the field operator  $\hat{\psi}(x)$ . By extending a technique originally developed for real scalar fields by DeWitt [16], we rederive the general S-Matrix element of the theory and compare this to the derivation presented in Section 3. We also show how our particle definition can be expressed in more conventional terms by discussing it’s relation to the method of Hamiltonian diagonalisation. Our definition consistently combines the conventional ‘Bogoliubov coefficient method’ with the ‘tunnelling method’, resolving the gauge inconsistencies [17] that trouble each of these methods. A retrospective overview is presented in Section 6.

## 2. THE STATE SPACE

### 2.1. Preliminaries

The Lagrangian density for the Dirac equation in an electromagnetic background  $A_\mu(x)$  is [18]

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu \overleftrightarrow{\nabla}_\mu - m)\psi(x) = \frac{i}{2}[\bar{\psi}\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma^\mu\psi] - m\bar{\psi}\psi \quad (1)$$

where  $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}I_4$ , ( $I_4$  is the  $4 \times 4$  identity matrix and  $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ ),  $\psi(x)$  is a 4-component spinor,  $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$ , and  $e$  is the charge of the fermion ( $e < 0$  for electrons). The *covariant derivative*  $\nabla_\mu$  is defined by  $\nabla_\mu\psi(x) \equiv \partial_\mu\psi(x) + ieA_\mu(x)\psi(x)$ . The Lagrangian (1) leads to the governing equation:

$$(i\gamma^\mu\nabla_\mu - m)\psi(x) = 0 \quad (2)$$

with a conserved inner product:

$$\langle\psi(\mathbf{x},t)|\phi(\mathbf{x},t)\rangle = \int d^3\mathbf{x} \psi^\dagger(\mathbf{x},t)\phi(\mathbf{x},t) \quad (3)$$

A ‘first quantized’ Dirac state is a spinor valued function of space, which evolves with time. The ‘first quantized’ state space  $\mathcal{H}$  is the set of all spinor-valued functions of  $\mathbb{R}^3$  whose norm under (3) is finite,  $\mathcal{H} \equiv L^2(\mathbb{R}^3)^4$ . We shall denote a first quantized state at time  $t$  by  $\psi(\mathbf{x}, t)$  or  $|\psi(t)\rangle$  or, where no ambiguity is possibly, simply  $\psi$  (there will be little need to distinguish between a state and its coordinate representation).

It is convenient to write (2) in ‘Hamiltonian form’ as:

$$i\nabla_0\psi(\mathbf{x}, t) = \hat{H}_1(\mathbf{x}, t)\psi(\mathbf{x}, t) \quad (4)$$

$$\text{where } \hat{H}_1(\mathbf{x}, t)\psi(\mathbf{x}, t) = (-i\gamma^0\gamma^k\nabla_k + m\gamma^0)\psi(\mathbf{x}, t) \quad (5)$$

and  $k$  is to be summed over  $k=1, 2, 3$ . This defines the (gauge covariant) *first quantized Hamiltonian operator*  $\hat{H}_1(\mathbf{x}, t)$ , which plays an important role below. The expectation value of  $\hat{H}_1(\mathbf{x}, t)$  in the state  $\psi(\mathbf{x}, t)$  is:

$$\langle\psi(\mathbf{x}, t)|\hat{H}_1(\mathbf{x}, t)|\psi(\mathbf{x}, t)\rangle = \int d^3\mathbf{x} \bar{\psi}(\mathbf{x}, t)(-i\gamma^k\nabla_k + m)\psi(\mathbf{x}, t) \quad (6)$$

$$= \int d^3\mathbf{x} T^0_{0,\psi}(x) \equiv H_{t_0}(\psi) \quad (7)$$

$$\text{where } T^\mu_{\nu,\psi}(x) = i\bar{\psi}\gamma^\mu\overleftrightarrow{\nabla}_\nu\psi - \delta^\mu_\nu\mathcal{L} \quad (8)$$

is the energy-momentum tensor. The expectation value of  $\hat{H}_1$  is identified as the spatial integral of the ‘00-component’ of the energy-momentum tensor. If we require that  $\psi(\mathbf{x}, t)$  is a solution of the Dirac equation, the  $\delta^\mu_\nu\mathcal{L}$  term and the  $\leftrightarrow$  in (8) both vanish and we can write  $T^0_0(x) = i\bar{\psi}\gamma^0\nabla_0\psi$ . Also notice that if the LHS of (4) had been  $i\frac{\partial\psi}{\partial t}$  rather than  $i\nabla_0\psi = i\frac{\partial\psi}{\partial t} - eA_0(x)\psi$ , then the RHS would have been  $\hat{H}_{ev} = \hat{H}_1 + eA_0(x)$ , which is clearly dependent on gauge.

## 2.2. The Full Fock Space Over $\mathcal{H}$

The *antisymmetric Fock Hilbert space* over the complex Hilbert space  $\mathcal{H}$  is denoted  $\mathcal{F}_\wedge(\mathcal{H})$  and is defined [19] in terms of the *antisymmetric Tensor Algebra* over  $\mathcal{H}$ . It is a natural and familiar construction by which a quantum theory of fermions can be formulated. We now define  $\mathcal{F}_\wedge(\mathcal{H})$ . Let  $\mathcal{H}$  be the Hilbert space in the previous Section, with inner product denoted by  $\langle | \rangle$ . Let  $\otimes^n\mathcal{H}$  denote the direct product of  $n$  copies of  $\mathcal{H}$ , and let  $\wedge^n\mathcal{H}$  denote the restriction of  $\otimes^n\mathcal{H}$  to those ‘states’ which are completely antisymmetric under changes in the order of the elements  $|\psi\rangle \in \mathcal{H}$  from which it is constructed. Given  $|\psi_1\rangle|\psi_2\rangle\dots|\psi_n\rangle \in \otimes^n\mathcal{H}$  we can define  $\psi_1 \wedge \psi_2 \wedge \dots \wedge \psi_n \in \wedge^n\mathcal{H}$  by:

$$\psi_1 \wedge \psi_2 \wedge \dots \wedge \psi_n \equiv \frac{1}{\sqrt{n!}} \sum_\sigma \text{sign}(\sigma) |\psi_{\sigma(1)}\rangle |\psi_{\sigma(2)}\rangle \dots |\psi_{\sigma(n)}\rangle \quad (9)$$

where  $\{\sigma(i), i = 1, \dots, n\}$  is a permutation of  $\{1 \dots n\}$ . This is simply the Slater determinant of the states  $|\psi_1\rangle \dots |\psi_n\rangle$ . The antisymmetric Fock Hilbert space  $\mathcal{F}_\wedge(\mathcal{H})$  is now given by:

$$\mathcal{F}_\wedge(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \wedge^n \mathcal{H}$$

where  $\wedge^0 \mathcal{H} \equiv \mathbb{C}$  and  $\wedge^1 \mathcal{H} \equiv \mathcal{H}$ . States which lie entirely within  $\wedge^r \mathcal{H}$  for some  $r$  are said to be of *grade*  $r$ .

A useful operation on  $\mathcal{F}_\wedge(\mathcal{H})$  is the ‘inner derivative’ (named by analogy with differential geometry)  $i_\psi : \wedge^n \mathcal{H} \rightarrow \wedge^{n-1} \mathcal{H}$ . This is defined by:

$$i_\psi : \phi_1 \wedge \dots \wedge \phi_n \rightarrow \sum_i (-)^{i+1} \langle \psi | \phi_i \rangle \phi_1 \wedge \dots \wedge \check{\phi}_i \wedge \dots \wedge \phi_n \quad (10)$$

where the check over  $\phi_i$  signifies that this state is omitted from the product. The relation  $i_\psi : \mathcal{F}_\wedge(\mathcal{H}) \rightarrow \mathcal{F}_\wedge(\mathcal{H})$  is obtained from (10) by imposing linearity, together with the additional convention  $i_\psi \lambda = 0$  for  $\lambda \in \wedge^0 \mathcal{H}$ . It is clear that  $i_\psi(i_\psi |F\rangle) = 0$  for all  $|F\rangle \in \mathcal{F}_\wedge(\mathcal{H})$ , and that:

$$i_\psi(\phi \wedge |F\rangle) = \langle \psi | \phi \rangle |F\rangle - \phi \wedge (i_\psi |F\rangle) \quad (11)$$

The operation  $i_\psi$  is denoted as  $a(\psi)$  by Ottesen [19], and plays the role of an annihilation operator. Here  $i_\psi$  will play a similar, although not identical role.

Finally, the inner product on  $\mathcal{F}_\wedge(\mathcal{H})$  is given by:

$$\langle \psi_1 \wedge \dots \wedge \psi_n | \phi_1 \wedge \dots \wedge \phi_m \rangle = \delta_{nm} \det[\langle \psi_i | \phi_j \rangle] \quad (12)$$

where  $\langle \psi_i | \phi_j \rangle$  refers to the inner product on  $\mathcal{H}$ . (For states  $\lambda, \mu \in \wedge^0 \mathcal{H}$  define  $\langle \lambda | \mu \rangle = \lambda \mu$  and  $\langle \lambda | F_n \rangle = 0$  for any state  $|F_n\rangle$  of grade  $n > 0$ .) This agrees with the inner product defined in terms of Slater determinants. Although we use the notation  $\langle | \rangle$  to refer to both the inner product on  $\mathcal{H}$  and the inner product on  $\mathcal{F}_\wedge(\mathcal{H})$ , it will be clear from the context which is meant.

### 2.3. Operators on Fock Space

Let  $\hat{A}_1 : \mathcal{H} \rightarrow \mathcal{H}$  be an operator on the space of Dirac states. We wish to construct from it an operator which can act on all of state space. There are two useful ways of doing this: *Hermitian extension*  $\hat{A}_H : \mathcal{F}_\wedge(\mathcal{H}) \rightarrow \mathcal{F}_\wedge(\mathcal{H})$ , and *Unitary extension*  $\hat{A}_U : \mathcal{F}_\wedge(\mathcal{H}) \rightarrow \mathcal{F}_\wedge(\mathcal{H})$  (outlined also in Ottesen [19]). These are defined by:

$$\hat{A}_H : \psi_1 \wedge \psi_2 \wedge \dots \wedge \psi_N \rightarrow \sum_{i=1}^N \psi_1 \wedge \dots \wedge (\hat{A}_1 \psi_i) \wedge \psi_{i+1} \wedge \dots \wedge \psi_N \quad (13)$$

$$\hat{A}_U : \psi_1 \wedge \psi_2 \wedge \dots \wedge \psi_N \rightarrow (\hat{A}_1 \psi_1) \wedge (\hat{A}_1 \psi_2) \wedge \dots \wedge (\hat{A}_1 \psi_N) \quad (14)$$

If  $\hat{A}_1$  is (anti)hermitian with respect to the inner product (3) on  $\mathcal{H}$ , then  $\hat{A}_H$  is (anti)hermitian with respect to the inner product (12) on  $\mathcal{F}_\wedge(\mathcal{H})$ . If  $\hat{A}_1$  is unitary, then so is  $\hat{A}_U$ . Also  $(e^{\hat{A}_1})_U = e^{\hat{A}_H}$ , so that if  $\hat{U}_1 = e^{\hat{A}_1}$  on  $\mathcal{H}$  then  $\hat{U}_U = e^{\hat{A}_H}$  on  $\mathcal{F}_\wedge(\mathcal{H})$ .

#### Some Simple Properties

1.  $(\hat{A} + \hat{B})_H = \hat{A}_H + \hat{B}_H$ ,  $[\hat{A}_H, \hat{B}_H] = [\hat{A}, \hat{B}]_H$  and  $(\hat{A}\hat{B})_U = \hat{A}_U\hat{B}_U$ .
2.  $[\hat{A}_H, \psi_\wedge] = (\hat{A}_1\psi)$  and  $[\hat{A}_H, i_\psi] = -i_{\hat{A}_1^\dagger\psi}$
3. If  $\psi_1, \psi_2, \dots, \psi_N$  are all eigenstates of  $\hat{A}_1$  with eigenvalues  $\lambda_1, \dots, \lambda_N$ , then  $\psi_1 \wedge \psi_2 \cdots \wedge \psi_N$  is an eigenstate of  $\hat{A}_H$  with eigenvalue  $\sum_{i=1}^N \lambda_i$ .
4. If  $\psi_1, \psi_2, \dots, \psi_N$  are orthonormal and  $|F\rangle \equiv |\psi_1 \wedge \psi_2 \cdots \wedge \psi_N\rangle$  then

$$\langle F | \hat{A}_H | F \rangle = \sum_{i=1}^N \langle \psi_i | \hat{A}_1 | \psi_i \rangle \quad (15)$$

$$\langle F | (\hat{A}_H)^2 | F \rangle = \sum_{i=1}^N \langle \psi_i | \hat{A}_1^2 | \psi_i \rangle + 2 \sum_{i < j} (\langle \psi_i | \hat{A}_1 | \psi_i \rangle \langle \psi_j | \hat{A}_1 | \psi_j \rangle - \langle \psi_i | \hat{A}_1 | \psi_j \rangle \langle \psi_j | \hat{A}_1 | \psi_i \rangle) \quad (16)$$

$$\langle F | (\hat{A}_H)^2 | F \rangle - (\langle F | \hat{A}_H | F \rangle)^2 = \sum_i \langle \psi_i | \hat{A}_1^2 | \psi_i \rangle - \sum_{i,j} |\langle \psi_i | \hat{A}_1 | \psi_j \rangle|^2 \quad (17)$$

## 2.4. Evolution of States

Define the evolution operator  $\hat{U}_1(t, t_0)$  on  $\mathcal{H}$  by

$$\hat{U}_1(t, t_0) |\psi_{t_0}\rangle \equiv |\psi_{t_0}(t)\rangle \quad (18)$$

where  $|\psi_{t_0}\rangle$  represents the chosen initial conditions  $\psi_{t_0}(\mathbf{x})$  at time  $t_0$ , and  $|\psi_{t_0}(t)\rangle$  represents the solution  $\psi_{t_0}(\mathbf{x}, t)$  of the Dirac equation satisfying these initial conditions.

We consider only QFT in an (external) electromagnetic background, so that we ignore direct particle-particle interactions and just work within the ‘zeroth order Hartree Fock’ approximation. This assumes that the evolution operator on  $\mathcal{F}_\wedge(\mathcal{H})$  is just the unitary extension of the evolution operator on  $\mathcal{H}$ , or that  $\hat{H}(t) = \hat{H}_{1,H}(t)$ , which is the natural generalisation of the equation that appears in multiparticle quantum mechanics textbooks [20, 21] as  $\hat{H}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \hat{H}(\mathbf{x}_1) + \dots + \hat{H}(\mathbf{x}_n)$ . The action of  $\hat{U}(t, t_0)$  is given by:

$$\hat{U}(t, t_0) : \psi_{1,t_0} \wedge \dots \wedge \psi_{n,t_0} \rightarrow \psi_{1,t_0}(t) \wedge \dots \wedge \psi_{n,t_0}(t) \quad (19)$$

The multiparticle solution is simply the Slater determinant of the appropriate ‘first quantized’ solutions. This construction preserves grade, and implies that the unitarity of  $\hat{U}(t, t_0)$  follows immediately from the unitarity of the first quantized Dirac equation.

We have now a state space, an evolution equation (19) and a conserved inner product (12). This is all we need to calculate arbitrary S-Matrix elements (from (19) and (12)), arbitrary expectation values (from (15)), and even fluctuations in these expectation values (from (17)). However, the theory is not invested with physical meaning until the states of the system can be specified in terms of their physical properties. Accordingly we now turn our attention to a particle interpretation.

### 3. A PARTICLE INTERPRETATION OF STATE SPACE

#### 3.1. The Positive/Negative Energy Split and the Vacuum State

Consider the action of  $\hat{H}_1(t_0)$  on  $\mathcal{H}$ , at some fixed time  $t_0$ . Since  $\hat{H}_1(t_0)$  is Hermitian we can parametrise  $\mathcal{H}$  in terms of the eigenvectors of  $\hat{H}_1(t_0)$ . From this we can define  $\mathcal{H}^\pm(t_0)$  such that:

$\mathcal{H}^+(t_0)$  is the span of the positive spectrum of  $\hat{H}_1(t_0)$

$\mathcal{H}^-(t_0)$  is the span of the negative spectrum of  $\hat{H}_1(t_0)$

$\mathcal{H}^+(t_0)$  is the set of all positive energy states, and  $\mathcal{H}^-(t_0)$  is the set of all negative energy states as defined at time  $t_0$ .  $\mathcal{H}^\pm(t_0)$  can alternatively be defined such that the projection operators  $\hat{P}^\pm(t_0) : \mathcal{H} \rightarrow \mathcal{H}^\pm(t_0)$  are orthogonal projections satisfying:

$$H_{t_0}(\hat{P}^+(t_0)\psi) \geq H_{t_0}(\psi) \geq H_{t_0}(\hat{P}^-(t_0)\psi) \quad (20)$$

(in the notation of (7)) for all  $\psi \in \mathcal{H}$ . These definitions are equivalent (as shown in [22]) but since (20) does not refer explicitly to the spectrum of  $\hat{H}_1(t_0)$  it is often more useful, as in section 4.1. Also, we have implicitly assumed that  $\hat{H}_1(t)$  has no zero energy eigenstates (for any  $t$ ). This is true for a large class of observers, but no longer holds when particle horizons are present. In the presence of particle horizons we have a third space  $\mathcal{H}^0(t)$ , consisting of zero energy eigenstates. This case is explained in detail in [23]; the simplest example is the Unruh effect, which will be discussed briefly in Section 4.1.

We can now define the *vacuum at time*  $t_0$ ,  $|\text{vac}_{t_0}\rangle$  as follows:

$$\begin{aligned} |\text{vac}_{t_0}\rangle &\text{ is the Slater determinant of any basis of } \mathcal{H}^-(t_0), \\ &\text{ normalised so that } \langle \text{vac}_{t_0} | \text{vac}_{t_0} \rangle = 1. \end{aligned}$$

This specifies  $|\text{vac}_{t_0}\rangle$  up to an arbitrary phase factor. It is the state in which all negative energy degrees of freedom are full, and hence is a concrete manifestation of the Dirac Sea.

To illustrate this, suppose temporarily that  $\mathcal{H}$  contains only  $N$  positive and  $N$  negative energy degrees of freedom ( $N \rightarrow \infty$  contains no complications). Let  $\{u_{i,t_0}; i=1, \dots, N\}$  be an orthonormal basis for  $\mathcal{H}^+(t_0)$  at some time  $t_0$ , and let  $\{v_{i,t_0}; i=1 \dots N\}$  be an orthonormal basis for  $\mathcal{H}^-(t_0)$ . The vacuum at time  $t_0$  can be written as:

$$|\text{vac}_{t_0}\rangle = v_{1,t_0} \wedge \dots \wedge v_{N,t_0} \quad (21)$$

This state is independent of the choice of basis for  $\mathcal{H}^-(t_0)$  (up to a phase factor) because of the complete antisymmetry of the Slater determinant. If  $\hat{H}_1(t)$  depends on time then so does the space  $\mathcal{H}^-(t)$ , and the vacuum  $|\text{vac}_t\rangle$  will differ at different times. If  $\{u_{i,t_1}; i=1, \dots, N\}$  and  $\{v_{i,t_1}; i=1, \dots, N\}$  are orthonormal bases for  $\mathcal{H}^+(t_1)$  and  $\mathcal{H}^-(t_1)$  respectively for some time  $t_1 > t_0$ , then we may write the vacuum at time  $t_1$  as:

$$|\text{vac}_{t_1}\rangle = v_{1,t_1} \wedge \dots \wedge v_{N,t_1} \quad (22)$$

The evolved state  $|\text{vac}_{t_0}(t_1)\rangle$ , obtained by evolving  $|\text{vac}_{t_0}\rangle$  from time  $t_0$  to time  $t_1$  is simply:

$$|\text{vac}_{t_0}(t_1)\rangle = v_{1,t_0}(t_1) \wedge \dots \wedge v_{N,t_0}(t_1) \quad (23)$$

where  $v_{i,t_0}(t_1)$  denotes the state obtained from  $v_{i,t_0}$  by evolution to time  $t_1$ , and will not, in general, be contained in  $\mathcal{H}^-(t_1)$ . We can now calculate quantities such as the probability that  $|\text{vac}_{t_0}(t_1)\rangle$  will still be in the vacuum state:

$$\mathcal{P}_{\text{vac} \rightarrow \text{vac}} = |\langle \text{vac}_{t_1} | \text{vac}_{t_0}(t_1) \rangle|^2 \quad (24)$$

$$\text{where } \langle \text{vac}_{t_1} | \text{vac}_{t_0}(t_1) \rangle = \det[\langle v_{i,t_1} | v_{j,t_0}(t_1) \rangle] \quad \text{from(12)} \quad (25)$$

In the case of Dirac theory in an electromagnetic background, the time dependence of  $\hat{H}(t)$  arises entirely from the time dependence of the background  $A^\mu(\mathbf{x}, t)$ , so that the state  $|\text{vac}_{t_0}\rangle$  represents the ‘vacuum in the



presence of a background  $A^\mu(\mathbf{x}, t_0)$ , while  $|\text{vac}_{t_1}\rangle$  represents the ‘vacuum in the presence of a background  $A^\mu(\mathbf{x}, t_1)$ ’. The state  $|\text{vac}_{t_0}(t_1)\rangle$ , sometimes called the *evolved vacuum*, is not actually a vacuum state.

### 3.2. Particle States Built On The Vacuum

Let  $u_{t_0} \in \mathcal{H}^+(t_0)$ , and let  $v_{t_0} \in \mathcal{H}^-(t_0)$ .

Then a *one-electron state at time  $t_0$*  is of the form:  $u_{t_0} \wedge |\text{vac}_{t_0}\rangle$

while a *one-positron state at time  $t_0$*  is of the form:  $i_{v_{t_0}} |\text{vac}_{t_0}\rangle$

As expected, electrons are represented by the presence of positive energy degrees of freedom, and positrons are represented by the absence of negative energy degrees of freedom (note that  $v_{t_0} \wedge |\text{vac}_{t_0}\rangle = 0 = i_{u_{t_0}} |\text{vac}_{t_0}\rangle$  for all  $u_{t_0} \in \mathcal{H}^+(t_0)$  and  $v_{t_0} \in \mathcal{H}^-(t_0)$ ). States having higher numbers of particles are constructed in the obvious way. These states are related to the conventional constructions by defining creation and annihilation operators as:

$$a(u_{t_0}) = i_{u_{t_0}} \qquad a^\dagger(u_{t_0}) = u_{t_0} \wedge \qquad (26)$$

$$b(v_{t_0}) = v_{t_0} \wedge \qquad b^\dagger(v_{t_0}) = i_{v_{t_0}} \qquad (27)$$

It is routine to verify that  $a^\dagger(u_{t_0})$  is indeed the Hermitian conjugate of  $a(u_{t_0})$ , and similarly for  $b^\dagger(v_{t_0})$  and  $b(v_{t_0})$ . The Canonical Anticommutation Relations (CARs) follow directly from (11), while equations such as

$$[\hat{P}_\mu, a_\lambda^\dagger(\mathbf{p})] = p_\mu a_\lambda^\dagger(\mathbf{p}) \qquad [\hat{Q}, a_\lambda^\dagger(\mathbf{p})] = a_\lambda^\dagger(\mathbf{p})$$

are all contained in Property 2 of Section 2.2.

It is useful to introduce an orthonormal basis for  $\mathcal{F}_\wedge(\mathcal{H})$  in ‘standard form’ at some time  $t_0$ . For this purpose, let  $\{u_{i,t_0}; i = 1, \dots, N\}$ ,  $\{v_{i,t_0}; i = 1, \dots, N\}$  be orthonormal bases for  $\mathcal{H}^+(t_0)$  and  $\mathcal{H}^-(t_0)$  respectively, as in Section 3.1. Introduce the symbol  $| \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} _{t_0} \rangle$  to denote an ‘in’ state, of  $m$  particles (in states  $u_{i_1} \dots u_{i_m}$  with  $i_1 < i_2 < \dots < i_m$  by convention), and  $n$  antiparticles (corresponding to the absence of states  $v_{j_1} \dots v_{j_n}$ ), prepared at time  $t_0$ . This state is given by:

$$| \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} _{t_0} \rangle \equiv (-)^J u_{i_1, t_0} \wedge \dots \wedge u_{i_m, t_0} \wedge v_{1, t_0} \wedge \dots \wedge v_{j_1, t_0} \dots \wedge v_{j_n, t_0} \dots \wedge v_{N, t_0} \qquad (28)$$

$$= a_{i_1, t_0}^\dagger \dots a_{i_m, t_0}^\dagger b_{j_1, t_0}^\dagger \dots b_{j_n, t_0}^\dagger |\text{vac}_{t_0}\rangle \qquad (29)$$

where the check over  $v_{j,t_0}$  signifies that this degree of freedom is missing from the state, and  $J = \frac{n}{2}(n+1) + \sum_{k=1}^n j_k$  is included as a sign convention. This state evolves into

$$\begin{aligned} |(i_1 i_2 \dots i_m)_{(j_1 j_2 \dots j_n)}(t)\rangle &= (-)^J u_{i_1, t_0}(t) \wedge \dots \wedge u_{i_m, t_0}(t) \wedge v_{1, t_0}(t) \wedge \dots \\ &\quad \dots \check{v}_{j_1, t_0}(t) \dots \wedge \check{v}_{j_n, t_0}(t) \dots \wedge v_{N, t_0}(t) \end{aligned} \quad (30)$$

which we can write loosely as  $a_{i_1, t_0}^\dagger(t) \dots a_{i_m, t_0}^\dagger(t) b_{j_n, t_0}^\dagger(t) \dots b_{j_1, t_0}^\dagger(t) |\text{vac}_{t_0}(t)\rangle$  where  $a_{i, t_0}^\dagger(t)$  is shorthand for  $u_{i, t_0}(t) \wedge$  and  $b_{j, t_0}^\dagger(t)$  is shorthand for  $\check{v}_{j, t_0}(t)$ . However,  $a_{i, t_0}^\dagger(t)$  does *not* represent the creation of a particle at time  $t$ , since  $u_{i, t_0}(t)$  is not in  $\mathcal{H}^+(t)$ . Similarly,  $b_{j, t_0}^\dagger(t)$  does not represent the creation of an antiparticle.

We now define the *Time-dependent Bogoliubov Coefficients* by:

$$\alpha_{ij}(t, t_0) = \langle u_{i, t} | u_{j, t_0}(t) \rangle \quad \gamma_{ij}(t, t_0) = \langle v_{i, t} | u_{j, t_0}(t) \rangle \quad (31)$$

$$\beta_{ij}(t, t_0) = \langle u_{i, t} | v_{j, t_0}(t) \rangle \quad \epsilon_{ij}(t, t_0) = \langle v_{i, t} | v_{j, t_0}(t) \rangle \quad (32)$$

Then the *Bogoliubov conditions*

$$\begin{bmatrix} \alpha^\dagger \alpha + \gamma^\dagger \gamma & \alpha^\dagger \beta + \gamma^\dagger \epsilon \\ \beta^\dagger \alpha + \epsilon^\dagger \gamma & \beta^\dagger \beta + \epsilon^\dagger \epsilon \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \alpha \alpha^\dagger + \beta \beta^\dagger & \alpha \gamma^\dagger + \beta \epsilon^\dagger \\ \gamma \alpha^\dagger + \epsilon \beta^\dagger & \gamma \gamma^\dagger + \epsilon \epsilon^\dagger \end{bmatrix} \quad (33)$$

follow from unitarity of the ‘first quantized’ evolution matrix

$$\mathbf{S}_1(t_1, t_0) = \begin{bmatrix} \alpha(t_1, t_0) & \beta(t_1, t_0) \\ \gamma(t_1, t_0) & \epsilon(t_1, t_0) \end{bmatrix}$$

Note that our labelling of the Bogoliubov coefficients differs from that used in Manogue’s [24] asymptotic treatment (Manogue’s are the complex conjugates of those used here). From (12), (28) and (30) we have immediately:

$$\begin{aligned} &\langle (i'_1 i'_2 \dots i'_m)_{(j'_1 j'_2 \dots j'_n)}(t_1) | (i_1 i_2 \dots i_m)_{(j_1 j_2 \dots j_n)}(t_0) \rangle \\ &= (-)^{J-J'} \det \left[ \begin{array}{c} \left[ \begin{array}{ccc} \alpha_{i'_1 i_1} & \dots & \alpha_{i'_1 i_m} \\ \vdots & & \vdots \\ \alpha_{i'_m i_1} & \dots & \alpha_{i'_m i_m} \end{array} \right] \left[ \begin{array}{ccc} \beta_{i'_1 1} & \binom{j_1 \dots j_n}{\text{missing}} & \beta_{i'_1 N} \\ \vdots & & \vdots \\ \beta_{i'_m 1} & \binom{j_1 \dots j_n}{\text{missing}} & \beta_{i'_m N} \end{array} \right] \\ \left[ \begin{array}{ccc} \gamma_{1 i_1} & \dots & \gamma_{1 i_m} \\ \binom{j'_1 \dots j'_n}{\text{missing}} & \binom{j'_1 \dots j'_n}{\text{missing}} & \binom{j'_1 \dots j'_n}{\text{missing}} \\ \gamma_{N i_1} & \dots & \gamma_{N i_m} \end{array} \right] \left[ \begin{array}{ccc} \epsilon_{11} & \binom{j_1 \dots j_n}{\text{missing}} & \epsilon_{1N} \\ \binom{j'_1 \dots j'_n}{\text{missing}} & \binom{j_1 \dots j_n}{\text{missing}} & \binom{j'_1 \dots j'_n}{\text{missing}} \\ \epsilon_{N1} & \binom{j_1 \dots j_n}{\text{missing}} & \epsilon_{NN} \end{array} \right] \end{array} \right] \end{aligned} \quad (34)$$

if  $m - n = m' - n'$  (charge conservation), and zero otherwise. This is a completely general formula for an arbitrary S-Matrix element, and it is remarkable that its derivation is so easy. This is in stark contrast to conventional formulations (see Section 5). The reason is the concrete representation of states as given by (28), and the simple evolution equation which allows us to deduce (30). In most approaches to QFT in a classical background [25, 26, 16] the states are defined only implicitly, by equations like (29), and the creation/annihilation operators are defined implicitly, by the CAR's. The derivation of S-Matrix elements must then proceed by a much more round-about method.

Some other textbook approaches do present a concrete representation of state space [27, 28], and also use the antisymmetric tensor algebra  $\mathcal{F}_\wedge(\mathcal{H}')$  over some 'first quantised' state space  $\mathcal{H}'$ . However, they do not simply take  $\mathcal{H}'$  to be the space of Dirac states, but rather require that it represent the 'set of all one-particle states'. Then  $\wedge^n \mathcal{H}'$  represents the set of all  $n$ -particle states, while  $\wedge^0 \mathcal{H}'$  represents the physical vacuum. Solutions of the Dirac equation must then be modified in order to construct these 1-particle states. For example, Thaller ([27], page 275) uses the set of 'positive energy solutions and charge conjugates of negative energy solutions'  $\mathcal{H}' \equiv \mathcal{H}^+ + \hat{C}\mathcal{H}^-$ , but treats only those cases where this choice can be made independently of time, while Wald ([28], page 103) uses a construction that no longer makes reference to positive or negative energy states. Such modifications of  $\mathcal{H}$  destroy the simplicity of the evolution equation (19) and require that grade violating terms be included to represent pair creation. This is why, even with a concrete representation of state space, such approaches still find it easier to describe those states in terms of creation/annihilation operators and a field operator rather than referring to the states directly, and why (34) is derived using methods like those in Section 5. Also, the requirement that  $\mathcal{H}'$  be time independent either restricts the applications of the theory or implies that  $\mathcal{H}'$  can no longer be interpreted as representing 1-particle states, removing any motivation for such a choice of  $\mathcal{H}'$ .

Before we examine expectation values and vacuum subtraction, it is convenient to factor out  $\langle \text{vac}_{t_1} | \text{vac}_{t_0}(t_1) \rangle = \det(\epsilon(t_1, t_0))$  from some of the S-

Matrix elements presented in (34). Consider for example  $\langle (i'_1 i'_2 \dots i'_{n'})_{j'_1 j'_2 \dots j'_{n'}} | \text{vac}_{t_0}(t_1) \rangle$ .

This can be written as:

$$\langle (i'_1 i'_2 \dots i'_{n'})_{j'_1 j'_2 \dots j'_{n'}} | \text{vac}_{t_0}(t_1) \rangle = (-)^{J'} \det \begin{bmatrix} \vec{\beta}_{i'_1} \\ \vdots \\ \vec{\beta}_{i'_{n'}} \\ \vec{\epsilon}_1 \\ (j'_1 \dots j'_{n'}) \\ \vec{\epsilon}_N \end{bmatrix}$$

where  $\vec{\beta}_i$  represents the  $i^{\text{th}}$  row of  $\beta$ , and similarly for  $\vec{\epsilon}_i$ . By multiplying this on the right by the matrix  $\begin{bmatrix} \vec{\epsilon}_{j'_1}^{-1} & \dots & \vec{\epsilon}_{j'_{n'}}^{-1} & \vec{\epsilon}_1^{-1} & (j'_1 \dots j'_{n'}) \\ & & & & \vec{\epsilon}_N^{-1} \end{bmatrix}$  (having determinant  $\frac{(-1)^{\sum 1^{j'_k-1}}}{\det(\epsilon)}$ ) we have

$$\begin{aligned} \langle (i'_1 i'_2 \dots i'_{n'})_{j'_1 j'_2 \dots j'_{n'}} | \text{vac}_{t_0}(t_1) \rangle &= (-)^{\frac{n'}{2}(n'-1)} \det(\epsilon) \det \begin{bmatrix} V_{i'_1 j'_1} & \dots & V_{i'_1 j'_{n'}} \\ \vdots & & \vdots \\ V_{i'_{n'} j'_1} & \dots & V_{i'_{n'} j'_{n'}} \end{bmatrix} \\ &= (-)^{\frac{n'}{2}(n'-1)} \det(\epsilon) \det(\mathbf{V}_{out}) \end{aligned} \quad (35)$$

where  $\mathbf{V} \equiv \beta \epsilon^{-1}$  and  $\mathbf{V}_{out}$  is constructed from  $\mathbf{V}$  using only those matrix entries relevant to the desired out state. Similarly:

$$\langle \text{vac}_{t_1} | (i_1 i_2 \dots i_m)_{j_1 j_2 \dots j_n} \rangle (t_1) = (-)^{\frac{n}{2}(n-1)} \det(\epsilon) \det \begin{bmatrix} \Lambda_{j_1 i_1} & \dots & \Lambda_{j_1 i_n} \\ \vdots & & \vdots \\ \Lambda_{j_n i_1} & \dots & \Lambda_{j_n i_n} \end{bmatrix} \quad (36)$$

where  $\Lambda \equiv \epsilon^{-1} \gamma$ .

The general case is more arduous, but routine. Rather than present it here we shall consider, as an example, the S-Matrix element between two 1-electron states,  $\langle (i')_{t_1} | (i)_{t_0} \rangle (t_1) = \langle \text{vac}_{t_1} | a_{i',t_1}^\dagger a_{i,t_0}^\dagger | \text{vac}_{t_0} \rangle (t_1)$ . We first expand  $u_{i,t_1}$  in terms of  $\{u_{j,t_0}(t_1), v_{j,t_0}(t_1)\}$ , as  $u_{i,t_1} = \sum_j \{\alpha_{ij}^*(t_1, t_0) u_{j,t_0}(t_1) + \beta_{ij}^*(t_1, t_0) v_{j,t_0}(t_1)\}$ . From the definition (26), this immediately yields the familiar equation:

$$a_{i,t_1} = \sum_j \{\alpha_{ij}(t_1, t_0) a_{j,t_0}(t_1) + \beta_{ij}(t_1, t_0) b_{j,t_0}^\dagger(t_1)\} \quad (37)$$

From this, we can write:

$$\langle (i')_{t_1} | (i)_{t_0} (t_1) \rangle = \langle \text{vac}_{t_1} | \alpha_{i'i} + \sum_j \beta_{i'j} b_{j,t_0}^\dagger(t_1) a_{i,t_0}^\dagger(t_1) | \text{vac}_{t_0}(t_1) \rangle \quad (38)$$

$$= \det(\epsilon) (\alpha - \beta \epsilon^{-1} \gamma)_{i'i} = \overline{\alpha_{ii'}^{-1}} \det(\epsilon) \quad (39)$$

where we have used  $\alpha - \beta \epsilon^{-1} \gamma = (\alpha^\dagger)^{-1}$ , which follows from the Bogoliubov conditions. The two terms in (38) represent distinct contributions to the ‘evolved 1-particle state’  $u_{i,t_0}(t_1) \wedge | \text{vac}_{t_0}(t_1) \rangle$ . The first contribution corresponds to the positive energy component of  $u_{1,t_0}(t_1)$  acting on the vacuum component of  $| \text{vac}_{t_0}(t_1) \rangle$ , while the second contribution comes from the negative energy component of  $u_{1,t_0}(t_1)$  destroying the antiparticle of one of the two-particle components of  $| \text{vac}_{t_0}(t_1) \rangle$ . Similarly, the S-matrix element between two antiparticle states is:

$$\langle (j')_{t_1} | (j)_{t_0} (t_1) \rangle = (\epsilon^{-1})_{jj'} \det(\epsilon) \quad (40)$$

We can use the Bogoliubov conditions to deduce the relations

$$\begin{aligned} \Lambda^\dagger &= -\alpha^{-1} \beta & \mathbf{V}^\dagger &= -\gamma \alpha^{-1} \\ \epsilon^\dagger \epsilon &= (1 + \Lambda \Lambda^\dagger)^{-1} & \alpha \alpha^\dagger &= (1 + \mathbf{V} \mathbf{V}^\dagger)^{-1} \end{aligned} \quad (41)$$

Since  $\alpha^{-1}$  and  $\epsilon^{-1}$  respectively represent the relative particle/particle and antiparticle/antiparticle transition amplitudes, it follows that (42), along with the relations  $\mathbf{V}^\dagger \alpha = -\epsilon \Lambda$  and  $\alpha \Lambda^\dagger = -\mathbf{V} \epsilon$  (these follow from (41) and the definitions of  $\mathbf{V}$ ,  $\Lambda$ ) constitute the fermionic version of the relativistic generalisation of the Optical Theorem, that has been given by DeWitt [16, 29].

### 3.3. Vacuum Subtraction and Physical Operators

Consider the expectation value, in the physical vacuum at time  $t_0$ , of the Hermitian extension of an operator  $\hat{A}_1(t_0): \mathcal{H} \rightarrow \mathcal{H}$ . From (15) we have:

$$\langle \text{vac}_{t_0} | \hat{A}_H(t_0) | \text{vac}_{t_0} \rangle = \sum_i \langle v_{i,t_0} | \hat{A}_1(t_0) | v_{i,t_0} \rangle$$

where  $\{v_{i,t_0}\}$  constitute an orthonormal basis for  $\mathcal{H}^-(t_0)$ . This is not zero, nor even finite in general. At this point we need to introduce a scheme analogous to normal ordering. Accordingly, given an operator  $\hat{A}_1(t): \mathcal{H} \rightarrow \mathcal{H}$  corresponding to an observable which is zero in the vacuum we define the *physical extension*  $\hat{A}_{\text{phys}}(t): \mathcal{F}_\wedge(\mathcal{H}) \rightarrow \mathcal{F}_\wedge(\mathcal{H})$  of  $\hat{A}_1(t)$  by:

$$\hat{A}_{\text{phys}}(t) = \hat{A}_H(t) - \langle \text{vac}_t | \hat{A}_H(t) | \text{vac}_t \rangle \hat{1} \quad (43)$$

So  $\langle \text{vac}_t | \hat{A}_{\text{phys}}(t) | \text{vac}_t \rangle$  is now zero by construction.

In Section 5.2 we show that this vacuum subtraction is equivalent to normal ordering with respect to the particle interpretation at the time of measurement. This choice is also made in previous ‘Hamiltonian diagonalisation’ procedures, and uniquely guarantees the positive definiteness of  $\hat{H}_{\text{phys}}(t)$  while maintaining  $\langle \text{vac}_t | \hat{H}_{\text{phys}}(t) | \text{vac}_t \rangle = 0$ . Unfortunately, experience has shown that vacuum subtraction does not on its own always return a finite expectation value of  $\hat{H}_{\text{phys}}(t)$  in the *evolved vacuum*,  $\langle \text{vac}_{t_0}(t) | \hat{H}_{\text{phys}}(t) | \text{vac}_{t_0}(t) \rangle$ . Further renormalisation is often required. Techniques for the renormalisation of quantities ‘after vacuum subtraction’ are presented in, for example, Grib et al.[30], while many other techniques are presented in [25] or [26].

We now treat two important examples of ‘physical extension’.

1. Consider the unit operator on  $\mathcal{H}$ . Its physical extension clearly satisfies:

$$\begin{aligned} \hat{1}_{\text{phys}} | \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} \rangle_{t_0}(t) &= \{ \text{grade}(| \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} \rangle_{t_0}(t)) - \text{grade}(| \text{vac}_t \rangle) \} | \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} \rangle_{t_0}(t) \\ &= (m - n) | \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} \rangle_{t_0}(t) \end{aligned}$$

at all times  $t$  (including  $t = t_0$ ). This clearly represents charge;  $\hat{Q} = e \hat{1}_{\text{phys}}$ . Charge conservation therefore follows directly from the fact that evolution is grade preserving. That the physical extension of the unit operator should represent charge is clearly appropriate, since the norm  $\langle \psi | \hat{1}_1 | \psi \rangle$  of a state  $\psi \in \mathcal{H}$  is well known to be the conserved ‘charge’ of the Dirac Lagrangian conjugate to changes in phase.

2. The number operator is the most important example of an operator depending explicitly on the split of  $\mathcal{H}$  into  $\mathcal{H}^+(t_0)$  and  $\mathcal{H}^-(t_0)$ . It is the physical extension of the operator  $\hat{N}_1(t_0) : \mathcal{H} \rightarrow \mathcal{H}$  defined by  $\hat{N}_1(t_0) = \hat{P}^+(t_0) - \hat{P}^-(t_0)$ . Clearly  $\hat{N}_1(t_0)$  commutes with  $\hat{H}_1(t_0)$ , but does not in general commute with time evolution (since it does not commute with  $\hat{H}_1(t)$  for  $t \neq t_0$ ).  $\hat{N}_{\text{phys}}(t_0)$  inherits both of these properties. Therefore the number operator  $\hat{N}_{\text{phys}}(t_0)$  represents a well-defined physical observable, but is not conserved. When acting on states in standard form, it gives:

$$\hat{N}_{\text{phys}}(t_0)|{}_{(j_1 j_2 \dots j_n)}^{(i_1 i_2 \dots i_m)}\rangle = (m+n)|{}_{(j_1 j_2 \dots j_n)}^{(i_1 i_2 \dots i_m)}\rangle \quad (44)$$

so that it is positive definite and has integer eigenvalues. We can expand  $\hat{N}_{\text{phys}}(t_0)$  in modes as:

$$\hat{N}_{\text{phys}}(t_0) = \sum_k (\hat{N}_{k,t_0}^+ + \hat{N}_{k,t_0}^-)$$

where  $\{u_{k,t_0}\}$  and  $\{v_{k,t_0}\}$  are orthonormal bases of  $\mathcal{H}^\pm(t_0)$ ,  $\hat{N}_{k,t_0}^+ = u_{k,t_0} \wedge (i_{u_{k,t_0}})$  is the physical extension of  $|u_{k,t_0}\rangle\langle u_{k,t_0}|$  and  $\hat{N}_{k,t_0}^- = i_{v_{k,t_0}}(v_{k,t_0} \wedge \cdot)$  is the physical extension of  $-|v_{k,t_0}\rangle\langle v_{k,t_0}|$ . Similarly:

$$\hat{Q}_{\text{phys}} = e \sum_k (\hat{N}_{k,\tau}^+ - \hat{N}_{k,\tau}^-)$$

### 3.4. Expectation Values

Consider the expectation values of an operator  $\hat{A}_{\text{phys}}(t_1)$  which is the physical extension of some operator  $\hat{A}_1(t_1)$ . Consider first, for simplicity, its expectation value in the so-called ‘evolved vacuum’  $|\text{vac}_{t_0}(t_1)\rangle$ . From (15) and (43) we have

$$\langle \text{vac}_{t_0}(t_1) | \hat{A}_{\text{phys}}(t_1) | \text{vac}_{t_0}(t_1) \rangle = \sum_{i=1}^N \langle v_{i,t_0}(t_1) | \hat{A}_1(t_1) | v_{i,t_0}(t_1) \rangle - \sum_{i=1}^N \langle v_{i,t_1} | \hat{A}_1(t_1) | v_{i,t_1} \rangle \quad (45)$$

This can be expressed in terms of Bogoliubov coefficients by inserting factors of  $\hat{1} = \sum_i \{|u_{i,t_1}\rangle\langle u_{i,t_1}| + |v_{i,t_1}\rangle\langle v_{i,t_1}|\}$  on either side of  $\hat{A}_1(t_1)$  in the first term and rearranging, to give

$$\begin{aligned} & \langle \text{vac}_{t_0}(t_1) | \hat{A}_{\text{phys}}(t_1) | \text{vac}_{t_0}(t_1) \rangle \\ &= \text{Trace}(\beta^\dagger \mathbf{A}^{++} \beta + \beta^\dagger \mathbf{A}^{+-} \epsilon + \epsilon^\dagger \mathbf{A}^{-+} \beta + \epsilon^\dagger \mathbf{A}^{--} \epsilon) - \text{Trace}(\mathbf{A}^{--}) \\ &= \text{Trace}(\beta \beta^\dagger \mathbf{A}^{++} - \gamma \gamma^\dagger \mathbf{A}^{--} + \epsilon \beta^\dagger \mathbf{A}^{+-} + \beta \epsilon^\dagger \mathbf{A}^{-+}) \end{aligned} \quad (46)$$

where we have defined:

$$\begin{aligned} \mathbf{A}_{jk}^{++} &\equiv \langle u_{j,t_1} | \hat{A}_1(t_1) | u_{k,t_1} \rangle & \mathbf{A}_{jk}^{--} &\equiv \langle v_{j,t_1} | \hat{A}_1(t_1) | v_{k,t_1} \rangle \\ \mathbf{A}_{jk}^{+-} &\equiv \langle u_{j,t_1} | \hat{A}_1(t_1) | v_{k,t_1} \rangle & \text{and } \mathbf{A}_{jk}^{-+} &\equiv \langle v_{j,t_1} | \hat{A}_1(t_1) | u_{k,t_1} \rangle \\ & & &= \overline{\mathbf{A}_{kj}^{+-}} \text{ if } \hat{A}_1 \text{ is Hermitian} \end{aligned} \quad (47)$$

The subtraction  $(\epsilon\epsilon^\dagger)_{kj} - \delta_{kj} = -(\gamma\gamma^\dagger)_{kj}$ , which is used to pass to (46) relies on the fact that we are vacuum subtracting with respect to the vacuum *at the time of measurement*.

As a simple example of (46) we can calculate the number of particles in the evolved vacuum by using  $\hat{N}_1(t_1) = \hat{P}^+(t_1) - \hat{P}^-(t_1)$ , so that  $N_{jk}^{++} = \delta_{jk} = -N_{jk}^{--}$  and  $N_{jk}^{+-} = 0 = N_{jk}^{-+}$ . This gives:

$$\begin{aligned} N_{\text{vac},t_0}(t_1) &\equiv \langle \text{vac}_{t_0}(t_1) | \hat{N}_{\text{phys}}(t_1) | \text{vac}_{t_0}(t_1) \rangle = \text{Trace}(\beta\beta^\dagger + \gamma\gamma^\dagger) \\ &= 2\text{Trace}(\beta\beta^\dagger) \end{aligned} \quad (48)$$

where we have used the Bogoliubov conditions in the last stage. Conservation of charge follows by considering  $\hat{A}_1 = \hat{1}$ . Also, we can now write equation (24) as:

$$\begin{aligned} \mathcal{P}_{|\text{vac}_{t_0}\rangle \rightarrow |\text{vac}_{t_1}\rangle} &= |\det(\epsilon(t_1, t_0))|^2 = \det(\epsilon(t_1, t_0)\epsilon^\dagger(t, t_0)) \quad (49) \\ &= \exp(\text{Trace}(\log(1 - \gamma\gamma^\dagger))) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \text{Trace}((\gamma\gamma^\dagger)^n)\right) \quad (50) \end{aligned}$$

In an electromagnetic background  $\gamma$  is first order in the coupling constant  $e$ , so that  $\gamma\gamma^\dagger$  is second order. Hence (50) allows us to write:

$$\mathcal{P}_{|\text{vac}_{t_0}\rangle \rightarrow |\text{vac}_{t_1}\rangle} = \exp\left(-\frac{1}{2}N_{\text{vac},t_0}(t_1)\right) + O(e^4) \quad (51)$$

which relates the probability of vacuum decay to the expected pair creation. This relation is used in Itzykson and Zuber [31] to find  $N_{\text{vac},-\infty}(\infty)$  (to this order in  $e$ ) without using Bogoliubov coefficients. However Itzykson and Zuber [31] justify the result on physical grounds, without formal proof. Perturbative calculations now involve simply the expansion of the ‘first quantized’ solutions  $u_{i,t_0}(t), v_{i,t_0}(t)$  in powers of the coupling constant  $e$ , so as to generate expansions of the Bogoliubov coefficients. This is considered in detail in [22].

The derivation of  $\langle F_{t_0}(t_1) | \hat{A}_{\text{phys}}(t_1) | F_{t_0}(t_1) \rangle$  for an arbitrary state  $|F_{t_0}(t_1)\rangle$  is identical to the derivation of (46), and gives:



$$\begin{aligned}
 & \langle ({}_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m})_{t_0}(t_1) | \hat{A}_{\text{phys}}(t_1) | ({}_{j_1 j_2 \dots j_n}^{i_1 i_2 \dots i_m})_{t_0}(t_1) \rangle \\
 &= \sum_{k=1}^m \langle u_{i_k, t_0}(t_1) | \hat{A}_1(t_1) | u_{i_k, t_0}(t_1) \rangle - \sum_{k=1}^n \langle v_{j_k, t_0}(t_1) | \hat{A}_1(t_1) | v_{j_k, t_0}(t_1) \rangle \\
 & \quad + \langle \text{vac}_{t_0}(t_1) | \hat{A}_{\text{phys}}(t_1) | \text{vac}_{t_0}(t_1) \rangle \tag{52}
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k=1}^m (\alpha^\dagger \mathbf{A}^{++} \alpha + \alpha^\dagger \mathbf{A}^{+-} \gamma + \gamma^\dagger \mathbf{A}^{-+} \alpha + \gamma^\dagger \mathbf{A}^{--} \gamma)_{i_k i_k} \\
 & \quad - \sum_{k=1}^n (\beta^\dagger \mathbf{A}^{++} \beta + \beta^\dagger \mathbf{A}^{+-} \epsilon + \epsilon^\dagger \mathbf{A}^{-+} \beta + \epsilon^\dagger \mathbf{A}^{--} \epsilon)_{j_k j_k} \\
 & \quad + \langle \text{vac}_{t_0}(t_1) | \hat{A}_{\text{phys}}(t_1) | \text{vac}_{t_0}(t_1) \rangle \tag{53}
 \end{aligned}$$

### *Anomalies and Fluctuations*

Now let  $\hat{A}_1(t)$  represent a quantity that is conserved at the level of the Dirac equation. In this case  $\langle v_{i, t_0}(t_1) | \hat{A}_1(t_1) | v_{i, t_0}(t_1) \rangle = \langle v_{i, t_0} | \hat{A}_1(t_0) | v_{i, t_0} \rangle$ , so that (46) becomes:

$$\langle \text{vac}_{t_0}(t_1) | \hat{A}_{\text{phys}} | \text{vac}_{t_0}(t_1) \rangle = \text{Trace}(\mathbf{A}^{--}(t_0) - \mathbf{A}^{--}(t_1))$$

This can be non-zero even when  $\hat{A}_1$  is independent of time, because of the varying particle interpretation. Herein lies an elegant physical description of quantum anomalies. Although the expectation value of  $\hat{A}_H$  in any given state does not change with time, the portion of it attributable to the vacuum may change, as the definition of the vacuum changes. Since the amount that would be measured experimentally is the amount left after subtraction of the vacuum this can change with time. As an example, an elegant treatment of the axial anomaly in an external electromagnetic background, which appeals to a similar physical mechanism as above, is given in [32] and in [15].

Fluctuations in expectation values can be calculated using (17). Consider fluctuations of some quantity  $\hat{A}(t_1)$  in the ‘evolved vacuum’. Define

$$\hat{P}_{t_0}^-(t_1) \equiv \sum_i |v_{i, t_0}(t_1)\rangle \langle v_{i, t_0}(t_1)| = \hat{U}_1(t_1, t_0) \hat{P}^-(t_0) \hat{U}_1^\dagger(t_1, t_0)$$

and similarly for  $\hat{P}_{t_0}^+(t_1)$ . Then we can write:

$$\langle \text{vac}_{t_0}(t_1) | (\hat{A}_{\text{phys}})^2 | \text{vac}_{t_0}(t_1) \rangle - \langle \text{vac}_{t_0}(t_1) | \hat{A}_{\text{phys}} | \text{vac}_{t_0}(t_1) \rangle^2 \quad (54)$$

$$= \sum_i \langle v_{i,t_0}(t_1) | \hat{A}_1(t_1)^2 - \hat{A}_1(t_1) \hat{P}_{t_0}^-(t_1) \hat{A}_1(t_1) | v_{i,t_0}(t_1) \rangle \quad (55)$$

$$= \text{Trace}(\hat{P}_{t_0}^-(t_1) \hat{A}_1(t_1) \hat{P}_{t_0}^+(t_1) \hat{A}_1(t_1)) \quad (56)$$

where the trace here is  $2N$ -dimensional. It follows that the size of the fluctuations in the evolved vacuum are determined by the commutator of  $\hat{A}_1(t_1)$  and  $\hat{P}_{t_0}^-(t_1)$ . If we consider fluctuations in the physical vacuum at time  $t_1$ ,  $|\text{vac}_{t_1}\rangle$  (by putting  $t_0 = t_1$  above), we observe that any operator commuting with the Hamiltonian (and hence with  $\hat{P}^-(t_1)$ ) has zero fluctuations in the physical vacuum. This reflects the fact that the physical vacuum, which is constructed from the spectrum of the Hamiltonian, must be an eigenstate of any operator that commutes with the Hamiltonian.

#### 4. ARBITRARY OBSERVERS

Implicit in the construction presented so far is the assumption that the ‘in-state’ is prepared on the spacelike Cauchy surface  $t = t_0$  for some  $t_0$ , and that the out state is to be measured on the spacelike Cauchy surface  $t = t_1$  for some  $t_1 > t_0$ . We will show in this Section that this amounts to assuming that the system is being observed by an inertial observer, and described in their rest frame. We also show how the particle interpretation can be generalised to an arbitrary observer, by using their ‘radar time’, and we will demonstrate that this definition depends only on the motion of the observer, not on their choice of coordinates or their choice of gauge. The nonlocal nature of this definition will also be discussed.

#### 4.1. An Observer and The Corresponding Particle Interpretation

Consider an observer travelling on path  $\gamma : x^\mu = x^\mu(\tau)$  with proper time  $\tau$ , and define:

$\tau^+(x) \equiv$  (earliest possible) proper time at which a null geodesic leaving point  $x$  could intercept  $\gamma$ .

$\tau^-(x) \equiv$  (latest possible) proper time at which a null geodesic could leave  $\gamma$ , and still reach point  $x$ .

$\tau(x) \equiv \frac{1}{2}(\tau^+(x) + \tau^-(x)) =$  ‘radar time’.

$\rho(x) \equiv \frac{1}{2}(\tau^+(x) - \tau^-(x)) =$  ‘radar distance’.

$\Sigma_{\tau_0} \equiv \{x : \tau(x) = \tau_0\} =$  observer’s ‘hypersurface of simultaneity at time  $\tau_0$ ’.

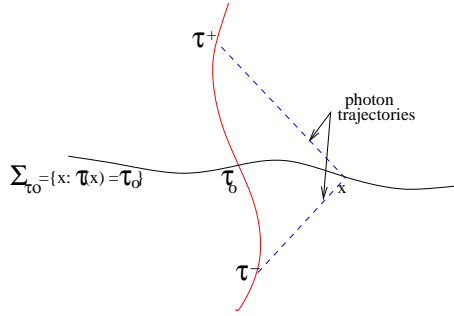


FIG. 1. Schematic of the definition of ‘radar time’  $\tau(x)$ .

This is a simple generalisation of the definition made popular by Bondi in his work on special relativity and k-calculus [33, 2, 34]. It generalises immediately to gravitational backgrounds [23, 22]. It is the only possible construction which agrees with proper time on the observer’s path, and is also invariant under ‘time-reversal’ - that is, under reversal of the sign of the observer’s proper time.

We can now define the ‘time-translation’ vector field:

$$k_\mu(x) \equiv \frac{\frac{\partial \tau}{\partial x^\mu}}{\eta^{\sigma\nu} \frac{\partial \tau}{\partial x^\sigma} \frac{\partial \tau}{\partial x^\nu}}$$

This represents the perpendicular distance between neighbouring hypersurfaces of simultaneity, since it is normal to these hypersurfaces, and satisfies  $k^\mu(x) \frac{\partial \tau}{\partial x^\mu} = 1$ . Now use the identity  $i k^\mu \gamma^\mu \nabla_\mu = i k^\mu \nabla_\mu + \sigma^{\mu\nu} k_\mu \nabla_\nu$  (where

$\not{k} \equiv k_\mu \gamma^\mu$  and  $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$  to write (2) as:

$$i k^\mu \nabla_\mu \psi = -\sigma^{\mu\nu} k_\mu \nabla_\nu \psi + m \not{k} \psi \quad (57)$$

From this we can define the ‘Hamiltonian on  $\Sigma_\tau$ ’  $\hat{H}_{\text{nh}}(\tau)$  by:

$$\hat{H}_{\text{nh}}(\tau) : \psi \rightarrow -\sigma^{\mu\nu} k_\mu \nabla_\nu \psi + m \not{k} \psi \quad (58)$$

$\hat{H}_{\text{nh}}(\tau)$  is not in general Hermitian! (hence the subscript *nh* here). At first sight this seems to disagree with unitarity on  $\mathcal{H}$ . In fact there is no inconsistency, because the inner product now depends (via the volume element on  $\Sigma_t$ ) explicitly on  $\tau$ , and because (57) is no longer of the form  $i \frac{d}{dt} |\psi(t)\rangle = \hat{H}_1(t) |\psi(t)\rangle$  so that the standard equivalence proof of unitary evolution and a Hermitian Hamiltonian no longer applies.

To investigate the relation between  $\hat{H}_{\text{nh}}(\tau_0)$  and the energy momentum tensor, define:

$$H_{\tau_0}(\psi) \equiv \int_{\Sigma_{\tau_0}} T_{\mu\nu}(\psi(x)) k^\mu d\Sigma^\nu \quad (59)$$

By substituting the energy momentum tensor (8) into this we see that:

$$H_{\tau_0}(\psi) = \Re[\langle \psi | \hat{H}_{\text{nh}}(\tau_0) | \psi \rangle_{\Sigma_{\tau_0}}] = \langle \psi | \hat{H}_1(\tau_0) | \psi \rangle_{\Sigma_{\tau_0}} \quad (60)$$

$$\text{where } \hat{H}_1(\tau_0) \equiv \frac{1}{2} \{ \hat{H}_{\text{nh}}(\tau_0) + \hat{H}_{\text{nh}}^\dagger(\tau_0) \} \quad (61)$$

It follows that we can define the projection operators  $\hat{P}^\pm(\tau_0)$  and the spaces  $\mathcal{H}^\pm(\tau_0)$  by requiring that  $H_{\tau_0}(\hat{P}^+(\tau_0)\psi) \geq H_{\tau_0}(\psi) \geq H_{\tau_0}(\hat{P}^-(\tau_0)\psi)$  for all  $\psi$ , just as in (20). Clearly this definition depends only on the background and the motion of the observer, and not on the choice of coordinates or of gauge. It is equivalent to defining:

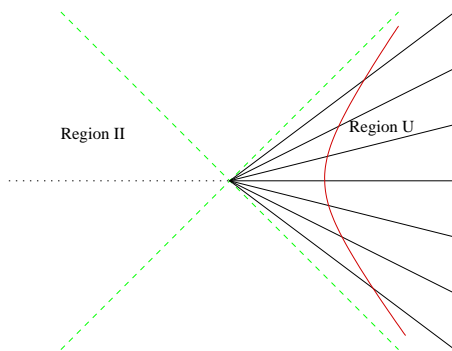
$\mathcal{H}^+(\tau_0)$  is the span of the positive spectrum of  $\hat{H}_1(\tau_0)$

$\mathcal{H}^-(\tau_0)$  is the span of the negative spectrum of  $\hat{H}_1(\tau_0)$

We can now define  $\hat{N}_1(\tau) = \hat{P}^+(\tau) - \hat{P}^-(\tau)$ , as in Section 3.3. We define Bogoliubov coefficients  $\alpha(\tau_1, \tau_0), \beta(\tau_1, \tau_0), \dots$ , S-Matrix elements  $\langle (i_1' i_2' \dots i_m')_{j_1' j_2' \dots j_n'}^{\tau_1} | (i_1 i_2 \dots i_m)_{j_1 j_2 \dots j_n}^{\tau_0}(\tau_1) \rangle$  and so on entirely as before, and all formulae of the previous Sections still apply with  $\tau$  in place of  $t$ , and ‘on  $\Sigma_\tau$ ’ in place

of ‘at time  $t$ ’. This construction is easily generalised to encompass gravitational backgrounds; see [23, 22]. In the next Section it is shown that, in the conventional approach to background QFT, this definition corresponds to Hamiltonian diagonalisation of the second quantized Hamiltonian obtained by substituting the field operator  $\hat{\psi}(x)$  into expression (59) for  $H_{\tau_0}(\psi)$ . This definition can be seen as a generalisation of Gibbons’ approach [1] to arbitrary observers.

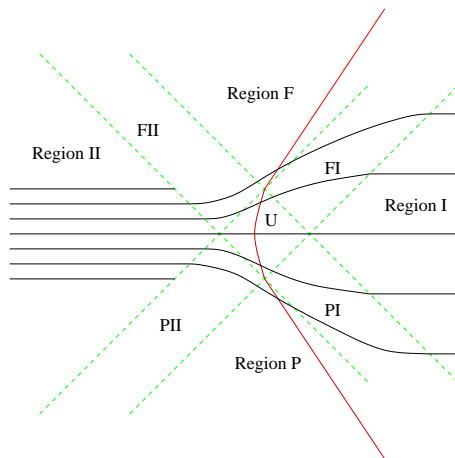
It is clear from the definition of radar time that  $\Sigma_{\tau_1}$  lies to the future of  $\Sigma_{\tau_0}$  (for  $\tau_1 > \tau_0$ ) except at the observer’s particle horizon (supposing one exists), at which point the various  $\Sigma_{\tau}$  converge. The domain of  $\tau(x)$  is not necessarily all of spacetime - only that part with which the observer can communicate (i.e. send and receive signals). The simplest illustration of this fact involves a uniformly accelerating observer.



**FIG. 2.** Hypersurfaces of simultaneity of a uniformly accelerating observer.

For a uniformly accelerating observer,  $\rho(x) = \frac{1}{2a} \log(a^2(z^2 - t^2))$  and  $\tau(x) = \frac{1}{2a} \log\left(\frac{z+t}{z-t}\right)$ , which are simply Rindler coordinates, covering only the region U in Figure 2. Also,  $k = z \frac{\partial}{\partial t} - t \frac{\partial}{\partial z}$  (or  $= \frac{\partial}{\partial \tau}$  in Rindler coordinates), which is the Killing vector field used to define positive/negative frequency modes in conventional derivations of the Unruh effect. For the significance of the dotted line in region II, consider the ‘Finite Acceleration Time’ case shown in Figure 3.

In the limit as this ‘Acceleration Time’ approaches infinity, this case approaches that of a uniformly accelerating observer. In this limit the hypersurfaces of simultaneity (which are all Cauchy) all approach this dotted line in region II, and  $k_{\mu}(x) \rightarrow 0$  there. A particle horizon forms as shown at the origin of Figure 2, and  $\mathcal{H}^0$  is no longer empty but rather consists of any states which are zero throughout region U. These states are ‘unmeasurable’



**FIG. 3.** Hypersurfaces of simultaneity of an observer undergoing uniform acceleration for a finite period of time (in Region U here) who is otherwise inertial.

by this accelerating observer, and must be traced out in derivations of the Unruh effect. This is explained in further detail in [36], and Figure 3 is also discussed in [35].

#### 4.2. Nonlocality of The Particle Definition

The particle definition presented in this paper, along with the definition of vacuum that accompanies it, are nonlocal in that they depend on the observer’s foliation of spacetime into ‘space at time  $\tau$ ’. We can easily see this since declaring a system to be ‘vacuum at time  $\tau$ ’ involves asserting that “at time  $\tau$  there were no particles anywhere”. This statement requires knowledge of a whole spacelike hypersurface defining space ‘at time  $\tau$ ’. We might have hoped, however, to define a local ‘particle density’  $N(x)$  with respect to which the ‘vacuum at time  $\tau$ ’ could be defined by requiring that  $N(x|_{\Sigma_\tau}) = 0$  in that state. But this too is impossible, since no local particle density can be consistent with the Unruh effect.

To investigate this nonlocality, consider the simple case of the number operator at time  $\tau$  for an inertial observer in Minkowski space. By working

in that observer's rest frame we identify  $\tau = t$ , and we can write [37]:

$$\hat{N}_1\psi(\mathbf{x}, t) \equiv \hat{P}^+\psi(\mathbf{x}, t) - \hat{P}^-\psi(\mathbf{x}, t) \quad (62)$$

$$= \int d^3\mathbf{y} G(\mathbf{x} - \mathbf{y})\psi(\mathbf{y}, t) \quad (63)$$

$$\text{where } G(\mathbf{x} - \mathbf{y}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \frac{\bar{\mathbf{p}} + m\gamma^0}{\sqrt{|\mathbf{p}|^2 + m^2}} \quad (64)$$

where  $\bar{\mathbf{p}} \equiv \sum_{k=1}^3 p^k \gamma_k \gamma_0$ . To derive this we simply expand an arbitrary state  $\psi(\mathbf{x})$  in terms of positive and negative energy plane wave states and substitute into (63). (The physical extension of this can be written in terms of a field operator as  $\hat{N} =: \int d^3\mathbf{x} d^3\mathbf{y} \hat{\psi}(\mathbf{x}) G(\mathbf{x} - \mathbf{y}) \hat{\psi}(\mathbf{y})$ : and is the standard number operator used for free Dirac field theory.) It is straightforward to calculate  $G(\mathbf{x} - \mathbf{y})$ , and we get

$$G(\mathbf{x}) = \frac{m^2}{2\pi^2|\mathbf{x}|} \left( K_1(m|\mathbf{x}|)\gamma^0 + iK_2(m|\mathbf{x}|)\frac{\bar{\mathbf{x}}}{|\mathbf{x}|} \right)$$

where  $K_n(m|\mathbf{x}|)$  are Bessel functions, which fall off exponentially for  $|\mathbf{x}| \gg \lambda_c$ . Hence, as we suspected, the particle definition is nonlocal on small scales, with the nonlocal contribution becoming negligible on scales larger than  $\lambda_c$ . This nonlocality need cause no concern, since it can be related directly to the motion of the particle detector. Indeed, the appearance at the heart of the interpretation of QFT of the observer and their hypersurface of simultaneity is quite natural. An ability to consider nonlocal issues within a relativistic context clearly requires (as we have done) that a choice of hypersurface be associated with any measurement, while consistency requires that this hypersurface be specified in terms of the motion of an observer.

On the other hand, the equation governing the evolution of states (given by (2) and (19)) remains local. The nonlocality discussed in this Section relates only to the calculation of expectation values in these states, and does not affect their evolution.

## 5. RELATIONSHIP TO CONVENTIONAL METHODS

In previous Sections we have shown that QFT in electromagnetic backgrounds can be performed with no mention of the 'field operator'  $\hat{\psi}(\mathbf{x})$ , provided we use a suitable representation of the states involved. But if we choose, we can for instance define the field operator just as in multiparticle

quantum mechanics [20, 21]:

$$\hat{\psi}(\mathbf{x}) = \sum_i \psi_i(\mathbf{x}) i_{\psi_i} \quad (65)$$

where the  $\{\psi_i(\mathbf{x})\}$  form an arbitrary orthonormal basis for  $\mathcal{H}$ . This automatically satisfies the Canonical Anticommutation Relations in view of the definition of  $i_{\psi}$ . It is also independent of the choice of basis, so that it does not depend on the split of solution space into ‘+ve/–ve energy solutions’. Operators  $\hat{A}_H : \mathcal{F}_\wedge(\mathcal{H}) \rightarrow \mathcal{F}_\wedge(\mathcal{H})$  can then be written as bilinears of the field operator, and the entire Canonical ‘second quantization’ procedure can be set up.

However, in conventional methods we do not have a concrete representation of state space, so that we cannot *derive* properties of the field operator (such as the Canonical Anticommutation Relations). Rather we must *require* these properties, and then use them to deduce the properties of the creation and annihilation operators and hence of the state space. In this Section we review this Canonical construction, and compare the conventional derivation of S-Matrix elements and expectation values with the derivations given in Section 3. Our presentation will follow that used (for real scalar fields) in Section 6 of DeWitt [16]. Interestingly, although the scalar field S-Matrix elements presented in [16] have been studied and applied extensively since their publication, we believe that the derivation presented in Section 5.2 represents the first treatment of fermionic systems along these lines. Although this derivation is new, it is in keeping with the conventional ‘canonical’ approach, and is the simplest available ‘conventional’ derivation. These results were also obtained, by very different methods, by Schwinger [38], while some special cases have also been derived by various other authors [39, 40, 41, 24, 42]. The equivalent derivation for charged bosonic systems is presented in [43].

### 5.1. Preliminaries

The field operator  $\hat{\psi}(x)$  is now defined (working now in the Heisenberg picture) by

$$\hat{\psi}(x) = \sum_i \{u_{i,\tau_0}(x) a_{i,\tau_0,h} + v_{i,\tau_0}(x) b_{i,\tau_0,h}^\dagger\} \quad (66)$$

where the  $u_{i,\tau_0}(x)$  are ‘states which at time  $\tau_0$  represent particles’, while  $v_{i,\tau_0}(x)$  are ‘states which at time  $\tau_0$  represent antiparticles’; the subscript  $h$  denotes that these operators annihilate or create Heisenberg-picture states.



Such an interpretation was previously only possible in general in the asymptotic limit  $\tau_0 \rightarrow \pm\infty$ , and sometimes a particle interpretation was completely abandoned [28] and the  $u_{i,\tau_0}(x)$ ,  $v_{i,\tau_0}(x)$  chosen arbitrarily. Now, of course, we will require that the  $u_{i,\tau_0}(x)$ ,  $v_{i,\tau_0}(x)$  be defined as in the previous Section.

Since we can no longer define creation and annihilations operators, by equations such as (26) and (27), we must deduce their properties from the properties of  $\hat{\psi}(x)$ . By requiring that  $\hat{\psi}(x)$  satisfy the CAR's we can deduce the standard anticommutation relations between the creation and annihilation operators. Next, by requiring that  $\hat{\psi}(x)$  be independent of  $\tau_0$  we can deduce relations such as

$$a_{i,\tau_1,h} = \sum_j \{ \alpha_{ij} a_{j,\tau_0,h} + \beta_{ij} b_{j,\tau_0,h}^\dagger \} \quad (67)$$

$$b_{i,\tau_1,h} = \sum_j \{ \gamma_{ij}^* a_{j,\tau_0,h}^\dagger + \epsilon_{ij}^* b_{j,\tau_0,h} \} \quad (68)$$

which are the Heisenberg picture equivalents of equations such as (37).

The (Heisenberg-picture) 'in' vacuum  $|\text{vac}_{\tau_0}, h\rangle$  is then defined implicitly by the requirement that

$$a_{i,\tau_0,h} |\text{vac}_{\tau_0}, h\rangle = 0 = b_{i,\tau_0,h} |\text{vac}_{\tau_0}, h\rangle \quad \langle \text{vac}_{\tau_0}, h | \text{vac}_{\tau_0}, h \rangle = 1 \quad (69)$$

and the 'in Fock space' is defined by acting on  $|\text{vac}_{\tau_0}, h\rangle$  with all possible combinations of creation operators, and taking the span of the result. The 'out' vacuum and the 'out Fock space' are defined similarly. Since these are Heisenberg picture states it follows that  $|\text{vac}_{\tau_0}, h\rangle$  represents not just  $|\text{vac}_{\tau_0}\rangle$ , but represents  $|\text{vac}_{\tau_0}(\tau)\rangle$  for all  $\tau$ , so that for instance  $\langle \text{vac}_{\tau_1} | \text{vac}_{\tau_0} \rangle \leftrightarrow \langle \text{vac}_{\tau_1}(\tau) | \text{vac}_{\tau_0}(\tau) \rangle$ , which is independent of  $\tau$  in view of conservation of the inner product.

We now show that equations (66) - (69) suffice for us to deduce the general S-Matrix element and expectation value of the theory without requiring a more concrete representation of the states involved. However these derivations are more difficult, and conceptually more obscure, than those given in Section 3.

## 5.2. S-Matrix Elements

Define the arbitrary 'in' and 'out' states:

$$\begin{aligned} & | \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} \rangle_{\tau_0}, h \rangle \equiv a_{i_1, \tau_0, h}^\dagger \dots a_{i_m, \tau_0, h}^\dagger b_{j_n, \tau_0, h}^\dagger \dots b_{j_1, \tau_0, h}^\dagger | \text{vac}_{\tau_0}, h \rangle \\ \text{and } & | \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} \rangle_{\tau_1}, h \rangle \equiv a_{i_1, \tau_1, h}^\dagger \dots a_{i_m, \tau_1, h}^\dagger b_{j_n, \tau_1, h}^\dagger \dots b_{j_1, \tau_1, h}^\dagger | \text{vac}_{\tau_1}, h \rangle \end{aligned}$$

We wish to calculate  $\langle \binom{i'_1 i'_2 \dots i'_m}{j'_1 j'_2 \dots j'_m}{}_{\tau_1}, h | \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n}{}_{\tau_0}, h \rangle$  using only the formalism set up in this Section. To do this, begin by defining:

$$c_v \equiv \langle \text{vac}_{\tau_1}, h | \text{vac}_{\tau_0}, h \rangle \quad (70)$$

$$V \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} \equiv \frac{1}{c_v} \langle \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n}{}_{\tau_1}, h | \text{vac}_{\tau_0}, h \rangle \quad (71)$$

$$\Lambda \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} \equiv \frac{1}{c_v} \langle \text{vac}_{\tau_1}, h | \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n}{}_{\tau_0}, h \rangle \quad (72)$$

From the definitions of  $V$  and  $\Lambda$  we have:

$$|\text{vac}_{\tau_0}, h\rangle = c_v \sum_{n,m=0}^{\infty} \sum_{\substack{i_1 < i_2 < \dots < i_m \\ j_1 < j_2 < \dots < j_n}} V \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} | \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n}{}_{\tau_1}, h \rangle \quad (73)$$

$$|\text{vac}_{\tau_1}, h\rangle = c_v \sum_{n,m=0}^{\infty} \sum_{\substack{i_1 < i_2 < \dots < i_m \\ j_1 < j_2 < \dots < j_n}} \Lambda^* \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} | \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n}{}_{\tau_0}, h \rangle \quad (74)$$

Acting on (74) with  $a_{i,\tau_1,h}$  and using (67) gives:

$$\begin{aligned} 0 = & \sum_{n,m=0}^{\infty} \sum_{\substack{i_1 < i_2 < \dots < i_m \\ j_1 < j_2 < \dots < j_n}} \left\{ \sum_{r=1}^m (-)^{r-1} \alpha_{ii_r} \Lambda^* \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} | \binom{i_1 \dots \check{i}_r \dots i_m}{j_1 j_2 \dots j_n}{}_{\tau_0}, h \rangle \right. \\ & \left. + \sum_j (-)^m \beta_{ij} \Lambda^* \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n} | \binom{i_1 i_2 \dots i_m}{j_1 j_2 \dots j_n j}{}_{\tau_0}, h \rangle \right\} \quad (75) \end{aligned}$$

valid for all  $i$ . Consideration of the coefficient of  $| \binom{i_1}{j_1}{}_{\tau_0}, h \rangle$  in (75) gives us

$$0 = \sum_{i_1} \alpha_{ii_1} \Lambda^* \binom{i_1}{j_1} + \beta_{ij_1} \Lambda^* ()$$

where  $\Lambda() = \frac{c_v}{c_v} = 1$ . So, if we define the matrix  $\mathbf{\Lambda} \equiv \epsilon^{-1} \boldsymbol{\gamma}$  as in (36), so that  $\mathbf{\Lambda}^\dagger = -\boldsymbol{\alpha}^{-1} \boldsymbol{\beta}$  (from (33)), then we have  $\Lambda \binom{i_1}{j_1} = \Lambda_{j_1 i_1}$ . By considering the coefficient of  $| \binom{i_1 i_2 \dots i_{m-1}}{j_1 j_2 \dots j_n}{}_{\tau_0}, h \rangle$  in (75) it follows after some work that

$$\sum_l \alpha_{il} \Lambda^* \binom{i_1 \dots i_{m-1} l}{j_1 j_2 \dots j_n} = (-)^{n+m} \sum_{r=1}^n (-)^r \beta_{ij_r} \Lambda^* \binom{i_1 i_2 \dots i_{m-1}}{j_1 \dots \check{j}_r \dots j_n}$$

Upon multiplying through by  $\alpha_{i_m i}^{-1}$ , summing over  $i$  and conjugating, we have

$$\Lambda_{j_1 j_2 \dots j_n}^{(i_1 i_2 \dots i_m)} = (-)^{n+m} \sum_{r=1}^n (-)^{r+1} \Lambda_{j_r i_m} \Lambda_{j_1 \dots j_r \dots j_n}^{(i_1 i_2 \dots i_{m-1})} \quad (76)$$

For  $n=m$  this gives (by induction)

$$\Lambda_{j_1 j_2 \dots j_n}^{(i_1 i_2 \dots i_n)} = (-)^{\frac{n}{2}(n-1)} \det \begin{bmatrix} \Lambda_{j_1 i_1} & \cdots & \Lambda_{j_1 i_n} \\ \vdots & & \vdots \\ \Lambda_{j_n i_1} & \cdots & \Lambda_{j_n i_n} \end{bmatrix} \quad (77)$$

in agreement with (36). By considering the coefficient of  $|(i_1 \dots i_{m-1})_{\tau_0}, h\rangle$  in (75) we have  $\Lambda_{j_1 j_2 \dots j_n}^{(i_1 i_2 \dots i_m)} = 0$  whenever  $m > 0$ , while it can be deduced that  $\Lambda_{j_1 j_2 \dots j_n}^{(i_1 i_2 \dots i_m)} = 0$  by acting on (74) with  $b_{i, t_1, h}$  and looking at the coefficient of  $|(j_1 j_2 \dots j_{n-1})_{\tau_0}, h\rangle$ . Combined with (76), this gives  $\Lambda_{j_1 j_2 \dots j_n}^{(i_1 i_2 \dots i_m)} = 0$  for  $m \neq n$ , as required for agreement with (36). Similarly, by acting on (73) with  $a_{i, \tau_0, h}$  and  $b_{i, \tau_0, h}$  and extracting components we can deduce that

$$V_{j_1 j_2 \dots j_n}^{(i_1 i_2 \dots i_m)} = \delta_{mn} (-)^{\frac{n}{2}(n-1)} \det \begin{bmatrix} V_{i_1 j_1} & \cdots & V_{i_1 j_n} \\ \vdots & & \vdots \\ V_{i_n j_1} & \cdots & V_{i_n j_n} \end{bmatrix} \quad (78)$$

To complete the rederivation of (35) and (36) we now calculate  $c_v$ , using

$$\begin{aligned} 1 &= \sum_{n, m=0}^{\infty} \sum_{\substack{i_1 < i_2 \dots < i_m \\ j_1 < j_2 \dots < j_n}} |(i_1 i_2 \dots i_m)_{\tau_1}, h\rangle \langle (i_1 i_2 \dots i_m)_{\tau_1}, h| \\ \text{to get} \quad 1 &= \sum_{n=0}^{\infty} \sum_{\substack{i_1 < i_2 \dots < i_n \\ j_1 < j_2 \dots < j_n}} |\langle (i_1 i_2 \dots i_n)_{\tau_1}, h | \text{vac}_{\tau_0}, h \rangle|^2 \\ &= |c_v|^2 \left( 1 + \sum_{n=1}^{\infty} \sum_{\substack{i_1 < i_2 \dots < i_n \\ j_1 < j_2 \dots < j_n}} \left| \det \begin{bmatrix} V_{i_1 j_1} & \cdots & V_{i_1 j_n} \\ \vdots & & \vdots \\ V_{i_n j_1} & \cdots & V_{i_n j_n} \end{bmatrix} \right|^2 \right) \\ &= |c_v|^2 \det(1 + \mathbf{V} \mathbf{V}^\dagger) \quad (79) \end{aligned}$$

where we have used the matrix identity:

$$\det(1 + \mathbf{V}\mathbf{V}^\dagger) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{i_1 < i_2 < \dots < i_n \\ j_1 < j_2 < \dots < j_n}} |\det \begin{bmatrix} V_{i_1 j_1} & \dots & V_{i_1 j_n} \\ \vdots & & \vdots \\ V_{i_n j_1} & \dots & V_{i_n j_n} \end{bmatrix}|^2$$

This identity follows from the Cauchy-Binet formula, applied (in the notation of Horn and Johnson [44], pg 22) to  $A = [V, I_N] = B^\dagger$  with  $\alpha = \beta = \{1, \dots, N\}$  and  $r = N$ . Hence

$$|c_v|^2 = \det(1 + \mathbf{V}\mathbf{V}^\dagger)^{-1} \quad (80)$$

$$= \det(\boldsymbol{\alpha}\boldsymbol{\alpha}^\dagger) \quad \text{from (33)} \quad (81)$$

$$= |\det(\boldsymbol{\alpha}(\tau_1, \tau_0))|^2 = |\det(\boldsymbol{\epsilon}(\tau_1, \tau_0))|^2 \quad (82)$$

where the Bogoliubov conditions have been used in the last step. So we can successfully deduce that  $|c_v| = |\det(\boldsymbol{\epsilon}(\tau_1, \tau_0))|$  without a concrete representation of the vacuum state. However, this derivation is technically much more difficult than that in Section 3.2; it requires (rather than derives) unitarity of the full multiparticle S-Matrix; and it obscures the fact that  $\det(\boldsymbol{\epsilon}(\tau_1, \tau_0))$  is the inner product between ‘in’ and ‘out’ Dirac Seas.

### 5.3. Expectation Values and Vacuum Subtraction

Consider now the relation between Hermitian extension and conventional ‘second quantization’. We consider a ‘first quantized’ operator  $\hat{A}_1(\tau): \mathcal{H} \rightarrow \mathcal{H}$ , which in general is a differential operator on  $\Sigma_\tau$ . By inserting factors of  $1 = \sum_i |u_{i,\tau_0}(\tau)\rangle\langle u_{i,\tau_0}(\tau)| + |v_{i,\tau_0}(\tau)\rangle\langle v_{i,\tau_0}(\tau)|$  to either side of  $\hat{A}_1(\tau)$  we can write:

$$\begin{aligned} \hat{A}_1(\tau) &= \sum_{ij} \langle u_{i,\tau_0}(\tau) | \hat{A}_1(\tau) | u_{j,\tau_0}(\tau) \rangle | u_{i,\tau_0}(\tau) \rangle \langle u_{j,\tau_0}(\tau) | \\ &\quad + \langle u_{i,\tau_0}(\tau) | \hat{A}_1(\tau) | v_{j,\tau_0}(\tau) \rangle | u_{i,\tau_0}(\tau) \rangle \langle v_{j,\tau_0}(\tau) | \\ &\quad + \langle v_{i,\tau_0}(\tau) | \hat{A}_1(\tau) | u_{j,\tau_0}(\tau) \rangle | v_{i,\tau_0}(\tau) \rangle \langle u_{j,\tau_0}(\tau) | \\ &\quad + \langle v_{i,\tau_0}(\tau) | \hat{A}_1(\tau) | v_{j,\tau_0}(\tau) \rangle | v_{i,\tau_0}(\tau) \rangle \langle v_{j,\tau_0}(\tau) | \end{aligned}$$

Since under Hermitian extension  $|u_{i,\tau_0}(\tau)\rangle\langle u_{j,\tau_0}(\tau)| \rightarrow a_{i,\tau_0}^\dagger(\tau)a_{j,\tau_0}(\tau)$ ,  $|u_{i,\tau_0}(\tau)\rangle\langle v_{j,\tau_0}(\tau)| \rightarrow a_{i,\tau_0}^\dagger(\tau)b_{j,\tau_0}^\dagger(\tau)$  etc.. (in the notation introduced in

Section 3.2), we can now write  $\hat{A}_H(\tau)$  as:

$$\begin{aligned}
 \hat{A}_H(\tau) = & \sum_{ij} \{ \langle u_{i,\tau_0}(\tau) | \hat{A}_1(\tau) | u_{j,\tau_0}(\tau) \rangle a_{i,\tau_0}^\dagger(\tau) a_{j,\tau_0}(\tau) \\
 & + \langle u_{i,\tau_0}(\tau) | \hat{A}_1(\tau) | v_{j,\tau_0}(\tau) \rangle a_{i,\tau_0}^\dagger(\tau) b_{j,\tau_0}^\dagger(\tau) \\
 & + \langle v_{i,\tau_0}(\tau) | \hat{A}_1(\tau) | u_{j,\tau_0}(\tau) \rangle b_{i,\tau_0}(\tau) a_{j,\tau_0}(\tau) \\
 & + \langle v_{i,\tau_0}(\tau) | \hat{A}_1(\tau) | v_{j,\tau_0}(\tau) \rangle b_{i,\tau_0}(\tau) b_{j,\tau_0}^\dagger(\tau) \} \quad (83)
 \end{aligned}$$

In the conventional approach to second quantization the pre-normal-ordered ‘second quantized’ operator  $\hat{A}_{naive}(\tau)$  is obtained by substituting the field operator  $\hat{\psi}(x)$  from (66) into the integral expression for the ‘first quantized expectation value’  $\langle \psi(\tau) | \hat{A}_1(\tau) | \psi(\tau) \rangle$ . This gives:

$$\begin{aligned}
 \hat{A}_{naive}(\tau) = & \sum_{ij} \langle u_{i,\tau_0}(\tau) | \hat{A}_1(\tau) | u_{j,\tau_0}(\tau) \rangle a_{i,\tau_0,h}^\dagger a_{j,\tau_0,h} \\
 & + \langle u_{i,\tau_0}(\tau) | \hat{A}_1(\tau) | v_{j,\tau_0}(\tau) \rangle a_{i,\tau_0,h}^\dagger b_{j,\tau_0,h}^\dagger \\
 & + \langle v_{i,\tau_0}(\tau) | \hat{A}_1(\tau) | u_{j,\tau_0}(\tau) \rangle b_{i,\tau_0,h} a_{j,\tau_0,h} \\
 & + \langle v_{i,\tau_0}(\tau) | \hat{A}_1(\tau) | v_{j,\tau_0}(\tau) \rangle b_{i,\tau_0,h} b_{j,\tau_0,h}^\dagger \quad (84)
 \end{aligned}$$

which is clearly the Heisenberg picture version of (83). By construction, neither of these operators depends on the choice of  $\tau_0$ . It is also worth noting that although (83) and (84) are equivalent, the expression (83) is seldom very useful in our formalism. It is usually more convenient to work directly from the definition (13). Likewise, in conventional multiparticle quantum mechanics (see [20] for instance) expressions like (83) appear (with only the  $u_i(x)$ 's and  $a_i$ 's), but are seldom useful. This is partly because writing down (83) demands a complete set of solutions to the (first quantized) governing equation; the usefulness of (83) then relies on the matrix elements taking a convenient form in this set. Indeed, even writing down the field operator  $\hat{\psi}(x)$  requires this complete set of solutions. By contrast, the definition (13), which is precisely the definition used in multiparticle quantum mechanics, is well-defined whether or not we have such a complete set.

We can now calculate the expectation value of  $\hat{A}_{naive}(\tau)$  in the state  $|\text{vac}_{\tau_{in}}, h\rangle$ , representing the state that ‘at time  $\tau_{in}$ ’ was vacuum. To do this choose  $\tau_0 = \tau_{in}$  in (84), so that

$$\langle \text{vac}_{\tau_{in}}, h | \hat{A}_{naive}(\tau) | \text{vac}_{\tau_{in}}, h \rangle = \sum_i \langle v_{i,\tau_{in}}(\tau) | \hat{A}_1(\tau) | v_{i,\tau_{in}}(\tau) \rangle \quad (85)$$

This is simply the first term in (45), which followed directly from (15). Although  $\hat{A}_{\text{naive}}(\tau)$  is independent of the choice of  $\tau_0$ , the normal ordering process clearly depends on  $\tau_0$ . We denote *normal ordering with respect to the  $\tau'$  expansion of  $\hat{\psi}(x)$*  by  $:\ :_{\tau'}$ . Then:

$$\begin{aligned} :\hat{A}_{\text{naive}}(\tau):_{\tau'} &= \sum_{ij} \langle u_{i,\tau'}(\tau) | \hat{A}_1(\tau) | u_{j,\tau'}(\tau) \rangle a_{i,\tau',h}^\dagger a_{j,\tau',h} \\ &\quad + \langle u_{i,\tau'}(\tau) | \hat{A}_1(\tau) | v_{j,\tau'}(\tau) \rangle a_{i,\tau',h}^\dagger b_{j,\tau',h}^\dagger \\ &\quad + \langle v_{i,\tau'}(\tau) | \hat{A}_1(\tau) | u_{j,\tau'}(\tau) \rangle b_{i,\tau',h} a_{j,\tau',h} \\ &\quad - \langle v_{i,\tau'}(\tau) | \hat{A}_1(\tau) | v_{j,\tau'}(\tau) \rangle b_{j,\tau',h}^\dagger b_{i,\tau',h} \end{aligned} \quad (86)$$

$$= \hat{A}_{\text{naive}}(\tau) - \sum_i \langle v_{i,\tau'}(\tau) | \hat{A}_1(\tau) | v_{i,\tau'}(\tau) \rangle \hat{1} \quad (87)$$

Comparison of (87) with (45) reveals that vacuum subtraction corresponds to the choice  $\tau' = \tau$ , and so to normal ordering *at the time of measurement*:

$$\hat{A}_{\text{phys}}(\tau) \Leftrightarrow :\hat{A}_{\text{naive}}(\tau):_{\tau} \quad (88)$$

This is also the choice used in the ‘Hamiltonian diagonalisation’ procedure discussed in the next Section. From now on we shall assume that  $\tau' = \tau$ . Expression (86) is obtained upon normal ordering by ‘moving the creation operators to the left, and changing appropriate signs’, while (87) acknowledges that this is the same as subtracting an appropriate multiple of the unit operator. Since (87) still allows us to express  $\hat{A}_{\text{naive}}(\tau)$  in terms of an arbitrary  $\tau_0$  it is generally more convenient than (86), which involves expressing  $\hat{A}_{\text{naive}}(\tau)$  in terms of  $\tau_0 = \tau' (= \tau \text{ now})$ .

It will be convenient for later to write the operator  $\hat{H}_{\text{naive}}(\tau)$  (obtained by substituting  $\hat{\psi}(x)$  into (59)) expressed in terms of  $\tau_0 = \tau$ . This can be written as:

$$\begin{aligned} \hat{H}_{\text{naive}}(\tau) &= \sum_{ij} \mathbf{H}_{ij}^{++}(\tau) a_{i,\tau,h}^\dagger a_{j,\tau,h} + \mathbf{H}_{ij}^{+-}(\tau) a_{i,\tau,h}^\dagger b_{j,\tau,h}^\dagger \\ &\quad + \mathbf{H}_{ij}^{-+}(\tau) b_{i,\tau,h} a_{j,\tau,h} + \mathbf{H}_{ij}^{--}(\tau) b_{i,\tau,h} b_{j,\tau,h}^\dagger \end{aligned} \quad (89)$$

Since  $\{u_{j,\tau_0}(\tau)\}$  and  $\{v_{j,\tau_0}(\tau)\}$  are chosen to satisfy the particle definition introduced in Section 4, we clearly have  $\mathbf{H}^{-+}(\tau) = 0 = \mathbf{H}^{+-}(\tau)$ , so that  $\hat{H}_{\text{naive}}(\tau)$  contains no  $a_{i,\tau,h}^\dagger b_{j,\tau,h}^\dagger$ , or  $b_{i,\tau,h} a_{j,\tau,h}$  terms. It also follows that

$\mathbf{H}^{++}(\tau)$  and  $-\mathbf{H}^{--}(\tau)$  are positive definite. Conversely, it is straightforward to show that if  $\mathbf{H}^{-+}(\tau) = 0 = \mathbf{H}^{+-}(\tau)$ , (and  $\mathbf{H}^{++}(\tau)$  and  $-\mathbf{H}^{--}(\tau)$  are positive definite) then the  $\{u_{j,\tau_0}(\tau)\}$  span  $\mathcal{H}^+(\tau)$ , and  $\{v_{j,\tau_0}(\tau)\}$  span  $\mathcal{H}^-(\tau)$ . This verifies that the particle definition given in Section 4 is equivalent to Hamiltonian diagonalisation of the Hamiltonian obtained by substituting  $\hat{\psi}(x)$  into (59). We now discuss Hamiltonian diagonalisation.

#### 5.4. On Hamiltonian Diagonalisation

The idea of using ‘Hamiltonian diagonalisation’ as a prescription for defining particle/antiparticle states dates back to Imamura [49] in 1960, based on an analogy with Bogoliubov’s work on superfluidity [50] and superconductivity [51]. It was developed in more detail by Grib and Mamaev [52, 53] and Grib, Mamayev and Mostepanenko [30] during the 1970’s. The basic idea is to expand a second quantized Hamiltonian  $\hat{H}(\tau_0)$  in terms of creation/annihilation operators (just as in (89)), and choose the basis solutions that ‘represent particle/antiparticle states at time  $\tau_0$ ’ by requiring that  $\hat{H}(\tau_0)$  contain no terms proportional to  $b_{i,\tau_0}a_{j,\tau_0}$  or  $a_{i,\tau_0}^\dagger b_{j,\tau_0}^\dagger$ . Application of this requirement at two different times leads to two different mode-expansions of  $\hat{\psi}(x)$ , related by Bogoliubov coefficients, which can then be used to calculate S-matrix elements, vacuum expectation values etc.

This procedure clearly depends on the choice of second quantized Hamiltonian. As pointed out by Fulling [26], previous prescriptions for choosing a Hamiltonian [54, 30, 55] were far from unique, since they depended on an arbitrarily chosen ‘time’ coordinate. The prescription of Gibbons [1], which amounts to diagonalising  $\int T_{\nu\mu}k^\nu d\Sigma^\mu$  where  $k^\mu$  is a timelike Killing vector field, is less arbitrary, since it depends only on the choice of timelike Killing vector field. However, this restricts us to spacetimes which admit timelike Killing vector fields, and restricts us to considering observers travelling on the integral curves of those Killing vector fields. Our particle definition, which amounts to diagonalising the Hamiltonian obtained by substituting  $\hat{\psi}(x)$  into (59), is precisely the definition needed to fix these problems. This Hamiltonian, which has not been studied before, is the first Hamiltonian that is defined directly in terms of the motion of an observer and that depends *only* on the choice of observer and the background present - not on the choice of coordinates, the choice of gauge or the detailed construction of the observer’s particle detector. It provides a unique particle interpretation for any chosen observer travelling in an arbitrary electromagnetic background. Generalisation to curved spacetimes is straightforward, and is presented in [23].

Another advantage of the prescription presented in Sections 3 and 4 is that it does more than provide a ‘Hamiltonian diagonalisation condition’ (equation (20)) which  $\mathcal{H}^\pm(\tau_0)$  must satisfy. It also provides a clear procedure for constructing  $\mathcal{H}^\pm(\tau_0)$  (in terms of eigenstates of the 1st quantized Hamiltonian  $\hat{H}_1(\tau)$ ) which allows us to tackle arbitrarily complicated situations. With conventional Hamiltonian diagonalisation prescriptions unless the spacetime allows separation of variables no clue is given how to satisfy the ‘Hamiltonian diagonalisation condition’. This need to separate variables also plagues the ‘adiabatic approximation’ techniques advocated by Fulling [26, 56] and others.

Another particle definition is that used by Fradkin, Gitman and Shvartsman [45]. As here, they use eigenstates of a ‘first quantized Hamiltonian’ at times  $t_{in}$  and  $t_{out}$  to define ‘in’ and ‘out’ states (though they do not consider non-inertial observers). However they use the spectrum of  $\hat{H}_{ev}(t)$  rather than  $\hat{H}_1(t)$  making their formulation gauge dependent! To see this one need only consider free field theory in the gauge  $A(x) = (A, 0, 0, 0)$ , where  $A$  is some constant, for which  $eA \gg m$ . A more telling example arises in the familiar case of a constant electric field, where if the gauge  $A = (Ez, 0, 0, 0)$  is chosen, then no pair creation occurs in Fradkin et al.’s [45] approach! Gauge dependence in the choice of particle interpretation is a problem that has troubled all previous ‘Bogoliubov coefficient’ approaches to quantum field theory in electromagnetic backgrounds [17]. The problem is usually ‘patched up’ by appealing to the tunnelling interpretation. If the gauge  $A = (Ez, 0, 0, 0)$  is chosen, particle creation can be described in terms of particles ‘tunnelling through the barrier that separates particle/antiparticle states’ (see [41, 46] or [47] for a description of these methods). This leaves us with two theories, each failing where the other succeeds. Here the two approaches arise naturally as parts of the same formalism. By using eigenstates of the gauge-covariant Hamiltonian  $\hat{H}_1 \neq \hat{H}_{ev}$  to define particle states, we see that particle creation in the background  $A^\mu(x)$  requires either: (1) that  $A^0(x) \neq 0$ , so that  $A^0(x)$  can act as a potential, w.r.t. which tunnelling may occur, or (2)  $\mathbf{A}(x) = (A^1(x), A^2(x), A^3(x))$  is time-dependent, so that eigenstates of  $\hat{H}_1(t)$  mix during evolution. Both effects contribute to the calculation of Bogoliubov coefficients [22]. An immediate consequence is that if a gauge can be chosen such that  $A^0 = 0$  and  $\mathbf{A}$  is time-independent, then particle creation will not occur. That is, *time-independent magnetic fields cannot create particles*. This result agrees with Schwinger’s calculation [48] in terms of an effective Lagrangian, and successfully resolves all of the inconsistency problems raised by Sriramkumar and Padmanabhan [17]. Another simple example is the case of electrostatic



fields, where a gauge can be chosen such that  $A^0$  is time-independent, while  $\mathbf{A} = 0$ . In this case the physical vacuum is constructed from negative energy solutions of the free Dirac equation, and stationary states are constructed from eigenstates of  $\hat{H}_{ev}$ . The difference between the physical vacuum and the ‘lowest energy stationary state’ accounts for vacuum effects such as Lamb screening.

## 6. DISCUSSION

We have presented above a formulation of fermionic quantum field theory in electromagnetic backgrounds that is precisely analogous to the methods used in multiparticle quantum mechanics. The only difference is that the particle interpretation requires us to consider the entire Dirac Sea, as well as any particles which may be present. Rather than working with a field operator  $\hat{\psi}(x)$  and an abstract Fock space constructed solely from creation and annihilation operators acting on some postulated vacuum  $|0\rangle$ , we work directly with the Schrödinger picture states of the system, described in terms of Slater determinants of solutions of the Dirac equation. As well as providing a conceptually transparent approach to the theory and an extremely simple derivation of the general S-Matrix element and expectation value of the theory, this approach also allows us to provide a consistent particle interpretation for all times, without requiring any ‘asymptotic niceness conditions’ on the ‘in’ and ‘out’ states. Other advantages include the ease with which unitarity of the S-Matrix follows from conservation of the Dirac inner product, insights into quantum anomalies, and the fact that Hermitian extension provides well-defined second quantized operators without requiring a complete set of orthonormal modes. We have used the concept of ‘radar time’ to generalise the particle interpretation to an arbitrarily moving observer, providing a definition of particle which depends only on the observer’s motion and on the background present, not on the choice of coordinates, the choice of gauge, or the detailed construction of the particle detector.

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