Electron scattering without spin sums

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Abstract

The Spacetime Algebra provides an elegant language for studying the Dirac equation. We show how to perform cross section calculations following a method suggested by Hestenes (1982). The S-matrix is replaced with an operator which rotates the initial states into the scattered states. The method neatly handles spin dependence by allowing the scattering operator to become a function of the initial spin. When the operator is independent of spin we can provide manifestly spin-independent results. Spin basis states are not needed, and neither are spin sums. Instead we deal with the spin orientation directly. We perform example calculations of spin dependence and polarization in Coulomb scattering to second order, and briefly consider more complicated calculations in QED.

Keywords: Spacetime Algebra, scattering, cross sections, spin

1 Introduction

Methods for calculating spinor cross sections are well known, however these often involve complicated abstract calculations with gamma matrices. In this paper we show how to calculate cross sections in a more transparent and intuitive way. Instead

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of using spin basis states, summing over spins and using spin projection operators, we instead incorporate the spin orientation directly. This greatly streamlines the calculation of spin dependent results, and makes it clear when results are independent of spin. We first consider single electron scattering, where our method is most naturally applied, and then briefly how we can handle multi-particle scattering.

The starting point for our approach is the Spacetime Algebra formulation of Dirac theory. The Spacetime Algebra (STA) is the geometric (Clifford) algebra of Minkowski spacetime, first developed by Hestenes (1966, 1975, 1982). The formulation of Dirac theory within the algebra replaces the matrices of the conventional theory with multivectors. The two formulations are entirely equivalent, but the STA approach brings out the geometric structure leading to more physically transparent calculations. We briefly summarize the STA formulation below and then explain a method for performing cross section calculations first demonstrated by Hestenes (1982). We extend and clarify this work, handling spin-dependence in a natural way.

2 Spacetime Algebra and the Dirac Equation

Throughout we shall make use of the Geometric Algebra. We present a brief summary of the STA below to clarify our notation and conventions. Full details of Geometric Algebra can be found elsewhere (Hestenes & Sobczyk, 1984; Hestenes, 1966).

We shall use the four orthogonal basis vectors of spacetime γ_μ, where $\gamma_0^2 = 1$, and $\gamma_k^2 = -1$ for $k = 1, 2, 3$. The Geometric Algebra has an associative product, and the basis vectors then satisfy the Dirac algebra

$$\gamma_\mu \cdot \gamma_\nu \equiv \tfrac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = \mathrm{diag}(+ - - -).$$

The antisymmetric part of the product defines the *outer product*

$$\gamma_\mu \wedge \gamma_\nu \equiv \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$

By repeated multiplication of the basis vectors we can build up the 16 basis elements of STA multivectors:

$$1 \qquad \{\gamma_\mu\} \qquad \{\gamma_\mu \wedge \gamma_\nu\} \qquad \{I\gamma_\mu\} \qquad I$$

scalar vectors bivectors pseudovectors pseudoscalar

We can do a space-time split of a vector into the γ_0 frame by multiplying by γ_0. For example the momentum p is split

$$p\gamma_0 = p\cdot\gamma_0 + p\wedge\gamma_0 = E + \boldsymbol{p} = p^0 + p^i\gamma_i\gamma_0.$$

Bold letters are now used for relative 3-vectors (spacetime bivectors). Restricted Lorentz transformations are spacetime rotations and can be performed by use of a *rotor*, which can be written $R = \pm\exp(B/2)$. Here B is a bivector in the plane of the rotation and $|B|$ determines the amount of rotation. The rotation of a multivector M is then given by

$$M \to RM\widetilde{R}.$$

In the STA spinors are represented using the even subalgebra which has the required eight degrees of freedom. The minimally coupled form of the Hestenes' Dirac equation is

$$\hat{\jmath}\nabla\psi\gamma_0 - m\psi = eA\psi\gamma_0.$$

Here $\hat{\jmath}$ is an operator that multiplies on the right by an arbitary reference spatial bivector $\mathbf{\Sigma}$ so that

$$\hat{\jmath}\psi \equiv \psi\mathbf{\Sigma}.$$

The spinor ψ can be decomposed as

$$\psi = \rho^{1/2}e^{I\beta/2}LU$$

where L and U are rotors for a boost and spatial rotation respectively, ρ is a scalar, and the β-factor determines the rest ratio of particles and anti-particles. So in the STA approach the spinor directly encodes a Lorentz transformation and a propability density. The rotor U rotates the arbitrary reference plane $\mathbf{\Sigma}$ into the *rest spin bivector* observable of the electron

$$\hat{S}^0 \equiv U\mathbf{\Sigma}\widetilde{U}$$

and the boost L gives the momentum

$$p = mL\gamma_0\widetilde{L}.$$

We can also boost the rest spin bivector to define the relativistic spin bivector

$$\hat{S} = L\hat{S}^0\widetilde{L} = \rho^{-1}\psi\mathbf{\Sigma}\widetilde{\psi}.$$

Positive and negative energy plane wave solutions are given as usual by

$$\psi^{(+)} = u(p)e^{-\hat{\jmath}p\cdot x} \quad \text{and} \quad \psi^{(-)} = v(p)e^{\hat{\jmath}p\cdot x}$$

and the energy projection operators are

$$\Lambda_{\pm}(\psi) = \frac{1}{2m}(m\psi \pm p\psi\gamma_0).$$

3 The Feynman Propagator

The Feynman propagator S_F is the Greens' function for the Dirac equation that propagates negative energy waves into the past and positive energy waves into the future. As a Greens' function it satisfies

$$\hat{\jmath}\nabla_x S_F(x-x')\psi(x')\gamma_0 - mS_F(x-x')\psi(x') = \delta^4(x-x')\psi(x')$$

and an integral solution to the Dirac equation is given by

$$\psi(x) = \psi_i(x) + e\int d^4x' S_F(x-x')A(x')\psi(x')\gamma_0 \tag{1}$$

where ψ_i satisfies the free-particle equation. Taking the Fourier transform we have

$$pS_F(p)\psi\gamma_0 - mS_F(p)\psi = \psi$$

where

$$S_F(x-x') = \int \frac{d^4p}{(2\pi)^4} S_F(p) e^{-\hat{\jmath}p\cdot(x-x')}.$$

Operating on both sides with the energy projection operator Λ_+ we can solve for the momentum space Feynman propagator:

$$\begin{aligned}(p^2-m^2)S_F(p)\psi &= p\psi\gamma_0 + m\psi \\ \Rightarrow S_F(p)\psi &= \frac{p\psi\gamma_0 + m\psi}{p^2 - m^2 + \hat{\jmath}\epsilon}. \end{aligned} \tag{2}$$

The $\hat{\jmath}\epsilon$ ensures that the contour integral is in the $\mathbf{\Sigma}$ plane and that it is causal—positive energy waves propagate into the future and negative energy waves into the past. Fourier transforming back and performing the integral over dE we have

$$\begin{aligned} &S_F(x-x')\psi = \\ &\quad -2m\hat{\jmath}\int \frac{d^3\boldsymbol{p}}{2E_p(2\pi)^3}[\theta(t-t')\Lambda_+(\psi)e^{-\hat{\jmath}p\cdot(x-x')} + \theta(t'-t)\Lambda_-(\psi)e^{\hat{\jmath}p\cdot(x-x')}] \end{aligned} \tag{3}$$

where $E = +\sqrt{\boldsymbol{p}^2 + m^2}$.

4 Electron Scattering

For scattering calculations we write the wavefunction as the sum of an incoming plane wave and a scattered beam, $\psi = \psi_i + \psi_{\text{diff}}$, where ψ_{diff} is the solution at asymptotically large times given by

$$\psi_{\text{diff}}(x) = -2m\hat{\jmath}e\int d^4x' \int \frac{d^3\boldsymbol{p}}{2E_p(2\pi)^3}\Lambda_+ \left[A(x')\psi(x')\gamma_0\right] e^{-\hat{\jmath}p\cdot(x-x')}.$$

This can be written as a sum over final states

$$\psi_{\text{diff}}(x) = \int \frac{d^3\boldsymbol{p}_f}{2E_f(2\pi)^3}\psi_f(x),$$

the final states being plane waves of the form

$$\psi_f(x) \equiv \psi_f e^{-\hat{\jmath} p_f \cdot x} \equiv -\hat{\jmath} e \int d^4x' [p_f A(x')\psi(x') + mA(x')\psi(x')\gamma_0] e^{-\hat{\jmath} p_f \cdot (x-x')}. \quad (4)$$

The Born series perturbative solution is generated by iterating (1). In the first order Born approximation this amounts to simply replacing $\psi(x')$ by $\psi_i(x')$. For plane waves of particles we have

$$\psi(x) = \psi e^{-\hat{\jmath} p \cdot x} \quad \text{and} \quad m\psi\gamma_0 = p\psi$$

so the final states become

$$\begin{aligned} \psi_f &= -\hat{\jmath} e \int d^4x' [p_f A(x') + A(x')p_i]\psi_i e^{\hat{\jmath} q \cdot x'} \\ &= -\hat{\jmath} e [p_f A(q) + A(q)p_i]\psi_i \end{aligned}$$

where $q \equiv p_f - p_i$.

More generally we define

$$\psi_f = S_{fi}\psi_i$$

where S_{fi} is the *scattering operator* which rotates and dilates the initial states into the final states. Here the f and i indices label the initial and final momenta and the initial spin, so in general $S_{fi} = S_{fi}(p_f, p_i, \hat{S}_i)$. However S_{fi} does not depend on the final spin—instead the final spin is determined from the initial spin by a rotation encoded in S_{fi}.

Since S_{fi} consists of a rotation and dilation it is convenient to decompose it as

$$S_{fi} = \rho_{fi}^{1/2} R_{fi}$$

where R_{fi} is a rotor. The cross section will be determined by the ρ_{fi} factor, as detailed in the next section. The rotor R_{fi} rotates states with momentum p_i into states with momentum p_f. It also relates the initial and final spins by

$$\hat{S}_f = R_{fi}\hat{S}_i\widetilde{R}_{fi}$$

so the rest spins are related by

$$\hat{S}_f^0 = \widetilde{L}_f \hat{S}_f L_f = \widetilde{L}_f R_{fi} \hat{S}_i \widetilde{R}_{fi} L_f = \widetilde{L}_f R_{fi} L_i \hat{S}_i^0 \widetilde{L}_i \widetilde{R}_{fi} L_f.$$

We therefore define the *rest spin scattering operator*

$$U_{fi} \equiv \widetilde{L}_f R_{fi} L_i$$

so that

$$\hat{S}_f^0 = U_{fi} \hat{S}_i^0 \widetilde{U}_{fi}.$$

The rest spin scattering operator and the cross section contain all the information about scattering of states with initial momentum p_i and spin $\hat{S}_i$ into final states with momentum p_f.

The form of the external line Feynman propagator (3) ensures that S_{fi} is of the form

$$S_{fi} = -\hat{\jmath}(p_f M + M p_i) \tag{5}$$

where in the Born approximation example $M = eA(q)$. However in general M can have some $\hat{\jmath}$-dependence in which case we can write

$$S_{fi}\psi_i = -\hat{\jmath}(p_f[M_r + \hat{\jmath}M_j] + [M_r + \hat{\jmath}M_j]p_i)\,\psi_i$$

where M_j and M_r are independent of $\hat{\jmath}$. Using $\hat{\jmath}\psi_i = \psi_i \mathbf{\Sigma} = \hat{S}_i \psi_i$ and the fact that $\hat{S}_i$ and p_i commute this can be written

$$S_{fi} = -\hat{\jmath}(p_f M + M p_i)$$

where now $M = M_r + M_j \hat{S}_i$ depends on the initial spin. We can therefore convert dependence on the 'imaginary' $\hat{\jmath}$ into dependence on the physical spin bivector.

5 Cross Sections

The scattering rate into the final states per unit volume per unit time is given by

$$W_{fi} = \frac{\rho_f}{2mVT}$$

where ρ_f is given simply by

$$\rho_f = |S_{fi}|^2 \rho_i = \rho_{fi} \rho_i.$$

The cross section is defined as

$$d\boldsymbol{\sigma} = \frac{W_{fi}}{\text{Target density} \times \text{Incident flux}}.$$

For elastic scattering we have

$$S_{fi} = -\hat{\jmath} 2\pi \delta(E_f - E_i) T_{fi}$$

in which case

$$|S_{fi}|^2 = 2\pi T \delta(E_f - E_i) |T_{fi}|^2.$$

With a target density of $1/V$ and an incident flux of $|\boldsymbol{J}_i| = \rho_i |\boldsymbol{p}_i|/m$ we have

$$d\boldsymbol{\sigma} = \frac{\pi}{|\boldsymbol{p}_i|} \delta(E_f - E_i) |T_{fi}|^2.$$

This is readily extended to positron scattering and to more complicated cases.

6 Coulomb Scattering

Coulomb scattering is a useful test case where the vector potential is given by

$$A(x) = \frac{-Ze}{4\pi |\boldsymbol{x}|} \gamma_0.$$

Taking the Fourier transform we have

$$A(q) = -\frac{2\pi Ze}{\boldsymbol{q}^2} \delta(E_f - E_i) \gamma_0$$

and $M = eA(q)$ in the first Born approximation. Writing

$$S_{fi} = -\hat{\jmath} 2\pi\delta(E_f - E_i)T_{fi}$$

and using energy conservation we have

$$T_{fi} = -\frac{Ze^2}{\boldsymbol{q}^2}(2E + \boldsymbol{q})$$

so that the formula for the cross section becomes

$$d\boldsymbol{\sigma} = \left(\frac{Ze^2}{\boldsymbol{q}^2}\right)^2 \frac{\pi}{|\boldsymbol{p}_i|}\delta(E_f - E_i)(4E^2 - \boldsymbol{q}^2)\frac{d^3\boldsymbol{p}_f}{2E_f(2\pi)^3}.$$

Using $d^3\boldsymbol{p}_f = |\boldsymbol{p}_f|E_f dE_f d\Omega_f$ we recover the Mott cross section

$$\left(\frac{d\sigma}{d\Omega_f}\right)_{\text{Mott}} = \frac{Z^2\alpha^2}{\boldsymbol{q}^4}(4E^2 - \boldsymbol{q}^2) = \frac{Z^2\alpha^2}{4\boldsymbol{p}^2\beta^2\sin^4(\theta/2)}\big(1 - \beta^2\sin^2(\theta/2)\big),$$

where $\boldsymbol{q}^2 = (\boldsymbol{p}_f - \boldsymbol{p}_i)^2 = 2\boldsymbol{p}^2(1 - \cos\theta)$ and $\beta = |\boldsymbol{p}|/E$. The derivation is manifestly independent of initial spin, so the cross section is spin independent. If we had instead used the conventional spin sums method this would have been far from clear.

The final and initial spins will be related by the rest spin scattering operator U_{fi}, where

$$U_{fi} \propto L_f L_i + \widetilde{L}_f \widetilde{L}_i \propto (E + m)^2 + \boldsymbol{p}_f\boldsymbol{p}_i.$$

If U_{fi} rotates by an angle δ in the $\hat{B}$ plane ($\hat{B}^2 = -1$) it is given by

$$U_{fi} = e^{\delta\hat{B}/2} = \cos(\delta/2) + \hat{B}\sin(\delta/2).$$

So we see that the rotation is in the $\boldsymbol{p}_f \wedge \boldsymbol{p}_i$ plane and by an angle δ given by

$$\tan(\delta/2) = \frac{|\langle U_{fi}\rangle_2|}{\langle U_{fi}\rangle} = \frac{|\boldsymbol{p}_f \wedge \boldsymbol{p}_i|}{(E + m)^2 + \boldsymbol{p}_f\cdot\boldsymbol{p}_i} = \frac{\sin\theta}{(E + m)/(E - m) + \cos\theta}.$$

7 Second Order Coulomb Scattering

Second order Coulomb scattering is interesting as it is spin-dependent, though the calculation is now rather more involved. To avoid problems with divergent integrals

the potential is replaced with the screened potential

$$A(x) = -\frac{e^{-\lambda|\boldsymbol{x}|}Ze}{4\pi|\boldsymbol{x}|}\gamma_0$$

and the Coulomb result found in the limit λ goes to zero (Dalitz, 1951; Itzykson & Zuber, 1980). For this potential the first order analysis above can be applied with M given by

$$eA(q) = -\frac{2\pi Ze^2}{\lambda^2 + \boldsymbol{q}^2}\delta(E_f - E_i)\gamma_0.$$

To iterate to second order (4) is used, with the substitution

$$\psi(x') = \psi_i e^{-\hat{\jmath}p_i\cdot x'} + e\int d^4x''\int \frac{d^4k}{(2\pi)^4}\frac{kA(x'') + A(x'')p_i}{k^2 - m^2 + \hat{\jmath}\epsilon}\psi_i e^{\hat{\jmath}x''\cdot(k-p_i)}e^{-\hat{\jmath}k\cdot x'}$$

giving the extra contribution to M

$$M' = e^2\int d^4x'\int d^4x''\int \frac{d^4k}{(2\pi)^4}A(x')\frac{kA(x'') + A(x'')p_i}{k^2 - m^2 + \hat{\jmath}\epsilon}e^{\hat{\jmath}x'\cdot(p_f-k)}e^{\hat{\jmath}x''\cdot(k-p_i)}.$$

Carrying out the x' and x'' integrations and using one of the resultant δ-functions we have

$$M = 2\pi\delta(E_f - E_i)M_T$$

where the extra contribution to M_T is

$$M_T' = e^2\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{a_0(p_f - k)a_0(k - p_i)}{k^2 - m^2 + \hat{\jmath}\epsilon}\gamma_0[k\gamma_0 + \gamma_0 p_i]$$

and

$$a_0(p) = \int d^3\boldsymbol{x}e^{-\boldsymbol{p}\cdot\boldsymbol{x}}\gamma_0\cdot A(\boldsymbol{x}) = \frac{-Ze}{\lambda^2 + \boldsymbol{p}^2}.$$

Using

$$k^2 - m^2 = \boldsymbol{p}_i^2 - \boldsymbol{k}^2$$

and the integrals

$$I_1 + \tfrac{1}{2}(\boldsymbol{p}_i + \boldsymbol{p}_f)I_2 \;=\; \int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1 + \boldsymbol{k}}{[(\boldsymbol{p}_f - \boldsymbol{k})^2 + \lambda^2][(\boldsymbol{p}_i - \boldsymbol{k})^2 + \lambda^2](\boldsymbol{p}_i^2 - \boldsymbol{k}^2 + \hat{\jmath}\epsilon)}$$

we have

$$M_T' = Z^2 e^4 \left[\gamma_0 \tfrac{1}{2}(\boldsymbol{p}_i + \boldsymbol{p}_f) I_2 + (p_i + \gamma_0 E) I_1\right].$$

In the limit $\lambda \to 0$ our total M_T to second order is therefore

$$M_T = \frac{-Ze^2}{\boldsymbol{q}^2}\gamma_0 + Z^2 e^4 \left[(E\gamma_0 - \tfrac{1}{2}[p_f + p_i]) I_2 + (p_i + \gamma_0 E) I_1\right]$$

where the integrals are (Itzykson & Zuber, 1980)

$$\begin{aligned} I_1 &= \frac{-\hat{\jmath}}{16\pi|\boldsymbol{p}|^3 \sin^2(\theta/2)} \ln \frac{2|\boldsymbol{p}| \sin(\theta/2)}{\lambda} \\ I_2 &= \frac{1}{16\pi|\boldsymbol{p}|^3 \cos^2(\theta/2)} \left\{ \frac{\pi[\sin(\theta/2) - 1]}{2\sin^2(\theta/2)} - \hat{\jmath} \ln \frac{\lambda}{2|\boldsymbol{p}|} \right\} + \frac{I_1}{\cos^2(\theta/2)}. \end{aligned}$$

We see that M has some $\hat{\jmath}$ dependence, so writing $I_1 = (A + C)\hat{\jmath}$ and $I_2 = B + C\hat{\jmath}$ where A, B, and C are scalars, and replacing the $\hat{\jmath}$-dependence with $\hat{S}_i$-dependence, this becomes

$$M_T = \gamma_0 \left[-\frac{Ze^2}{\boldsymbol{q}^2} + EZ^2 e^4 \left\{B + (2C + A)\hat{S}_i\right\}\right] + Z^2 e^4 \left[p_i(A\hat{S}_i - B) - \tfrac{1}{2} q (B + C\hat{S}_i)\right].$$

The term proportional to q cancels in the calculation of T_{fi}. Using the result that

$$p_f p_i + m^2 = E(2E + \boldsymbol{q}) - \boldsymbol{p}^2 - \boldsymbol{p}_f \boldsymbol{p}_i$$

we have

$$T_{fi} = (2E + \boldsymbol{q}) \left[-\frac{Ze^2}{\boldsymbol{q}^2} + 2EZ^2 e^4 (A + C)\hat{S}_i\right] + Z^2 e^4 (\boldsymbol{p}^2 + \boldsymbol{p}_f \boldsymbol{p}_i)(B - A\hat{S}_i).$$

Keeping terms up to α^3 the cross section is then governed by

$$|T_{fi}|^2 = (4E^2 - \boldsymbol{q}^2)\frac{Z^2 e^4}{\boldsymbol{q}^4} - \frac{4Z^3 e^6}{\boldsymbol{q}^2} \left[EB(\boldsymbol{p}^2 + \boldsymbol{p}_f \cdot \boldsymbol{p}_i) + mA(\boldsymbol{p}_i \wedge \boldsymbol{p}_f) \cdot \hat{S}_i^0\right]$$

where $\hat{S}_i^0$ is the initial rest spin. As expected the divergent parts of the integrals have cancelled out, and we are only left with the finite terms B and

$$A = \frac{\ln \sin(\theta/2)}{16\pi|\boldsymbol{p}|^3 \cos^2(\theta/2)}.$$

One can obtain the cross section for unpolarized scattering by averaging over the initial spin. This gives the spin-independent part of the cross section since the spin dependent part averages to zero.

8 Spin Dependence and Double Scattering

As an example of handling spin dependence we can work out the asymmetry parameter for double scattering from a Coulomb potential. The idea is that since the second order correction to Coulomb scattering is spin dependent the scattered beam will be partially polarized even with an unpolarized incident beam. The scattered beam can then impinge on a second target, which leads to an observable asymmetry in the scattered intensity. The asymmetry was first worked out by Mott (Mott, 1929; Mott, 1932).

The first thing we need to know is the spin after the first scattering. This is given by

$$\hat{S}_f = R_{fi}\hat{S}_i\widetilde{R}_{fi}$$

so we have

$$\hat{S}_f \quad \propto \quad T_{fi}\hat{S}_i\widetilde{T}_{fi} = \frac{Z^2e^4}{\boldsymbol{q}^4}(2E+\boldsymbol{q})\hat{S}_i(2E-\boldsymbol{q}) - \frac{2Z^3e^6A}{\boldsymbol{q}^2}\left\langle(\boldsymbol{p}^2+\boldsymbol{p}_f\boldsymbol{p}_i)(2E-\boldsymbol{q})\right\rangle_2$$

where we have only kept the lowest order terms in the spin dependent and spin-independent parts. The first term depends on the initial spin but the second term does not, so if we average over the initial spin the spin independent part will determine the final polarization. We define S^0 to be the polarization in the plane $\hat{S}^0$. This is a bivector in the plane of $\hat{S}^0$ with modulus equal to the polarization of the beam. Since the incoming beam is taken to be unpolarized the resultant polarization plane will be given by the spin-independent part of $\hat{S}_f$ deboosted to rest. To get the polarization we then just divide by the magnitude of the spin-dependent part:

$$\begin{aligned} S^0_f &= -\frac{2Ze^2\boldsymbol{q}^2A}{(4E^2-\boldsymbol{q}^2)}\widetilde{L}_f\left\langle(\boldsymbol{p}^2+\boldsymbol{p}_f\boldsymbol{p}_i)(2E-\boldsymbol{q})\right\rangle_2 L_f \\ &= \frac{2Ze^2\boldsymbol{q}^2A}{(4E^2-\boldsymbol{q}^2)}2m\boldsymbol{p}_i\wedge\boldsymbol{p}_f. \end{aligned}$$

So the beam after the first scattering is polarized in the scattering plane $\boldsymbol{p}_i\wedge\boldsymbol{p}_f$. The

spin-dependent part of the cross section for the second scattering is then given by

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega_f}\right)_{\text{spin}} &= -\frac{4Z^3e^6mA_2}{\boldsymbol{q}_2^2(2\pi)^2}(\boldsymbol{p}_f\wedge\boldsymbol{p}_2)\cdot S_f^0 \\ &= -\frac{64(2\pi)^2Z^4\alpha^4\boldsymbol{q}_1^2m^2A_1A_2}{\boldsymbol{q}_2^2(4E^2-\boldsymbol{q}_1^2)}(\boldsymbol{p}_f\wedge\boldsymbol{p}_2)\cdot(\boldsymbol{p}_i\wedge\boldsymbol{p}_f)\end{aligned}$$

where the 1 and 2 subscripts refer to the first are second scattering respectively (e.g. $\boldsymbol{q}_2=\boldsymbol{p}_2-\boldsymbol{p}_f$). We see that the asymmetry will depend on the cosine of the angle ϕ between the $\boldsymbol{p}_f\wedge\boldsymbol{p}_2$ and $\boldsymbol{p}_i\wedge\boldsymbol{p}_f$ planes. The asymmetry parameter δ is defined so that the final intensity depends on ϕ through the factor

$$1+\delta\cos\phi.$$

In the case where $\boldsymbol{p}_i\cdot\boldsymbol{p}_f=\boldsymbol{p}_f\cdot\boldsymbol{p}_2=0$ ($\boldsymbol{p}_i\cdot\boldsymbol{p}_2=-\boldsymbol{p}^2\cos\phi$) we find that the first non-zero contribution to the asymmetry factor is

$$\begin{aligned}\delta &= \frac{64(2\pi)^2Z^4\alpha^4m^2A^2}{(4E^2-\boldsymbol{q}^2)}\boldsymbol{p}^4\frac{\boldsymbol{q}^4}{Z^2\alpha^2(4E^2-\boldsymbol{q}^2)} \\ &= Z^2\alpha^2(\ln 2)^2\frac{\beta^2(1-\beta^2)}{(2-\beta^2)^2}\end{aligned}$$

in agreement with the answer quoted by (Dalitz, 1951). It is of course only the first approximation, and for large Z nuclei higher order corrections will be far from negligible.

9 Partial Spin Sums in QED

Much of the simplicity and elegance of the above method comes from the fact we were considering a single electron. As a more complicated example consider electron-muon scattering. For each of the fermion lines one has a scattering operator with an M of the form

$$M=eD_F\gamma^aJ_a$$

where D_F is the photon propagator and J_a is given by

$$J_a = e\langle \widetilde{u}_s \gamma_a \psi_2 \gamma_0 \rangle_S$$

Here $\langle \ldots \rangle_S$ denotes the scalar and Σ projection, and provides the usual complex structure, u_s is the normalized final state of the other particle, and ψ_2 is the incoming state of the other particle. Now we can proceed to calculate cross sections as before if we sum over the final spin of the other particle to get the result

$$|T_{fi}|^2 = \frac{2e^4\rho_1\rho_2}{m_1 m_2 q^4}\Big[p_1'\cdot p_2' p_1\cdot p_2 + p_2'\cdot p_1 p_2\cdot p_1' - m_1^2 p_2'\cdot p_2 \\ + m_2^2 p_1'\cdot p_1 + 2m_2^2 m_1^2 - [q\cdot(\hat{S}_1\wedge p_1)]\cdot[q\cdot(\hat{S}_2\wedge p_2)]\Big].$$

One could also calculate final polarizations and spins in the same way as before.

Whilst the scattering operator approach offers little advantage if one is just interested in unpolarized cross sections, it may still useful when one wants to consider spin dependent results. If you are just interested in the spin dependence of a particular fermion line the scattering operator approach works well once you have summed over the spins of the other particles. So in this approach one still has to perform a spin sum, but only over the spins of the other particles. One can of course introduce spin projection operators to single out particular spins of the other particles in the usual way.

10 Conclusions

We have seen how Hestenes' STA formulation of Dirac theory provides a useful and elegant method of performing cross section calculations. Spin is handled in a simple manner, and the logic of calculating cross sections is simplified considerably. We don't perform unnecessary spin sums and spin dependence is manifest in the spin bivector dependence of the scattering operator. It's a simple matter to calculate spin precessions, polarizations and spin dependent results, and the results are

automatically expressed in terms of physical spin bivectors and the other scattering parameters. We can perform unpolarized calculations simply by averaging over spins.

In the multiparticle case things are more complicated. We do not yet have a neat method for performing arbitrary spin dependent calculations, and still have to resort to spin sums over terms involving complex conserved currents. However we can still write down a scattering operator for any given fermion line, retaining the benefits of the scattering operator for calculations involving the spin of the particle.

For clarity we have only considered electron scattering, but all our results are easily extended to positron scattering and electron-positron annihilation (Lewis, 1999).

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