Surface Evolution and Representation using Geometric Algebra

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Summary. Recent developments in geometric algebra have shown that by moving from a projective to a conformal representation (5d representation of 3d space), one is able to extend the range of geometrical operations that can be carried out in an efficient and elegant way. For example, while in projective space one is able to intersect lines and planes in a simple fashion, in conformal space one is able to to intersect and represent spheres, lines, circles and planes. In addition, all the operations of Euclidean geometry (dilations, translations, rotations and inversions) are smoothly integrated with the projective representation.

The paper will use the conformal representation to look at the problems of surface representation and evolution, and of wavefront propagation from such surfaces.

1 Introduction

The mathematical language we will use throughout will be that of geometric algebra (GA). This language is based on the algebras of Clifford and Grassmann and the form we follow here is that developed by David Hestenes [1]. There are now many texts and useful introductions to GA, [2–5] so we do no more here than outline some aspects used in the problems we will discuss.

In a geometric algebra of n-dimensions, we have the standard *inner* product which takes two vectors and produces a scalar, plus an outer or *wedge* product that takes two vectors and produces a new quantity we call a *bivector* or oriented area. Similarly, the outer product between three vectors produces a *trivector* or oriented volume etc. Thus the algebra has basic elements which are oriented geometric objects of different orders. The highest order object in a given space is called the *pseudoscalar* with the unit pseudoscalar denoted by I, e.g. in 3d I is the unit trivector $e_1{\wedge}e_2{\wedge}e_3$ for basis vectors $\{e_i\}$. Multivectors are quantities which are made up of linear combinations of these different geometric objects. More fundamental than the inner or wedge products is the *geometric product* which can be defined between any multivectors – the geometric product, unlike the inner or outer products, is invertible. For vectors the inner and outer products are the symmetric and antisymmetric parts of the geometric product;

$$ab = a{\cdot}b + a{\wedge}b \tag{1}$$

In effect the manipulations within geometric algebra are keeping track of the objects of different grades that we are dealing with (much as complex number arithmetic does). For a general multivector M, we will use the notation $\langle M\rangle_r$ to denote the rth grade part of M.
In what follows we shall use the convention that vectors will be represented by non-bold lower case roman letters, while we use non-bold, upper case roman letters for 5d vectors and certain multivectors — exceptions to this are stated in the text. Unless otherwise stated, repeated indices will be summed over.

1.1 Rotations

If, in 3d, we consider a rotation to be made up of two consectutive reflections, one in the plane perpendicular to a unit vector m and the next in the plane perpendicular to a unit vector n, it can easily be shown [4] that we can represent this rotation by a quantity R we call a **rotor** which is given by

$$R = nm$$

Thus a rotor in 3D is made up of a scalar plus a bivector and can be written in one of the following forms

$$R = e^{-B/2} = \exp\left(-I\frac{\theta}{2}n\right) = \cos\frac{\theta}{2} - In\sin\frac{\theta}{2}, \tag{2}$$

which represents a rotation of θ radians about an axis parallel to the unit vector n in a right-handed screw sense. Here the bivector B represents the plane of rotation. Rotors act two-sidedly, ie. if the rotor R takes the vector a to the vector b then

$$b = Ra\tilde{R}$$

where $\tilde{R} = mn$ is the reversion of R (i.e the order of multiplication of vectors in any part of the multivector is reversed). We have that rotors must therefore satisfy the constraint that $R\tilde{R} = 1$. One huge advantage of this formulation is that rotors take the same form, i.e. $R = \pm\exp(B)$ in any dimension (we can define hyperplanes or bivectors in any space) and can rotate any objects, not just vectors; e.g.

$$\begin{aligned} R(a\wedge b)\tilde{R} &= \langle Rab\tilde{R}\rangle_2 = \langle Ra\tilde{R}Rb\tilde{R}\rangle_2 \\ &= Ra\tilde{R}\wedge Rb\tilde{R} \end{aligned} \tag{3}$$

gives the formula for rotating a bivector.

2 Conformal Geometry in Geometric Algebra

It has long been known that going to a 4d, projective, description of 3d Euclidean space can have various advantages – particularly when intersections of planes and lines are required. Such projective descriptions are used extensively in computer vision and computer graphics where rotations and translations can be described by a single 4×4 matrix and non-linear projective transformations become linear. Projective geometry fits very nicely into the geometric algebra framework and applications are given in [6,7]. In [1] conformal geometry was briefly discussed and recently, [8,9], the application of these ideas in which a 5d conformal space is used as the representation of 3d Euclidean space has been the subject of renewed interest. In this conformal space we have, as a subset, the projective geometry, but also the ability to extend to circles and spheres. Below we describe the basics of this conformal representation and outline how it can be of use in specific problems, e.g. reflecting a general wavefront from a spherical surface.

We start with the simplest formulation of Hestenes' conformal geometry work in 3d. However, unlike the treatments given in [1,8,9], we will use *reflection* as the key to *inversion* which will enable us to treat circles and spheres very easily and efficiently.

The notation we use will follow the original notation given in [1]. Let x be a vector in a space $\mathcal{A}(p,q)$, where the *signature* (p,q) implies that the space has a basis $\{e_i\}$, $i = 1,\ldots,n = p+q$ where $e_i^2 = +1$ for $i = 1,\ldots,p$ and $e_i^2 = -1$ for $i = p+q+1,\ldots,n$ – i.e. we take a general mixed signature space. Now extend this to a space $\mathcal{A}(p+1,q+1)$ via the inclusion of two additional basis vectors, e and $\bar{e}$, such that

$$e^2 = +1, \quad \bar{e}^2 = -1, \quad e\cdot\bar{e} = 0$$

Note that if $x \in \mathcal{A}(p,q)$, then $e\cdot x = \bar{e}\cdot x = 0$ since $e_i\cdot e = e_i\cdot\bar{e} = 0$ for $i = 1,\ldots,n$. We now introduce the vectors n and $\bar{n}$ where

$$n = e + \bar{e} \qquad \bar{n} = e - \bar{e} \tag{4}$$

$$\implies e = \frac{1}{2}(n+\bar{n}) \qquad \bar{e} = \frac{1}{2}(n-\bar{n}) \tag{5}$$

n and $\bar{n}$ are **null vectors** since

$$n^2 = (e+\bar{e})\cdot(e+\bar{e}) = e^2 + 2e\cdot\bar{e} + \bar{e}^2 = 1+0-1 = 0$$
$$\bar{n}^2 = (e-\bar{e})\cdot(e-\bar{e}) = e^2 - 2e\cdot\bar{e} + \bar{e}^2 = 1-0-1 = 0$$

Note also that

$$n\cdot\bar{n} = (e+\bar{e})\cdot(e-\bar{e}) = e^2 - \bar{e}^2 = 2$$
$$x\cdot n = 0 \quad \text{and} \quad x\cdot\bar{n} = 0$$

for $x \in \mathcal{A}(p,q)$. We now map a point x in $\mathcal{A}(p,q)$ to a point $F(x)$ in $\mathcal{A}(p+1,q+1)$ via the Hestenes ([1], p.302) representation

$$F(x) = -(x-e)n(x-e) \tag{6}$$

Substituting for $n = e+\bar{e}$ and using the fact that $\bar{e}\cdot x = 0 = \bar{e}\cdot e = n\cdot x$, it is not hard to rewrite this equation in terms of the null vectors as follows:

$$F(x) = x^2 n + 2x - \bar{n} \tag{7}$$

which is precisely the form which is used in the more recent 'horosphere' formulations of the conformal framework [8,9].

Now for any $x_i \in \mathcal{A}(p,q)$ we evaluate $[F(x)]^2$

$$\begin{aligned}[F(x)]^2 &= (x^2 n + 2x - \bar{n})\cdot(x^2 n + 2x - \bar{n}) \\ &= -x^2 n\cdot\bar{n} + 4x^2 = -4x^2 + 4x^2 = 0\end{aligned} \tag{8}$$

Thus we have mapped vectors in $\mathcal{A}(p,q)$ into **null vectors** in $\mathcal{A}(p+1,q+1)$ – this is precisely the horosphere construction.

More generally, it can be shown that any *null* vector in $\mathcal{A}(p+1,q+1)$ can be written as

$$X = \lambda(x^2 n + 2x - \bar{n}) \tag{9}$$

with λ a scalar. We can now use this to provide a projective mapping between $\mathcal{A}(p,q)$ and $\mathcal{A}(p+1,q+1)$: the family of null vectors, $\lambda(x^2 n + 2x - \bar{n})$, in $\mathcal{A}(p+1,q+1)$ are taken to correspond to the single point $x \in \mathcal{A}(p,q)$.

At this point it is interesting to see what happens when we take the inner product of any two such null vectors:

$$A\cdot B = \{a^2 n + 2a - \bar{n}\}\cdot\{b^2 n + 2b - \bar{n}\} = -2a^2 - 2b^2 + 4a\cdot b = -2(a-b)^2 \tag{10}$$

We see therefore that taking the inner product of two 5d representative vectors gives a scalar which is proportional to the distance between the 3d vectors. This is where the formulation can be related to the study of *distance geometry* [10].

We will be especially interested in **Conformal Transformations** in $\mathcal{A}(p,q)$, and we shall see later that these are represented by *rotors* and *reflections* in $\mathcal{A}(p+1,q+1)$. We now look at the operations of rotation and reflection more closely.

1. **Rotations**

   If $x \mapsto Rx\tilde{R}$ with $x \in \mathcal{A}(p,q)$ and R a rotor in $\mathcal{A}(p,q)$, then what happens when R acts on $F(x)$?

   $$RF(x)\tilde{R} = R(x^2 n + 2x - \bar{n})\tilde{R} = x^2 Rn\tilde{R} + 2Rx\tilde{R} - R\bar{n}\tilde{R}$$

   Since R is a rotor, it contains only even blades and therefore commutes with n and $\bar{n}$ ($e_i n = -ne_i$, so if we have an even number of es we have commutation), so that $Rn\tilde{R} = R\tilde{R}n = n$ and $R\bar{n}\tilde{R} = \bar{n}$. Thus we have

   $$RF(x)\tilde{R} = {x'}^2 n + 2x' - \bar{n} \tag{11}$$

   where $x' = Rx\tilde{R}$. That is, rotors in $\mathcal{A}(p,q)$ remain rotors in $\mathcal{A}(p+1,q+1)$, i.e.

   $$x \mapsto Rx\tilde{R} \qquad \Longrightarrow \qquad F(x) \mapsto F(Rx\tilde{R}) \tag{12}$$

2. **Inversions**

Here we have $x \mapsto \frac{x}{x^2}$ or equivalently, $x \mapsto x^{-1}$ (since $x^{-1} = \frac{x}{x^2}$). Firstly, we look at the properties of relection in e:

$$-ene = -ee\bar{n} = -\bar{n}$$

using the fact that $ne = \frac{1}{2}(e+\bar{e})e = \frac{1}{2}(e^2 + \bar{e}e) = \frac{1}{2}(e^2 - e\bar{e}) = e\bar{n}$. Similarly, we can show that the following reflection properties hold: $-ene = -\bar{n}$, $-e\bar{n}e = -n$ and $-exe = x$. Now we look at what happens to $F(x)$ under reflection in e

$$\begin{aligned} -eF(x)e &= -e(x^2n + 2x - \bar{n})e = -x^2\bar{n} + 2x + n \\ &= x^2\left[\frac{1}{x^2}n + 2\frac{x}{x^2} - \bar{n}\right] = x^2F(\frac{x}{x^2}) \end{aligned} \tag{13}$$

Therefore, we have that inversion in $\mathcal{A}(p,q)$ is brought about by reflection in e in $\mathcal{A}(p+1,q+1)$, i.e.

$$x \mapsto \frac{x}{x^2} \qquad \Longrightarrow \qquad F(x) \mapsto -eF(x)e = x^2F(\frac{x}{x^2}) \tag{14}$$

Note here that it is irrelevant whether we take $-e(...)e$ or $e(...)e$ as the reflection – henceforth we will use $e(...)e$ for convenience.

3. **Translations**

Here we wish to achieve a translation $x \mapsto x + a$; we will show that this is performed by a rotor $R = T_a = e^{\frac{na}{2}}$, where $a \in \mathcal{A}(p,q)$;

$$R = T_a = e^{\frac{na}{2}} = 1 + \frac{na}{2} + \frac{1}{2}\left(\frac{na}{2}\right)^2 + \ldots = 1 + \frac{na}{2} \tag{15}$$

since n is null and $an = -na$. Firstly we see how R acts on n, $\bar{n}$ and x.

$$\begin{aligned} Rn\tilde{R} &= \left(1 + \frac{na}{2}\right) n \left(1 + \frac{an}{2}\right) \\ &= n + \frac{1}{2}nan + \frac{1}{2}nan + \frac{1}{4}nanan = n \end{aligned} \tag{16}$$

again using $an = -na$ and $n^2 = 0$. Similarly we can show that

$$R\bar{n}\tilde{R} = \bar{n} - 2a - a^2n \tag{17}$$

$$Rx\tilde{R} = x + n(a{\cdot}x) \tag{18}$$

We can now see how the rotor acts on $F(x)$

$$\begin{aligned} RF(x)\tilde{R} &= \left(1 + \frac{na}{2}\right)(x^2n + 2x - \bar{n})\left(1 + \frac{an}{2}\right) \\ &= x^2n + 2(x + n(a{\cdot}x)) - (\bar{n} - 2a - a^2n) \\ &= (x+a)^2n + 2(x+a) - \bar{n} \\ &= {x'}^2n + 2x' - \bar{n} = F(x+a) \end{aligned} \tag{19}$$

where $x' = x + a$. Thus, translations in $\mathcal{A}(p,q)$ can be performed by the $\mathcal{A}(p+1,q+1)$ rotor $R = T_a$, $a \in \mathcal{A}(p,q)$ so that

$$x \mapsto x + a \qquad \Longrightarrow \qquad F(x) \mapsto F(x+a) \tag{20}$$

4. **Dilations**

To investigate how dilations are formed we start by considering the rotor $R = D_\alpha = e^{\frac{\alpha}{2}e\bar{e}}$ and the following relations which can easily be verified:

$$e\bar{e}n = -\bar{n} = -ne\bar{e} \quad \text{and} \quad e\bar{e}\bar{n} = \bar{n} = -\bar{n}e\bar{e} \tag{21}$$

Using these relations it is straightforward to show that $RF(x)\tilde{R}$ gives

$$\begin{aligned} D_\alpha F(x)\tilde{D}_\alpha &= e^{\frac{\alpha}{2}e\bar{e}}\{x^2n + 2x - \bar{n}\}e^{-\frac{\alpha}{2}e\bar{e}} \\ &= e^\alpha\left\{x'^2 n + 2x' - \bar{n}\right\} \end{aligned} \tag{22}$$

where $x' = e^{-\alpha}x$. The above can be verified by expanding $e^{-\frac{\alpha}{2}e\bar{e}}$ as $1 - \frac{\alpha}{2}e\bar{e} + \frac{1}{2!}\left(\frac{\alpha}{2}e\bar{e}\right)^2 + \ldots$ etc. and using the relations given in equation (21). Thus, dilations in $\mathcal{A}(p,q)$ can be performed by the $\mathcal{A}(p+1,q+1)$ rotor $R = D_\alpha$, so that

$$x \mapsto e^{-\alpha}x \qquad \Longrightarrow \qquad F(x) \mapsto e^\alpha F(e^{-\alpha}x) \tag{23}$$

2.1 Special Conformal Transformations

We have seen above that we are able to express rotations, inversions, translations and dilations in $\mathcal{A}(p,q)$ by rotations and reflections in $\mathcal{A}(p+1,q+1)$. This now leads us to consider *special conformal transformations* of the form

$$x \mapsto x\frac{1}{1+ax} \tag{24}$$

which is actually a combination of inversion, translation and inversion again:

$$\begin{aligned} x \underset{\text{inversion}}{\longmapsto} & \frac{x}{x^2} \\ \underset{\text{translation}}{\longmapsto} & \frac{x}{x^2} + a \\ \underset{\text{inversion}}{\longmapsto} & \frac{\frac{x}{x^2}+a}{(\frac{x}{x^2}+a)(\frac{x}{x^2}+a)} \\ = \quad & \frac{x + ax^2}{1 + 2a\cdot x + a^2x^2} = x\frac{1}{1+ax} \end{aligned} \tag{25}$$

The final line in the above expression shows us that $x\frac{1}{1+ax}$ is indeed a vector. As we have built up the special conformal transformation via inversions and translations, we know exactly how to construct the $\mathcal{A}(p+1,q+1)$ operator that performs such a transformation – the required rotor is

$$K_a = eT_ae = 1 - \frac{\bar{n}a}{2}, \qquad \text{so that} \qquad x \mapsto K_a x \tilde{K}_a \tag{26}$$

and

$$K_a x \tilde{K}_a = e\left\{T_a(exe)\tilde{T}_a\right\}e \tag{27}$$

Now, when we act on $F(x)$ with K_a we can use our previous results to obtain

$$K_a F(x) \tilde{K}_a = (1 + 2a{\cdot}x + a^2x^2)F\left(x\frac{1}{1+ax}\right) \tag{28}$$

which therefore tells us that

$$x \mapsto x\frac{1}{1+ax} \qquad \Longrightarrow \qquad F(x) \mapsto (1+2a{\cdot}x + a^2x^2)F\left(x\frac{1}{1+ax}\right) \tag{29}$$

2.2 Projective geometry in the conformal space

In this section we first consider the part $\lambda(2x-\bar{n})$ of the conformal representation of a point x and show it enables us to deal with projective geometry and the incidence of planes, lines etc. Indeed it is very similar to forming $\lambda(x+\gamma_o)$ (x in Euclidean 3-space) – in the conformal representation the signature of the bit we add on is irrelevant for projective geometry. Indeed it is generally better, if dealing only with projective geometry, that we do not have null structures present, which implies adding a 4th basis vector which gives a $\mathcal{A}(4,0)$ space. So, we need to ask ourselves if there is any advantage in going to a $\lambda(x^2n + 2x - \bar{n})$ representation.

The key fact here is that by enlarging the representation and employing the reflection formula to do inversions, we can now study the incidence relations of spheres, circles, lines and planes and not just lines and planes. A second, linked, advantage, is that in the conformal representation we can represent incidence relations via wedge products just as we can in the GA version of projective geometry. For example, in projective geometry, if a line, L, passes through two points a, b, whose (4d) homogeneous representations are A, B, we can write the line as a bivector, $L = A{\wedge}B$. Then, any X lying on the line L will satisfy

$$X{\wedge}L = 0$$

In the conformal representation, rotations, translations, dilations, inversions are all represented by rotors or reflections, which tells us that any incidence relations remain invariant in form under such operations – we can see this explicitly as follows.

Suppose we have the incidence relation

$$X \wedge Y \wedge \ldots \wedge Z = 0$$

where $X, Y, ..., Z \in \mathcal{A}(p+1, q+1)$. Then under reflections in e we have

$$X \wedge Y \wedge \ldots \wedge Z \mapsto (eXe) \wedge (eYe) \wedge \ldots \wedge (eZe) = e(X \wedge Y \wedge \ldots \wedge Z)e \tag{30}$$

where we have used the fact that $(eXe) \wedge (eYe) = \frac{1}{2}(eXeeYe - eYeeXe) = \frac{1}{2}e(XY - YX)e = e(X \wedge Y)e$, since $e^2 = 1$. Thus, if $X \wedge Y \wedge .... \wedge Z = 0$ then so too does $(eXe) \wedge (eYe) \wedge .... \wedge (eZe)$.

Similarly, under rotations we have

$$X \wedge Y \wedge \ldots \wedge Z \mapsto (RX\tilde{R}) \wedge (RY\tilde{R}) \wedge \ldots \wedge (RZ\tilde{R}) = R(X \wedge Y \wedge \ldots \wedge Z)\tilde{R} \tag{31}$$

where again we use $(RX\tilde{R}) \wedge (RY\tilde{R}) = \frac{1}{2}(RX\tilde{R}RY\tilde{R} - RY\tilde{R}RX\tilde{R}) = \frac{1}{2}R(XY - YX)\tilde{R} = R(X \wedge Y)\tilde{R}$, since $R\tilde{R} = 1$. Thus, if $X \wedge Y \wedge .... \wedge Z = 0$ then so too does $(RX\tilde{R}) \wedge (RY\tilde{R}) \wedge .... \wedge (RZ\tilde{R})$.

We see therefore that translations, rotations, dilations and inversions can now be brought into the context of projective geometry – a significant increase in the usefulness of the projective representation. It is now possible to build up a set of useful results in this projective conformal representation.

2.3 The equation of a line

Because the incidence relations are invariant under rotations and translations in the $\mathcal{A}(p, q)$ space, wlog we can consider a line in the direction e_1 passing through the origin.

Let three points on this line be x_1, x_2, x_3 with $\mathcal{A}(p+1, q+1)$ representations X_1, X_2, X_3. Now, the $\{X_i\}$ contain only the vectors $n, \bar{n}$ and e_1 (since $x_i = \lambda_i e_1$). Thus, if X is the representation of any other point on the line we have

$$X \wedge X_1 \wedge X_2 \wedge X_3 = 0 \tag{32}$$

This is because each of the above 5d vectors contains only the three vectors $n, \bar{n}$ and e_1 and therefore the wedge of four of them must be zero since each term will involve a wedge of two identical vectors. By invariance of the incidence relations under rotations and translations, we see that (32) is the equation of a line for *any* three general points X_1, X_2 and X_3 on the line. It is interesting to see how this parallels the projective case and also to note that we appear to need 3 points in this conformal representation to describe a line – we will return to this later.

2.4 The equation of a plane

Exactly the same sort of thing goes through here. By translational and rotational invariance we can, wlog, take the plane as that spanned by e_1 and e_2 and passing through the origin. If x lies in this plane then we can write

$$x = \lambda e_1 + \mu e_2$$

and its conformal representation, X, therefore only contains the vectors $n, \bar{n}, e_1, e_2$, i.e.

$$X = x^2 n + 2(\lambda e_1 + \mu e_2) - \bar{n}$$

Take $\Phi = X_1 \wedge X_2 \wedge X_3 \wedge X_4$, where $X_i,\ \ i = 1, ..., 4$ lie in the plane; following the same reasoning as given for the line, we see that for any X on Φ we must have $X \wedge \Phi = 0$, therefore

$$X \wedge X_1 \wedge X_2 \wedge X_3 \wedge X_4 = 0 \tag{33}$$

is the equation of the plane passing through points X_i, $i = 1, ..., 4$. Once again we note here that in the conformal representation we appear to require 4 points rather than 3 to specify the plane.

Extending this to higher dimensions we see that to specify an r-d hyperplane (where a line is $r = 1$, a plane is $r = 2$ etc) the equation is

$$X \wedge X_1 \wedge X_2 \wedge \ldots X_{r+1} \wedge X_{r+2} = 0 \tag{34}$$

where X_i, $i = 1, ..., r+2$ are conformal representations of the $r+2$ points x_i lying in the hyperplane.

2.5 The role of inversion

It may be thought strange that we need to specify $r+2$ points in order to determine an r-d hyperplane. For example, 2 points clearly suffice to determine a line, 3 points for a plane etc. So what is the role of the extra points?

We can best understand this, and the role inversion plays, by considering a simple example. Let the $\mathcal{A}(p, q)$ space be $\mathcal{A}(2, 0)$, i.e. the ordinary Euclidean plane with basis (e_1, e_2), $e_1^2 = 1,\ \ e_2^2 = 1$.

Let the line L be $x = 1$ i.e. $(1, y) : -\infty \leq y \leq +\infty$ and let $a = (x, y)$. Suppose we want to invert points on this line – we then obtain the set of points

$$a \mapsto \frac{a}{a^2} \qquad \Longrightarrow \qquad L \mapsto \left(\frac{1}{1+y^2}, \frac{y}{1+y^2}\right) \tag{35}$$

Parameterizing the original line as $x = 1, y = t;\ \ -\infty \leq t \leq +\infty$, the inversion produces $(x', y') = \left(\frac{1}{1+t^2}, \frac{t}{1+t^2}\right)$ – it is then easy to show that

$$(x' - \frac{1}{2})^2 + y'^2 = \left(\frac{1}{2}\right)^2$$

Hence the inversion produces a circle, centre $(\frac{1}{2}, 0)$ radius $\frac{1}{2}$.

$$\text{straight line} \quad \underset{\text{inversion}}{\longmapsto} \quad \text{circle}$$

Any three points on this line, X_i, $i = 1, 2, 3$ (conformal representation) therefore invert to give three points, X_i', $i = 1, 2, 3$ on this circle. Let the general point on the line be X; we know that

$$X \wedge X_1 \wedge X_2 \wedge X_3 = 0$$

Thus, if X' is a general point on the circle,, we know that

$$X' \wedge X_1' \wedge X_2' \wedge X_3' = 0$$

Recall that we see this by performing an inversion via reflection in e; i.e.

$$e(X \wedge X_1 \wedge X_2 \wedge X_3)e = eXe \wedge eX_1e \wedge eX_2e \wedge eX_3e$$

This gives a very useful form for the equation of a circle. Here, we derived it for a special case but since we know that we can dilate and translate as we wish, it must in fact be true for a completely general circle. Thus if X_i, $i = 1, 2, 3$ are *any* three points, the equation of the circle passing through these points is

$$X \wedge X_1 \wedge X_2 \wedge X_3 = 0 \tag{36}$$

If we now invert this equation via $e(...)e$ we will in general obtain another circle since $X_i' = eX_ie$ will be another three general points in the plane. This only fails if X_1', X_2', X_3' are collinear and this will occur if the original circle passes through the origin (as in the case we started with here). Recall that if $x = 0$, $X \equiv x^2 n + 2x - \bar{n} = -\bar{n}$, so that $\bar{n}$ represents the origin – by inversion, $e\bar{n}e = n$, we see therefore that we can associate n with the point at infinity – the inversion of the origin. What happens for the original circle passing through the origin is that the representation of the origin, any multiple of $\bar{n}$, is transformed by inversion to a multiple of n, the point at infinity. Thus, the equation of a line can always be written as

$$X \wedge n \wedge X_1 \wedge X_2 = 0 \tag{37}$$

where X_1 and X_2 are any two (finite) points on the line – we can see this by choosing the origin as one of the 3 points on the circle before inverting. This therefore explains the extra point we appeared to need in describing a line earlier – what is really going on is that

$$X \wedge X_1 \wedge X_2 \wedge X_3 = 0$$

describes a *circle* and therefore genuinely requires 3 points – while a line is just a special case of a circle which passes through the point at infinity.

2.6 Extension to higher dimensions

All of the previous section transfers immediately to higher dimensions and different signatures – although for indefinite metrics, hyper-hyperboloids have to be considered as well as hyperspheres. Here we just illustrate the extension from $\mathcal{A}(2,0)$ to $\mathcal{A}(3,0)$, i.e. Euclidean 2-space to Euclidean 3-space.

The special case we start from this time – from which everything else can be derived – is the $x=1$ plane, i.e. the set of points $(1,y,z): \quad -\infty \leq y \leq +\infty, \ -\infty \leq z \leq +\infty$. Inverting this plane we have

$$(1,y,z) \mapsto (x',y',z') = \left(\frac{1}{1+y^2+z^2}, \frac{y}{1+y^2+z^2}, \frac{z}{1+y^2+z^2}\right)$$

It is then not difficult to show that x',y',z' satisfy the following equation

$$\left(x'-\frac{1}{2}\right)^2 + y'^2 + z'^2 = \left(\frac{1}{2}\right)^2$$

which is the equation of a sphere, radius $\frac{1}{2}$ and centre $(\frac{1}{2},0,0)$. We already know that the equation of a plane is

$$X\wedge X_1\wedge X_2\wedge X_3\wedge X_4 = 0 \tag{38}$$

where X_i, $i=1,..,4$ are any 4 points on the plane. By inversion, translation, dilation and rotation we can now see that the equation of a sphere is given by the same equation

$$X\wedge X_1\wedge X_2\wedge X_3\wedge X_4 = 0$$

for X_i, $i=1,..,4$ any 4 points on the sphere. The arguments are precisely as before – since we can always translate and rotate our plane to the plane $x=1$ and we have shown that under inversion the plane $x=1$ gives a sphere, then we know that we must have the general equation for a sphere. Thus, as expected, it really does take 4 points to describe a sphere.

We shall now show how a plane is a special case of a sphere. Its inverse is a sphere passing through the origin – again, we see this using the arguments we have used previously. Equation (38) is the equation of a sphere passing through the 4 points X_i, $i=1,..,4$. We now invert this to give

$$X'\wedge X'_1\wedge X'_2\wedge X'_3\wedge X'_4 = 0$$

where $X'_i = eX_ie$. The X'_i are generally another set of general points, so we get another *sphere* through these new points. However, if the X'_i are coplanar then the above construction will not give a sphere – this occurs if the original sphere passes through the origin. In this case the equation of the original sphere can be written as

$$X\wedge n\wedge X_2\wedge X_3\wedge X_4 = 0$$

so that when we transform to give a *plane* we will get (since $ene = \bar{n}$)

$$X \wedge \bar{n} \wedge X_2' \wedge X_3' \wedge X_4' = 0 \tag{39}$$

That is, any plane passing through the points X_2', X_3', X_4' is given by equation (39). So indeed we only need 3 points to describe a plane – we can think of a plane as a sphere passing through the point at infinity.

Finally we consider a plane passing through the origin – we know that we can write this as

$$X \wedge n \wedge \bar{n} \wedge X_1 \wedge X_2 = 0$$

with X_1, X_2 lying on the plane. Now invert this to get

$$X \wedge \bar{n} \wedge n \wedge X_1' \wedge X_2' = 0$$

since $ene = \bar{n}$, $e\bar{n}e = n$. Under inversion we know that $x_1 \mapsto \frac{x_1}{x_1^2}$ and similarly for x_2; we therefore see that the plane is mapped onto itself under this inversion operation. Thus a plane passing through the origin is its own inverse since the null vectors $n, \bar{n}$ are just swapped under inversion. These results are all well known using a conventional approach of course, [11]. The novelty here is to show how easy they are to derive in the conformal approach using the key idea of reflection to perform inversion.

3 Intersections of Surfaces

In problems in computer graphics, robotics and inverse kinematics, large parts of the tasks involve intersecting lines, planes, circles, spheres, and indeed more general surfaces. In this section we will begin to put the formalism described so far to work in particular problems involving such intersections and hope to show that it provides a very elegant framework for carrying out these tasks.

Before looking at particular examples we will look briefly at representations of linear combinations of points. In order to take full advantage of the projective representation we should be able to consider linear combinations of points in the $\mathcal{A}(p,q)$ space. We know that a linear combination of two points a, b of the form

$$\lambda a + \mu b \quad \text{where} \quad \lambda + \mu = 1$$

gives another point on the line joining a and b. Similarly, a linear combination of 3 points, a, b, c of the form

$$\lambda a + \mu b + \nu c \quad \text{where} \quad \lambda + \mu + \nu = 1$$

gives another point on the plane containing a, b and c. The usual projective representation, where we go up just one dimension, has the advantage of still being linear in the representatives of the points e.g. if $x = \lambda a + \mu b + \nu c$ $(\lambda + \mu + \nu = 1)$ then its 4d projective representation, X, can also be written in the form

$$X = \lambda A + \mu B + \nu C$$

Here we note that we have insisted that the point X is 'normalised', i.e. that $X = x + e$ rather than some multiple of this. With the conformal representation, working in $\mathcal{A}(p+1, q+1)$, we appear to have lost this advantage of linearity. For example, if A and B are the $\mathcal{A}(p+1, q+1)$ representatives of a and b, then in general

$$\lambda A + (1-\lambda)B \neq \text{ a multiple of } F(\lambda a + (1-\lambda)b)$$

This is due to the presence of the $x^2 n$ term in the representation, which removes linearity. However, the following **is** true and is easy to show from the definition of $F(x)$:

$$F(\lambda a + (1-\lambda)b) = \lambda A + (1-\lambda)B + \frac{1}{2}\lambda(1-\lambda)A\cdot Bn \tag{40}$$

We therefore see that the departure from linear behaviour is given by the addition of a multiple of the point at infinity. This is relatively benign behaviour and means that many of the techniques we use in the GA version of projective geometry will still work here. For example, this gives us another way of seeing that the equation for a line passing through points a and b is $X\wedge n\wedge A\wedge B = 0$ – the wedging with n knocks out the non-linear term $\frac{1}{2}\lambda(1-\lambda)A\cdot Bn$ and we are left with the usual GA projective geometry result.

Precisely the same sort of thing goes through for a plane: let a, b, c define a plane and let

$$x = \alpha a + \beta b + \gamma c \quad \text{where} \quad \alpha + \beta + \gamma = 1$$

be a general point on the plane. Then it is easy to show that the representative of x, $X = F(x)$ satisfies

$$X = \alpha A + \beta B + \gamma C + \delta n \quad \text{where} \quad \delta = \frac{1}{2}(\alpha\beta A\cdot B + \alpha\gamma A\cdot C + \beta\gamma B\cdot C) \tag{41}$$

– again making it clear why the equation of the plane can be written as

$$X\wedge n\wedge A\wedge B\wedge C = 0$$

Note that in this section, and subsequent sections where we use the same multiples $(\alpha, \beta, \gamma$ etc) in $\mathcal{A}(p+1, q+1)$ space as in $\mathcal{A}(p, q)$ space, it is important that the representatives are taken as $F(a)$, $F(b)$ etc and not arbitrary multiples of these. What it amounts to is that we have to use **normalised** representatives (as referred to previously) satisfying

$$X\cdot n = -n\cdot\bar{n} = -2 \tag{42}$$

Working with these normalised points will also turn out to be useful shortly when we consider an alternative representation for spheres, circles etc.

3.1 Intersection of a line and a sphere

For reflection of a wavefront from a spherical surface, we need to find the intersection points of a sphere, S, and a line, L. Let the line be specified by $\mathcal{A}(3,0)$ points a and b and the sphere by $\mathcal{A}(3,0)$ points p, q, r and s. A general point, x, on the line is therefore given by

$$x = \lambda a + (1-\lambda)b$$

where λ is a scalar. Writing as usual the $\mathcal{A}(4,1)$ representations of the points as the corresponding captial letters, e.g. $x \mapsto X$ etc., the 4-vector representing the sphere can be written as

$$\Sigma = P \wedge Q \wedge R \wedge S$$

Thus, taking $x = \lambda a + (1-\lambda)b$ as a general point on the line, we can use equation (40) to write the intersection of the line and the sphere as

$$[\lambda A + (1-\lambda)B + \frac{1}{2}\lambda(1-\lambda)A \cdot Bn] \wedge \Sigma = 0 \tag{43}$$

If we write $\lambda = \frac{1}{2}\mu$ we obtain the following, symmetric form for the intersection equation

$$\left[-\frac{1}{2}\mu^2 A \cdot Bn + \mu(A-B) + \frac{1}{2}(A + B + \frac{1}{4}A \cdot Bn)\right] \wedge \Sigma = 0 \tag{44}$$

Multiplying this by $I = e_1 e_2 e_3 e \bar{e}$, we get the scalar quadratic in μ that we desire – this time with explicit coefficients.

3.2 Alternative representation for spheres and circles

We know that $X \wedge \Sigma = 0$ can be rewritten as

$$X \cdot (I\Sigma) = 0 \quad \Longrightarrow \quad X \cdot \Sigma^* = 0$$

where $\Sigma^* = \Sigma I^{-1}$ is the **dual** to Σ and is a vector. This therefore suggests a very useful alternative representation for a sphere (or a circle in one dimension down), which we now discuss.

We know that for any two normalised points A and B

$$A \cdot B = -2(a-b)^2 \tag{45}$$

Thus, if X is a point on a sphere and C is its centre we know that we can write

$$X \cdot C = -2(x-c)^2 \equiv -2\rho^2$$

where ρ is the radius of the sphere. For a normalised point X this therefore implies that

$$X \cdot (C - \rho^2 n) = 0$$

since $X \cdot n = -2$. Comparing this with $X \cdot \Sigma^*$ we see that provided we normalise Σ^* after taking the dual, then we will find

$$\Sigma^* = C - \rho^2 n \tag{46}$$

Thus, the vector Σ^* encodes, in a very neat fashion, the centre and radius of the sphere. As an immediate application, writing equation (44) in dual form and multiplying out, gives the following explicit equation for the intersection points of a line through a and b with the sphere centre c and radius ρ:

$$\mu^2 A \cdot B + \mu(A - B) \cdot C + \frac{1}{2}(A + B) \cdot C - \frac{1}{4} A \cdot B + 2\rho^2 = 0 \tag{47}$$

Whether one wishes to use this form or the form given in equation(44) depends on whether it is most useful to specify the sphere by 4 points lying on it or by its centre and radius. Note also that given a Σ^* (via taking the normalised form of the dual of $\Sigma = P \wedge Q \wedge R \wedge S$) we can immediately get the radius from

$$\begin{aligned} (\Sigma^*)^2 &= (C - \rho^2 n)^2 \\ &= -2\rho^2 C \cdot n = 4\rho^2 \end{aligned} \tag{48}$$

using the facts that $C^2 = 0$, $n^2 = 0$ and $C \cdot n = -2$. From this it then follows that $C = \Sigma^* + \frac{1}{4}(\Sigma^*)^2 n$. To summarise, from the vector form of the sphere, we can easily obtain the centre and radius as follows

$$(\Sigma^*)^2 = 4\rho^2 \tag{49}$$

$$C = \Sigma^* \left[1 + \frac{1}{4} \Sigma^* n \right] \tag{50}$$

4 Surface Evolution and Representation

Here our aims are threefold:

1. We would like to be able to represent a surface by piecewise spherical or planar patches – this would be useful in a variety of contexts.
2. We would like to be able to evolve a surface – here the idea is that a surface may have complicated features such as cusps or catastrophes, and we want to find a differentiable representation. We can then generate such a surface by evolution from some simple smooth starting point.
3. Wavefront propagation and reflection – this turns out to be linked with (2), and of course is useful in its own right.

These three are linked overall by the way geometric algebra helps with them, and particularly the conformal representation.

4.1 Surface representation

Here the machinery of the conformal representation turns out to be very useful. We start by triangulating the surface in the sense of putting down a grid of points where we identify groups of 3 points together with one interior point.

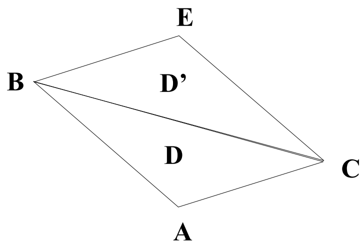

Fig. 1. Example of points for triangulation

If the interior point is to be taken as in the same plane as the other three points we will have a planar representation; if not, then we have a spherical representation. The conformal representation of the sphere is $S_1^* = A \wedge B \wedge C \wedge D$ say, so that S_1 is a vector and S_1^* is a 4-vector – note that if we compare with the notation used in previous sections, $\Sigma = S^*$. Now take another sphere, represented by the 4-vector S_2^* and the vector S_2, where $S_2^* = B \wedge C \wedge D' \wedge E$ where D and E are as shown in Figure 1. Now we find the equation of the intersection of these spheres. A point X on the intersection must simultaneously satisfy

$$X \cdot S_1 = 0 \quad \text{and} \quad X \cdot S_2 = 0$$

Now $X \cdot (S_1 \wedge S_2) = (X \cdot S_1) S_2 - (X \cdot S_2) S_1$ and we know $S_2 \neq \lambda S_1$ for any scalar λ (i.e. the spheres are distinct). Thus X lies on the intersection iff

$$X \cdot (S_1 \wedge S_2) = 0 \tag{51}$$

$$\Longleftrightarrow X \wedge [(S_1 \wedge S_2) I] = 0 \tag{52}$$

The intersection is thus the circle $C = (S_1 \wedge S_2) I$. This is a very neat way of finding the intersection in the case where the two spheres were originally specified by 4 points each (if the spheres were specified by the radius and centre, then we could do this fairly easily by conventional means). Indeed, given 4 points it is also not too hard to recover the conventional equation of a sphere in terms of its radius and centre, but things become slightly more complex when we are considering general 3d circles.

Given a general circle C (a trivector), how do we find its radius? (the radius of curvature of the intersection line in the above case). To do this we start with the unit circle in the x–y plane and take as three points on it

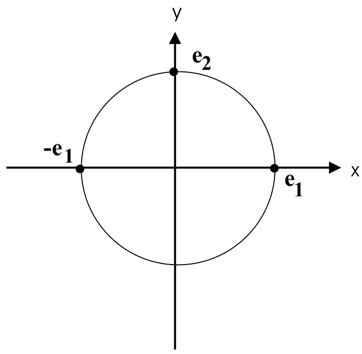

Fig. 2. Unit circle with three key points marked

those shown in Figure 2. Now for any unit length vector, $\hat{x}$ say, we know that $F(\hat{x}) = n + 2\hat{x} - \bar{n} = 2(\hat{x} + \bar{e})$, recalling the definition of $n, \bar{n}$ in equation 5. In particular we have

$$F(e_1)\wedge F(e_2)\wedge F(-e_1) = 16ie_3\wedge\bar{e} = 16ie_3\bar{e}$$

where $i = e_1e_2e_3$ and we have used the facts that $e_1\wedge e_2 = ie_3$ and $(ie_3)\cdot\bar{e} = 0$. Normalising, we can write $\hat{C} = ie_3\bar{e}$ ($\hat{C}$ meaning the unit circle, in this case, in the x–y plane). Next, we note that

$$\frac{-\hat{C}^2}{(n\wedge\hat{C})^2} = 1$$

since $(ie_3\bar{e})(ie_3\bar{e}) = 1$ and $n\wedge\hat{C} = ie_3e\bar{e}$. This therefore gives us a way of finding the radius of **any** circle centred on the origin. Let $C = D_\alpha\hat{C}\tilde{D}_\alpha$ where D_α is the dilatation rotor $D_\alpha = \mathrm{e}^{\frac{\alpha}{2}e\bar{e}}$ introduced earlier, which satisfied

$$D_\alpha F(x)\tilde{D}_\alpha = \mathrm{e}^{\alpha}F(\mathrm{e}^{-\alpha}x)$$

Now, $D_\alpha n\tilde{D}_\alpha = \mathrm{e}^{-\alpha}n$ (recall equations 22) and thus

$$n\wedge C = \mathrm{e}^{\alpha}D_\alpha(n\wedge\hat{C})\tilde{D}_\alpha$$

since $D_\alpha(n\wedge\hat{C})\tilde{D}_\alpha = D_\alpha n\tilde{D}_\alpha\wedge D_\alpha\hat{C}\tilde{D}_\alpha$. If we dilate the unit circle to a circle of radius ρ (so that $\rho = \mathrm{e}^{-\alpha}$) we thus find that

$$\frac{-C^2}{(n\wedge C)^2} = \rho^2 \tag{53}$$

Therefore, we can now position this circle anywhere we wish in 3d by applying the rotors for spatial rotations and translations. Neither of these involve further scale factors, so we deduce that $\frac{-C^2}{(n\wedge C)^2} = \rho^2$ is an identity for **any** circle C in 3d space. Note that $n \wedge C$ is the *plane* in which the circle lies (recall Section 1.4).

We now have the spheres and their intersections which can be used to represent the surface. The GA approach is again very useful here, since it enables us to retain full information for any point in space as to whether it is **inside** or **outside** any given spherical patch. This is because all orientation information is preserved and the representation for a hole differs from that of a sphere. Also for lines in space which we may wish to intersect with the surface, the fact that we can still use the $\lambda A + (1-\lambda)B$ type construct (even though working in the conformal geometry) means that we can tell where an intersection occurs relative to an ordered list of points on the line. For example, for the above construct

$$\begin{aligned}
\lambda < 0 \quad &\Longrightarrow \quad \text{intersection occurs past } B \\
0 < \lambda < 1 \quad &\Longrightarrow \quad \text{intersection occurs between } A \text{ and } B \\
\lambda > 1 \quad &\Longrightarrow \quad \text{intersection occurs before } A
\end{aligned}$$

Overall the GA conformal framework allows a systematic approach to be taken, based directly upon measured points on a surface and with fast look-up procedures to determine where a given space point or line lies with respect to the surface. This methodology is currently being implemented in a case of simulated surfaces coupled with ray-tracing to simulate radar reflection.

4.2 Surface evolution and wavefront propagation

We treat these two topics together since the particular technique we are investigating works in similar ways for each. Note a fuller account of the methods used here is in preparation – our main emphasis here will be on examples.

In [12] the problem of collisional avoidance in robotics was considered as an example of 'propagation' of surfaces. Suppose we have a surface $\mathcal{B}$ (we will describe shortly how we represent the surface in practice) which is the surface of a fixed obstacle. Let the surface of the robot interacting with this fixed surface be $\mathcal{A}$. In [12] it is shown that the collision avoidance problem for this robot is the same as the Huygens propagation of a wavefront $(-\mathcal{A})$ from the surface $\mathcal{B}$ (the minus just shows that we have to reverse the normal direction in moving between the two cases). If the wavefront $\mathcal{A}$ in fact just corresponds to reradiation of spherical wavelets ($\mathcal{A}$ is a sphere) then we have actual spherical wavefront propagation off $\mathcal{B}$, and the caustics (envelopes) of the wavefronts at successive times will be surfaces of constant phase in the geometric optics approximation. By using $\mathcal{A}$ (or $-\mathcal{A}$) to propagate $\mathcal{B}$, we 'evolve' $\mathcal{B}$ to successive new surfaces, which may be much more complicated than $\mathcal{B}$, and in particular by evolving an initially well-behaved surface we may be able to achieve one which exhibits the typical 'catastrophes' which occur for caustics in wavefront propagation. The method is still able to account for these in a fully differentiable manner however, since they are linked via a deterministic differentiable process to the initial differentiable surface.

In [12] a conformal geometry approach is employed to represent the surface. The key point of the method in [12] is to write the position p on the surface as a function of the set of normal directions m. In other words we regard the surface as 'indexed' by m and represent the position at a given value of m as $p[m]$ – the square brackets reflect the fact that this is not a single-valued function, since the same m may have several associated positions p if the object is not convex. Thus far, we do not need conformal geometry, but the next step is to write the representation as

$$\mathcal{R}(m) = m - n(p\cdot m) \tag{54}$$

Here, n is the null vector $e + \bar{e}$ introduced earlier. Thus, we adjoin to the normal m a multiple of a null vector given by the projection of p onto the normal. The neat feature of this is that we can now write

$$\mathcal{R}(m) = T_{-p} m \tilde{T}_{-p} \tag{55}$$

where $T_a = 1 + \frac{1}{2}na$, as earlier, is the conformal translation rotor. The result derived in [12] is that if we propagate a surface $\mathcal{B}(m)$ using the propagation function (wavefront) $\mathcal{A}(m)$ then the resulting surface, written as $\mathcal{A} \oplus \mathcal{B}(m)$ is described by the **composition** of the rotors corresponding to each surface individually:

$$T_{-p_{(\mathcal{A}\oplus\mathcal{B})}} = T_{-p_{\mathcal{A}}} T_{-p_{\mathcal{B}}}$$

This then suggests a spectral theory of surfaces with the indexing quantity as the normal direction and propagation as multiplication of 'direction spectra' (the translation rotors).

In any practical application of this formalism we have at some stage got to invert a given $\mathcal{R}(m)$ to find explicitly the current set of positions in the new surface. Since p enters into $\mathcal{R}(m)$ only via its projection on m, there is not sufficient information in $\mathcal{R}(m)$ itself to do this. In [12] this is solved by introducing derivatives of $\mathcal{R}(m)$. This enables $p[m]$ to be recovered, via quite a complicated inversion formula. Here we suggest a different technique, which achieves the required propagation much more simply and quickly, at least for the case of spherical wavefront propagation. We note that the problem we are addressing here is essentially the same as that considered in the *level set* method (see, e.g. [13]) applied in this instance to propagation with a constant velocity.

We explain the technique via two examples. First consider an initial surface consisting of a parabola in 2d as shown in Figure 3. We wish to carry out spherical Huygens propagation starting from this surface. It is easy to see that for a sphere, radius ρ and centre c, the representation is:

$$\mathcal{R}(m) = m - n(c\cdot m - \rho)$$

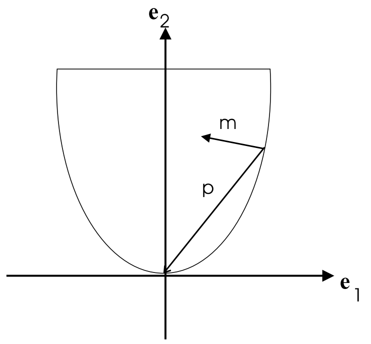

Fig. 3. Simple initial parabolic surface in 2d

(this is because the inward pointing normal, the one we are using by convention, is $m = \frac{(c-p)}{\rho}$ for a point on a sphere).

The Huygens construction effectively recentres each sphere on the points of the surface we wish to propagate and the construction in terms of composing rotors effectively tells us to add the ps at the same m (composition of rotors just corresponds to the sum of the translations). Thus the propagated $\mathcal{R}(m)$, starting from $\mathcal{R}(m) = m - (p{\cdot}m)n$ is

$$\mathcal{R}^{prop}(m) = m - [(p{\cdot}m) - \rho]n$$

For the specific example of propagating a parabola, we can write

$$p = te_1 + \frac{1}{2}t^2 e_2 \tag{56}$$

$$m = \frac{te_1 - e_2}{\sqrt{1+t^2}} \tag{57}$$

where t is a parameter (actually equal to the x-coordinate of a position). We find we can express $p{\cdot}m$ as $-\frac{1}{2}\frac{(m{\cdot}e_1)^2}{m{\cdot}e_2}$ and thus obtain

$$\mathcal{R}^{prop}(m) = m + n\left(\rho + \frac{1}{2}\frac{(m{\cdot}e_1)^2}{m{\cdot}e_2}\right) \tag{58}$$

It is an expression like this which needs to be inverted to find $p[m]$ using the differential $\mathcal{R}^{prop}(m)$ as in [12].

But this is actually much more complicated than we need. Instead one can note that propagating spherical wavefronts corresponds to geometric optics, and therefore just ray tracing, with rays normal to the wavefronts. All we need is to move out from any given position along the normal at that position by a distance ρ, in order to achieve the same effect as above. For example, in the parabola case, let us write

$$m = \cos\theta e_1 + \sin\theta e_2$$

and specify this as the inward normal by taking θ in the range $-\pi < \theta < 0$. (The link to the parameter t is $\cos\theta = t/\sqrt{1+t^2}$, $\sin\theta = -1/\sqrt{1+t^2}$). Then we have that

$$p = -\cot\theta e_1 + \frac{1}{2}\cot^2\theta e_2$$

and the propagated position is just

$$p - \rho m = (-\cot\theta - \rho\cos\theta)e_1 + (\frac{1}{2}\cot^2\theta - \rho\sin\theta)e_2$$

The surfaces found this way are plotted in Figure 4 for $\rho = 0, 1, 2$. We see

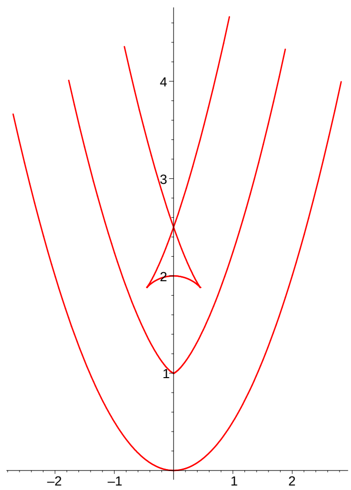

Fig. 4. Circular wavefront propagation of an initial parabola. Note the development of a 'swalowtail catastrophe'.

immediately that we have an initially smooth differentiable surface ($\rho = 0$), developing a cusp ($\rho = 1$) and then a swallowtail catastrophe ($\rho = 2$).

As a second example, also considered in [12], let us consider an initially cardioid-type surface, and propagate this. The initial cardioid is shown in Figure 5. The equation we use is

$$p_x = a\left(\frac{1}{2} + b\cos\theta + \frac{1}{2}\cos 2\theta\right) \tag{59}$$

$$p_y = a\left(b\sin\theta + \frac{1}{2}\sin 2\theta\right) \tag{60}$$

i.e. an expression in terms of circular harmonics. The corresponding inward pointing normal is found to be

$$m = \frac{-1}{\sqrt{1+2b\cos\theta+b^2}}\{(b\cos\theta+\cos 2\theta)e_1 + (b\sin\theta+\sin 2\theta)e_2\} \tag{61}$$

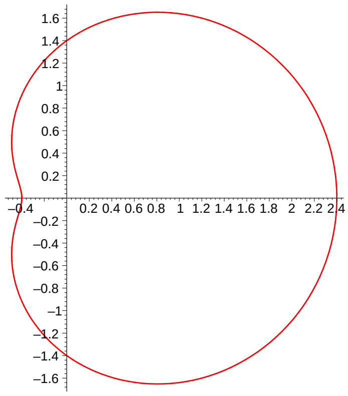

Fig. 5. Initial configuration of the cardioid-like shape used for circular wavefront propagation

Forming $p - \rho m$ as before, we get a succession of surfaces which initially collapse inwards, pass through each other and then eventually propagate outwards in a more or less circular fashion except for a swallowtail catastrophe. This matches what was found in [12] using the inversion approach.

One can also, of course, work in 3d. The above examples have been in 2d, but all of the formulae are 3-dimensional and may be applied to any initial surface for which we can find parametric representations of p and m. Figure 6 shows an example of a 3d cardioid surface which has been propagated in this fashion. Since the apparatus developed in [12] seems to be unnecessary at

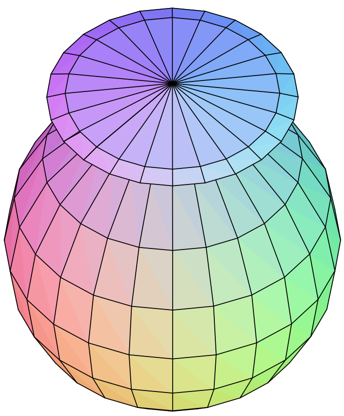

Fig. 6. A frame from the 3d development of cardioid propagation

least for this spherical wavefront propagation, it might be wondered where the conformal representation, or indeed geometric algebra, enters into this problem. The answer lies in what happens if we want to **reflect** a developing wavefront/surface off another surface. Figure 7 shows what happens when the developing cardioid figure reflects off a plane surface. This is in 2d, but

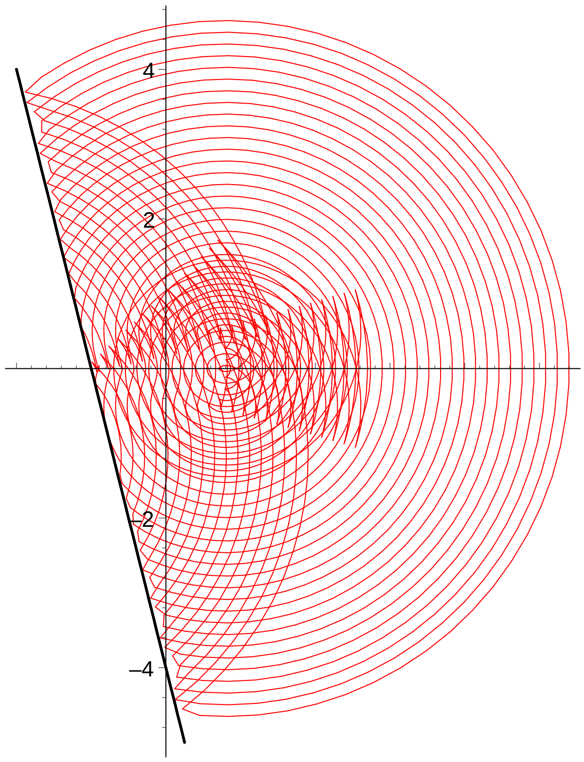

Fig. 7. Reflection of 'cardioid' wavefront from a plane sheet

we also have a 3d demonstration where an initially cylindrically symmetric propagating surface is reflected off an offset spherical surface, resulting in a quite general 3d shape, see Figure 8. The way in which these examples were computed used the GA formula for reflection in 3d;

$$a' = -mam$$

for the reflection of a vector a to a vector a' in the local normal m, together with the above results for the intersection of lines and planes and lines and spheres etc in the conformal geometry. The 'lines' in this case are the 'rays' coresponding to normals to the wavefront. The conformal approach enables one to have a straightforward synthetic algorithm for all these computations, which is basically very simple, but generates quite impressively sophisticated

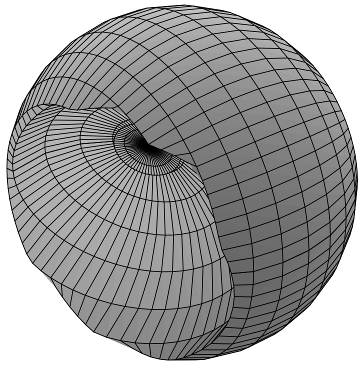

Fig. 8. Stage in propagation of an initially cylindrically symmetric cardioid-shaped 3d wavefront reflecting off an offset sphere

results. Extensions to non-uniform propagation velocity are currently being considered.

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