NEW TECHNIQUES FOR ANALYSING AXISYMMETRIC GRAVITATIONAL SYSTEMS. 1 VACUUM FIELDS

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Abstract

A new framework for analysing the gravitational fields in a stationary, axisymmetric configuration is introduced. The method is used to construct a complete set of field equations for the vacuum region outside a rotating source. These equations are under-determined. Restricting the Weyl tensor to type D produces a set of equations which can be solved, and a range of new techniques are introduced to simplify the problem. Imposing the further condition that the solution is asymptotically flat yields the Kerr solution uniquely. The implications of this result for the no-hair theorem are discussed. The techniques developed here have many other applications, which are described in the conclusions.

1 Introduction

The field equations of general relativity are remarkable both for their complexity and for the incredible variety of techniques that can be applied in generating solutions. These range from brute force computation through to advanced techniques based on symmetries and invariants [1, 2]. In this paper we introduce a new method for tackling the gravitational field equations for stationary, axisymmetric systems. The main applications of this scheme are to finding the gravitational fields inside and outside rotating sources. The method has strong similarities with the Newman–Penrose (NP) formalism [1], and also with coframe methods for analysing differential equations. The essential difference with the NP formalism is that our approach is based on a real orthonormal tetrad, as opposed to a complex null tetrad. The only complex structure we work with arises naturally in the structure of the bivector fields generated by the tetrad vectors.

The main advance in the work presented here is that the technique can be pushed right through to an explicit solution. This is possible with the introduction of a number of new methods. The closest precursor to these techniques is the work of Held [3], though we believe our current methods offer many improvements. In this paper we concentrate on the vacuum equations. We extract a complete set of vacuum equations, and find that these are under-determined for a general, axisymmetric source. To construct a unique solution we then impose the additional condition that the Riemann tensor is type D, and the solution is asymptotically flat. We are then able to show constructively that the Kerr is the unique solution for this case. This result does not rely on the existence of a horizon. The Carter–Robinson uniqueness theorem can therefore be interpreted as stating that the formation of a horizon is related to a restriction on the algebraic form of the Weyl tensor.

The Kerr solution occupies a unique place amongst the known exact solutions to the Einstein field equations. Many derivations of the solution have been found, a number of which are summarised in Chandrasekhar's classic text [4]. Few authors (apart from Chandrasekhar) have claimed that the derivation of the Kerr solution is simple, and many of the derivations found are highly opaque. Probably the most mysterious derivation of all is that of the complex coordinate transformation trick [5, 6]. In this derivation one starts with the Schwarzschild metric in advanced Eddington-Finkelstein coordinates, expressed in terms of a complex null tetrad. A complex coordinate transformation is then applied to yield a new metric, which turns out to be that of the Kerr solution. This is a trick because there is no reason to expect the complex coordinate transformation to generate a new vacuum solution. The justification for this is only revealed at the end of a detailed study of vacuum metrics of Kerr-Schild type, and is quite obscure [7].

The derivation presented here proceeds in a very different way to the usual 'metric' route adopted in general relativity (GR). In a typical GR derivation one starts with a metric of suitable symmetry containing a set of arbitrary scalar functions. The Einstein equations are then written as a set of coupled second-order differential equations in the metric functions. These equations are notoriously difficult to analyse, for many different reasons. One of the difficulties is that the freedom to perform coordinate transformations must be removed by restricting the form of the metric ('gauge-fixing'). If this is not done correctly, one has more unknowns than there are equations for. The problem is that it is often not clear *ab initio* how best to perform this gauge fixing. Ideally one would like the chosen coordinates to have some simple physical interpretation, but how to achieve this does not usually emerge until later in the solution process.

Our method avoids this problem by developing a first-order set of equations analogous to those obtained in the NP formalism. Gauge-fixing of the Lorentz group is performed at the level of the Riemann tensor, which makes it easier to ensure that the gauge choices are physically sensible and mathematically convenient. The resulting equations relate abstract derivatives of the terms in the spin connection to quadratic combinations of the same quantities. This provides a very clear way of expressing and analysing the non-linearities in the theory. Once the Riemann tensor has been found, some natural physical scalar fields start to emerge. Invariance under diffeomorphisms is then employed to give these physical fields simple expressions in terms of our chosen coordinates. This process lifts the status of the coordinates from arbitrary spacetime functions to local physically-measurable fields. We then introduce a series of new techniques which solve directly for the spin coefficients without computing any of the metric coefficients. The latter are then found by straightforward integration. This is a considerable advance on previous first-order formulations of GR, and opens up a number of new solution strategies. We do not yet possess a general method for solving all problems in our new formalism, but there are good reasons to believe that some very general techniques do exist. The derivation of the Kerr solution presented here provides a number of clues to how such general methods might be found.

One feature of our method, which it shares with the NP formalism, is that the early stages involve simple, repetitive algorithms for organising terms. This work is well suited to symbolic algebra packages such as Maple. The derivation of the equations is further simplified by employing the language of Clifford algebra [8, 9, 10, 11]. This is employed as a set of algebraic rules for manipulating vectors, rather than via a concrete matrix representation (such as the Dirac matrices). In this sense we are adopting the language of spacetime algebra [12, 10]. This approach to Clifford algebra is also well suited to implementation in symbolic algebra packages. The application of spacetime algebra clearly demonstrates the origin of a natural complex structure generated by the spacetime directed volume element, or pseudoscalar. This element squares to minus one, so unites the complex structure of the 2-spinor approach with duality transformations in tensor language. This is a considerable unification, and provides a number of insights into the complex structure underlying the Kerr solution. This also sheds some light on the origin of the mysterious complex coordinate transformation trick discovered by Newman and Janis [5].

This paper starts by introducing the variables we use to encode a stationary axisymmetric gravitating system. We then introduce a suitably general ansatz to describe axially-symmetric fields. The link between the gauge fields and the metric and Christoffel connection is explained. The complete set of vacuum equations is obtained, and we then restrict to looking for vacuum solutions of type D. Imposing this restriction causes the equations to simplify in a truly remarkable manner to a set of 8 coupled equations. These equations are then simplified by identifying various integrating factors. This enables us to make a range of deductions about the solution without introducing a definite coordinatisation. Finally, a concrete expression of the solution is produced, which reproduces the line element of the Kerr solution in Boyer–Lindquist coordinates. We end with a discussion of the insights the present formalism brings to the no hair theorem for black holes. This states that the the Kerr solution is the only possible vacuum axisymmetric solution outside a horizon, which is now interpreted in terms of a restriction on the algebraic type of the Weyl tensor at a horizon.

Summation convention and natural units ($c = G = 1$) are employed throughout. Greek letters are employed for coordinate indices and Latin for tetrad components. We work in a space with signature $(1, -1, -1, -1)$.

2 Field equations for axisymmetric systems

The equations derived here follow the route developed in an earlier paper [10], though the present derivation is streamlined and much of the less conventional terminology has been removed. A small price for this is that some results in the following derivation are stated without proof. Further details can be found in [10, 13]. Our starting point is the Clifford algebra of Minkowski spacetime. The (constant) generators of this are denoted by $\{\gamma_0 \cdots \gamma_3\}$ and satisfy

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\eta_{ij}, \qquad i, j = 0 \ldots 3, \tag{1}$$

where our convention is that $\eta_{ij} = \mathrm{diag}(1, -1, -1, -1)$. We suppress any mention of the identity matrix, which is superfluous for all calculations. Throughout we use Latin indices for tetrad frames and components, and Greek for coordinate indices. At various points we adopt the language of spacetime algebra and refer to the Clifford generators as 'vectors'.

The symmetric and antisymmetric parts of the Clifford product of two vectors define the inner and outer products, and are denoted with a dot and a wedge respectively:

$$\begin{aligned} \gamma_i \cdot \gamma_j &= \tfrac{1}{2}(\gamma_i\gamma_j + \gamma_j\gamma_i) = \eta_{ij} \\ \gamma_i \wedge \gamma_j &= \tfrac{1}{2}(\gamma_i\gamma_j - \gamma_j\gamma_i). \end{aligned} \tag{2}$$

A full (real) basis for the Clifford algebra is provided by

$$\begin{array}{ccccc} 1 & \{\gamma_i\} & \{\gamma_i\wedge\gamma_j\} & \{I\gamma_i\} & I \\ \text{1 scalar} & \text{4 vectors} & \text{6 bivectors} & \text{4 trivectors} & \text{1 pseudoscalar,} \\ \text{grade 0} & \text{grade 1} & \text{grade 2} & \text{grade 3} & \text{grade 4} \end{array} \tag{3}$$

where

$$I = \gamma_0\gamma_1\gamma_2\gamma_3. \tag{4}$$

The 4-vector I is the highest grade element in the algebra and is usually called the pseudoscalar, though directed volume element is perhaps more appropriate. The pseudoscalar squares to -1,

$$I^2 = -1, \tag{5}$$

and I anticommutes with all odd-grade elements and commutes with even-grade elements.

We next introduce a set of generators appropriate for the study of axisymmetric fields. With (r, θ, ϕ) denoting a standard set of polar coordinates we define

$$\begin{aligned} \gamma_t &= \gamma_0 \\ \gamma_r &= \sin\theta(\cos\phi\,\gamma_1 + \sin\phi\,\gamma_2) + \cos\theta\,\gamma_3 \\ \gamma_\theta &= \cos\theta(\cos\phi\,\gamma_1 + \sin\phi\,\gamma_2) - \sin\theta\,\gamma_3 \\ \gamma_\phi &= -\sin\phi\,\gamma_1 + \cos\phi\,\gamma_2. \end{aligned} \tag{6}$$

These vectors also form an orthonormal basis for Minkowski spacetime. Henceforth the Latin indices $i = 0 \ldots 3$ are assumed to run over the set $\{\gamma_t, \gamma_r, \gamma_\theta, \gamma_\phi\}$. The reciprocal frame is given by

$$\gamma^t = \gamma_t, \quad \gamma^r = -\gamma_r, \quad \gamma^\theta = -\gamma_\theta, \quad \gamma^\phi = -\gamma_\phi. \tag{7}$$

The (Clifford) product of two orthogonal vectors results in a bivector. There are six of these in total, and a useful basis for these is provided by

$$\begin{aligned} \sigma_r &= \gamma_r\gamma_t \\ \sigma_\theta &= \gamma_\theta\gamma_t \\ \sigma_\phi &= \gamma_\phi\gamma_t. \end{aligned} \tag{8}$$

These all have unit square,

$$\sigma_r\sigma_r = \sigma_\theta\sigma_\theta = \sigma_\phi\sigma_\phi = 1 \tag{9}$$

and distinct bivectors anticommute,

$$\sigma_r\sigma_\theta = -\sigma_\theta\sigma_r, \qquad \textit{etc.} \tag{10}$$

The three bivectors also satisfy the identity

$$\sigma_r\sigma_\theta\sigma_\phi = \gamma_t\gamma_r\gamma_\theta\gamma_\phi = \gamma_0\gamma_1\gamma_2\gamma_3 = I. \tag{11}$$

The bivector basis is completed by the dual bivectors

$$\begin{aligned} I\sigma_r &= -\gamma_\theta\gamma_\phi \\ I\sigma_\theta &= \gamma_r\gamma_\phi \\ I\sigma_\phi &= -\gamma_r\gamma_\theta. \end{aligned} \tag{12}$$

The algebra generated by the set $\{\sigma_r, \sigma_\theta, \sigma_\phi\}$ is isomorphic to the Pauli algebra. This is the reason for the notation. It should be remembered, however, that each of the $\{\sigma_r, \sigma_\theta, \sigma_\phi\}$ set anticommutes with γ_t.

The space of bivectors generates the Lie algebra of the group of Lorentz transformations. Under a proper orthochronous Lorentz transformation a vector a is mapped to the vector a' according to the rule

$$a \mapsto a' = Ra\tilde{R} \tag{13}$$

where R is an even-grade Clifford element satisfying

$$R\tilde{R} = 1. \tag{14}$$

The tilde on $\tilde{R}$ denote the *reverse* operation, which reverses the order of all products in a general element. The effect of this on the grade-r element M_r is

$$\tilde{M_r} = (-1)^{r(r-1)/2} M_r. \tag{15}$$

The object R is referred to as a *rotor*. Rotors form a group under the Clifford product, which gives a spin-1/2 representation of the proper orthochronous Lorentz group. All rotors can be written as $\pm\exp(B/2)$, where B is a bivector. The commutator of a bivector and a grade-r object results in a new object of grade-r. For this operation it is useful to define the commutator product by

$$M \times N = \tfrac{1}{2}(MN - NM), \tag{16}$$

where M and N are general Clifford elements. The space of bivectors is closed under the commutator product, and this generates the Lie algebra for the rotor group.

The key to our approach is to parameterise the gravitational fields via a coframe. We introduce the four vector fields

$$\begin{aligned} g^t &= a_1\gamma^t + \frac{a_2}{r\sin\theta}\gamma^\phi \\ g^r &= b_1\gamma^r + \frac{b_2}{r}\gamma^\theta \\ g^\theta &= \frac{c_1}{r}\gamma^\theta + c_2\gamma^r \\ g^\phi &= \frac{d_1}{r\sin\theta}\gamma^\phi + d_2\gamma^t, \end{aligned} \tag{17}$$

where all of the variables $(a_1 \dots d_2)$ are scalar functions of r and θ. The indices on the $\{g^t, g^r, g^\theta, g^\phi\}$ are to be read as coordinate indices, and we denote the set by $\{g^\mu\}$, $\mu = 0 \dots 3$. The reciprocal frame $\{g_\mu\}$ to the $\{g^\mu\}$ frame is defined by the equation

$$g_\mu g^\nu + g^\nu g_\mu = 2\delta_\mu^\nu. \tag{18}$$

The g_μ and γ^i vectors define a tetrad field in the obvious manner,

$$g_\mu \gamma^i + \gamma^i g_\mu = 2h_\mu{}^i. \tag{19}$$

The metric tensor $g_{\mu\nu}$ is derived from the g_μ vectors via

$$g_\mu g_\nu + g_\nu g_\mu = 2g_{\mu\nu}. \tag{20}$$

This results in the line element

$$\begin{aligned} ds^2 = {} & \frac{{d_1}^2 - {d_2}^2 r^2 \sin^2\theta}{(a_1 d_1 - a_2 d_2)^2} dt^2 + \frac{a_1 d_2 r^2 \sin^2\theta - d_1 a_2}{(a_1 d_1 - a_2 d_2)^2} 2dt\, d\phi - \frac{{a_1}^2 r^2 \sin^2\theta - {a_2}^2}{(a_1 d_1 - a_2 d_2)^2} d\phi^2 \\ & - \frac{{c_1}^2 + {c_2}^2 r^2}{(c_1 b_1 - c_2 b_2)^2} dr^2 - \frac{c_1 b_2 + b_1 c_2 r^2}{(c_1 b_1 - c_2 b_2)^2} 2dr\, d\theta - \frac{{b_1}^2 r^2 + {b_2}^2}{(c_1 b_1 - c_2 b_2)^2} d\theta^2. \end{aligned} \tag{21}$$

At first sight this looks unnecessarily complicated. Various nonlinearities are introduced, and there are redundant degrees of freedom in the g^μ vectors. But our method works directly with the g^μ and not with the line element. This gives greater control over the nonlinearities and the extra (gauge) degrees of freedom can be employed to advantage later in the calculation.

Our main differential operators are the coframe derivatives L_i,

$$L_i = \gamma_i \cdot g^\mu \partial_\mu, \tag{22}$$

where we employ the abbreviation

$$\partial_\mu = \frac{\partial}{\partial x^\mu}. \tag{23}$$

The coframe derivatives can be viewed as directional derivatives, with directions determined by the coframe. Some texts [2] prefer to use a partial-derivative notation for coframe derivatives. We have not chosen such a notation because coframe derivatives do not necessarily commute. Written out explicitly the L_i operators are

$$\begin{aligned} L_t &= a_1 \partial_t + d_2 \partial_\phi \\ L_r &= b_1 \partial_r + c_2 \partial_\theta \\ L_\theta &= \frac{1}{r}(c_1 \partial_\theta + b_2 \partial_r) \\ L_\phi &= \frac{1}{r \sin\theta}(d_1 \partial_\phi + a_2 \partial_t). \end{aligned} \tag{24}$$

The coframe derivatives satisfy a set of bracket identities

$$[L_i, L_j] = c_{ij}{}^k L_k. \tag{25}$$

The spin connection is encoded in a set of bivector fields ω_i. These are related to the ${c_{ij}}^k$ coefficients by

$$ {c_{ij}}^k = \gamma^k \cdot (L_i \gamma_j + \omega_i \times \gamma_j - L_i \gamma_i - \omega_j \times \gamma_i). \tag{26} $$

This equation embodies the content of the first structure equation.

Rather than solving this equation directly, our method involves introducing a general parameterisation for the connection terms. An appropriate parameterisation is defined by

$$ \begin{aligned} \omega_t &= -(T + IJ)\sigma_r - (S + IK)\sigma_\theta - d_2 \gamma_1 \gamma_2 \\ \omega_r &= -(\bar{S} + I\bar{K}) I\sigma_\phi + c_2 I \sigma_\phi \\ \omega_\theta &= -(\bar{G} + I\bar{J}) I\sigma_\phi + \frac{c_1}{r} I\sigma_\phi \\ \omega_\phi &= -(H + IK) I\sigma_r + (G + IJ) I\sigma_\theta - \frac{d_1}{r \sin\theta} \gamma_1 \gamma_2. \end{aligned} \tag{27} $$

This parameterisation introduces a set of 10 scalar functions (G, $\bar{G}$, J, $\bar{J}$, S, $\bar{S}$, K, $\bar{K}$, T, H). Each of these is a scalar function of r and θ. The reason for the labelling scheme will become apparent when the final set of equations is derived. Equation (26) now produces the 6 bracket relations

$$ \begin{aligned} &[L_t, L_r] = -T L_t - (K + \bar{K}) L_\phi & &[L_r, L_\theta] = -\bar{S} L_r - \bar{G} L_\theta \\ &[L_t, L_\theta] = -S L_t + (J - \bar{J}) L_\phi & &[L_r, L_\phi] = -(K - \bar{K}) L_t - G L_\phi \\ &[L_t, L_\phi] = 0 & &[L_\theta, L_\phi] = (J + \bar{J}) L_t - H L_\phi. \end{aligned} \tag{28} $$

The Christoffel connection coefficients are recovered from our various fields through the formula

$$ \Gamma^\lambda_{\mu\nu} = g^\lambda \cdot (\partial_\mu g_\nu + g_\mu \cdot \gamma^i \, \omega_i \times g_\nu). \tag{29} $$

Clearly this cannot be computed until the g_μ vectors are explicitly constructed, which is not achieved until the end of the calculation.

3 Gauge freedom

There are two distinct types of gauge freedom present in the setup described above. These correspond to the two symmetries on which the gauge treatment of gravity is based [10, 14]. The first is invariance under diffeomorphisms. This can be encoded in various different ways, but for our present purposes it is simplest to consider two new scalar coordinates $r'(r,\theta)$ and $\theta'(r,\theta)$. Suppose first that we replace r and θ with r' and θ' everywhere in the metric of equation (21). Clearly such a relabelling does not affect the solution. We then re-express the line element back in the original (r,θ) coordinate system. The result is that all of the fields $(a_1 \ldots d_2)$ have transformed to new fields. Four of these, a_1, a_2, d_1, d_2 transform according to the simple rule

$$ a_1(r,\theta) \mapsto a_1'(r,\theta) = a_1(r',\theta'). \tag{30} $$

Any field transforming in this manner is said to transform as a covariant scalar under diffeomorphisms. All of the 10 scalar functions introduced in the ω_i of equation (27) behave as covariant scalars.

The remaining variables in the g^μ vectors, b_1, b_2, c_1, c_2, pick up derivatives of the coordinate transformation under the diffeomorphism described above. The transformed variables are given by

$$\begin{pmatrix} b_1'(r,\theta) & c_2'(r,\theta) \\ b_2'(r,\theta) & c_1'(r,\theta)/r \end{pmatrix} = \begin{pmatrix} b_1(r',\theta') & c_2(r',\theta') \\ b_2(r',\theta') & c_1(r',\theta')/r' \end{pmatrix} \begin{pmatrix} \partial_{r'} r & \partial_{r'}\theta \\ \partial_{\theta'} r & \partial_{\theta'}\theta \end{pmatrix}. \tag{31}$$

It is simplest for our purposes to view the eight transformed variables $a_1' \dots d_2'$ as a new set of functions obtained from the original set by a gauge transformation. As such, the two sets are physically indistinguishable, and this gauge freedom must be fixed before we can write down a unique solution. The standard, metric-based approach to solving the Einstein equations usually attempts to fix this gauge freedom at the outset by restricting the terms in the metric. For example we could write the line element in the form [4]

$$ds^2 = e^{2\alpha}\, dt^2 - e^{2\psi}(d\phi - \omega\, dt)^2 - e^{2\mu_2}\, dr^2 - e^{2\mu_3}\, d\theta^2. \tag{32}$$

This leaves us with a set of 5 variables, with the freedom to fix the relation between μ_2 and μ_3 by a further coordinate transformation. Our strategy (in keeping with that of the NP formalism) is to leave this gauge unfixed until much later in the calculation. This is possible because the coframe derivatives transform as

$$L_r \mapsto L_r' = b_1(r',\theta') \frac{\partial}{\partial r'} + c_2(r',\theta') \frac{\partial}{\partial \theta'} \tag{33}$$

with a similar result holding for L_θ. Provided we formulate all equations in terms of coframe derivatives of covariant scalars, the entire structure will transform covariantly under diffeomorphisms. That is, the gauge freedom can be ignored until later in the problem. We then find that certain physical fields emerge and it is sensible to equate these with combinations of the chosen coordinates. This is the point at which the diffeomorphism gauge freedom is fixed. Working in this manner ensures that the chosen coordinates have a direct physical interpretation.

The second gauge freedom is that of applying a local Lorentz rotation to the g^μ vectors. This is the gravitational analogy of a phase transformation in quantum theory. Under a Lorentz rotation the g^μ vectors transform according to

$$g^\mu \mapsto R g^\mu \tilde{R} \tag{34}$$

where R is a local rotor. The reciprocal vectors g_μ transform the same way, and the metric derived by equation (20) is therefore invariant. Clearly this freedom is a gauge symmetry, as it does not change any physical quantity. This symmetry is removed if one works directly with terms in the metric, but our approach is to keep the symmetry explicit and use it to simplify our equations. The ω_i bivectors represent the connection for the gauge symmetry of equation (34). The combination

$$D_\mu = \partial_\mu + g_\mu \cdot \gamma^i\, \omega_i \times \tag{35}$$

defines the covariant derivative for objects transforming in the manner of equation (34). For our axisymmetric setup the degrees of freedom in the Lorentz gauge are reduced from six to two. These are a rotation in the $I\sigma_\phi$ plane and a boost in the σ_ϕ direction. The rotors describing these can be combined to form the single rotor

$$R = \exp(\alpha\sigma_\phi/2)\exp(\beta I\sigma_\phi/2) = \exp(wI\sigma_\phi/2), \tag{36}$$

where $w = \beta - I\alpha$ is an arbitrary function of (r, θ). The freedom to apply this rotor will be used to simplify the Riemann tensor.

One final (non-gauge) freedom present is that the equations are unchanged if all variables are scaled by a constant amount, $a_1 \mapsto ca_1$, $G \mapsto cG$, *etc.* This freedom is often employed to fix the asymptotic behaviour so that the g_μ vectors map to a flat space coordinate frame at spatial infinity.

4 The Riemann tensor and vacuum fields

The essential covariant object to construct is the Riemann tensor. The information contained in this is compactly summarised in a set of six bivectors. These are defined by

$$\mathsf{R}_{ij} = L_i\omega_j - L_j\omega_i + \omega_i\times\omega_j - c_{ij}{}^k\omega_k. \tag{37}$$

Calculating each of the terms R_{ij} is simply a matter of organisation, and is well suited to a symbolic algebra package. A useful set of intermediate results is provided by the identities

$$\begin{aligned} L_t\sigma_r + \omega_t\times\sigma_r &= (S+IK)I\sigma_\phi & L_t\sigma_\theta + \omega_t\times\sigma_\theta &= -(T+IJ)I\sigma_\phi \\ L_r\sigma_r + \omega_r\times\sigma_r &= (\bar{S}+I\bar{K})\sigma_\theta & L_r\sigma_\theta + \omega_r\times\sigma_\theta &= -(\bar{S}+I\bar{K})\sigma_r \\ L_\theta\sigma_r + \omega_\theta\times\sigma_r &= (\bar{G}+I\bar{J})\sigma_\theta & L_\theta\sigma_\theta + \omega_\theta\times\sigma_\theta &= -(\bar{G}+I\bar{J})\sigma_r \\ L_\phi\sigma_r + \omega_\phi\times\sigma_r &= (G+IJ)\sigma_\phi & L_\phi\sigma_\theta + \omega_\phi\times\sigma_\theta &= (H+IK)\sigma_\phi. \end{aligned} \tag{38}$$

On substituting the form for the ω_i into the definition of R_{ij} we now obtain the Riemann tensor in terms of abstract derivatives as

$$
\begin{aligned}
\mathsf{R}_{rt} &= \Big(-L_r(T+IJ)+(S+IK)(\bar{S}+I\bar{K})+T(T+IJ)+I(K+\bar{K})(H+IK)\Big)\sigma_r \\
&\quad + \Big(-L_r(S+IK)-(\bar{S}+I\bar{K})(T+IJ)+T(S+IK)-I(K+\bar{K})(G+IJ)\Big)\sigma_\theta \\
\mathsf{R}_{\theta t} &= \Big(-L_\theta(S+IK)-(\bar{G}+I\bar{J})(T+IJ)+S(S+IK)+I(J-\bar{J})(G+IJ)\Big)\sigma_\theta \\
&\quad + \Big(-L_\theta(T+IJ)+(\bar{G}+I\bar{J})(S+IK)+S(T+IJ)-I(J-\bar{J})(H+IK)\Big)\sigma_r \\
\mathsf{R}_{\phi t} &= -\Big((G+IJ)(T+IJ)+(S+IK)(H+IK)\Big)\sigma_\phi \\
\mathsf{R}_{r\theta} &= -\Big(L_r(\bar{G}+I\bar{J})-L_\theta(\bar{S}+I\bar{K})+\bar{G}(\bar{G}+I\bar{J})+\bar{S}(\bar{S}+I\bar{K})\Big)I\sigma_\phi \\
\mathsf{R}_{r\phi} &= \Big(L_r(G+IJ)-(\bar{S}+I\bar{K})(H+IK)+G(G+IJ)+I(K-\bar{K})(S+IK)\Big)I\sigma_\theta \\
&\quad - \Big(L_r(H+IK)+(\bar{S}+I\bar{K})(G+IJ)+G(H+IK)-I(K-\bar{K})(T+IJ)\Big)I\sigma_r \\
\mathsf{R}_{\theta\phi} &= -\Big(L_\theta(H+IK)+(\bar{G}+I\bar{J})(G+IJ)+H(H+IK)+I(J+\bar{J})(T+IJ)\Big)I\sigma_r \\
&\quad + \Big(L_\theta(G+IJ)-(\bar{G}+I\bar{J})(H+IK)+H(G+IJ)-I(J+\bar{J})(S+IK)\Big)I\sigma_\theta .
\end{aligned}
\tag{39}
$$

The standard coordinate expression of the Riemann tensor is obtained from the preceding bivectors by forming the contraction

$$R^\mu{}_{\nu\rho\sigma} = (g^\mu \wedge g_\nu)\cdot \mathsf{R}_{ij}\, \gamma^i \cdot g_\sigma \, \gamma^j \cdot g_\rho . \tag{40}$$

As with the Christoffel connection, these coefficients cannot be written down until an explicit form is obtained for the g^μ vectors.

By working with a general form for the spin connection we have no guarantee that the Riemann tensor satisfies all of its required symmetries. The first of these to consider arises from the assumption that there is no torsion present. In this case the Riemann tensor satisfies

$$\gamma^i \wedge \mathsf{R}_{ij} = \tfrac{1}{2}(\gamma^i \mathsf{R}_{ij} + \mathsf{R}_{ij}\gamma^i) = 0. \tag{41}$$

This says that a trivector vanished for each value of the index j, so corresponds to a set of 16 constraints. The Riemann tensor is encoded in six bivector-valued functions, giving a total of 36 scalar degrees of freedom. Equation (41) reduces this to the familiar 20 degrees of freedom of a general Riemann tensor.

In this paper we are interested in vacuum solutions to the field equations. For these the contraction of the Riemann tensor must also vanish,

$$\gamma^i \times \mathsf{R}_{ij} = \tfrac{1}{2}(\gamma^i \mathsf{R}_{ij} - \mathsf{R}_{ij}\gamma^i) = 0. \tag{42}$$

This can be combined with equation (41) to obtain

$$\gamma^i \mathsf{R}_{ij} = 0. \tag{43}$$

This equation is satisfied by all Weyl tensors. Equation (42) reduces the number of degrees of freedom in the vacuum Riemann tensor from 20 to 10 — the expected number for vacuum solutions. The compact combination of the symmetry and contraction information in equation 43 is unique to the Clifford algebra formulation.

Before proceeding, it is useful to adopt a slightly different notation for the Riemann tensor. We define a linear map from bivectors to bivectors by writing

$$\mathsf{R}(B) = \tfrac{1}{2} B \cdot (\gamma^j \wedge \gamma^i)\, \mathsf{R}_{ij}. \tag{44}$$

Equation (39) then shows that the Riemann tensor has the general form

$$\begin{aligned} \mathsf{R}(\sigma_r) &= \alpha_1 \sigma_r + \beta_1 \sigma_\theta & \qquad \mathsf{R}(I\sigma_r) &= \alpha_4 I\sigma_r + \beta_4 I\sigma_\theta \\ \mathsf{R}(\sigma_\theta) &= \alpha_2 \sigma_\theta + \beta_2 \sigma_r & \mathsf{R}(I\sigma_\theta) &= \alpha_5 I\sigma_\theta + \beta_5 I\sigma_r \\ \mathsf{R}(\sigma_\phi) &= \alpha_3 \sigma_\phi & \mathsf{R}(I\sigma_\phi) &= \alpha_6 I\sigma_\phi, \end{aligned} \tag{45}$$

where each of the α_i and β_i are scalar + pseudoscalar combinations. We can now clearly see how a complex structure emerges based on the spacetime pseudoscalar. On writing out of the four equations (43) in full, and pre-multiplying each equation by γ_j, we obtain the four equations

$$\begin{aligned} \sigma_r \mathsf{R}(\sigma_r) + \sigma_\theta \mathsf{R}(\sigma_\theta) + \sigma_\phi \mathsf{R}(\sigma_\phi) &= 0 \\ \sigma_r \mathsf{R}(\sigma_r) - I\sigma_\theta \mathsf{R}(I\sigma_\theta) - I\sigma_\phi \mathsf{R}(I\sigma_\phi) &= 0 \\ -I\sigma_r \mathsf{R}(I\sigma_r) + \sigma_\theta \mathsf{R}(\sigma_\theta) - I\sigma_\phi \mathsf{R}(I\sigma_\phi) &= 0 \\ -I\sigma_r \mathsf{R}(I\sigma_r) - I\sigma_\theta \mathsf{R}(I\sigma_\theta) + \sigma_\phi \mathsf{R}(\sigma_\phi) &= 0. \end{aligned} \tag{46}$$

Summing the final three equations and employing the first we obtain

$$I\sigma_r \mathsf{R}(I\sigma_r) + I\sigma_\theta \mathsf{R}(I\sigma_\theta) + I\sigma_\phi \mathsf{R}(I\sigma_\phi) = 0. \tag{47}$$

Substituting this back into the final three equations we obtain the simple relation

$$\mathsf{R}(I\sigma_i) = I\mathsf{R}(\sigma_i), \qquad (i = r, \theta, \phi). \tag{48}$$

This equation shows that the vacuum Riemann tensor (and the Weyl tensor in general) is linear on the pseudoscalar. This is the origin of the complex structure often employed in analysing the Weyl tensor. Equation (48) then sets

$$\alpha_1 = \alpha_4, \quad \alpha_2 = \alpha_5, \quad \alpha_3 = \alpha_6, \quad \beta_1 = \beta_4 \quad \beta_2 = \beta_5. \tag{49}$$

Equation (47) says that, viewed as a complex linear function, the vacuum Riemann tensor is symmetric and traceless. The most general form of $\mathsf{R}(B)$ allowed for vacuum axisymmetric solutions therefore has the form

$$\begin{aligned} \mathsf{R}(\sigma_r) &= \alpha_1 \sigma_r + \beta \sigma_\theta \\ \mathsf{R}(\sigma_\theta) &= \alpha_2 \sigma_\theta + \beta \sigma_r \\ \mathsf{R}(\sigma_\phi) &= -(\alpha_1 + \alpha_2) \sigma_\phi. \end{aligned} \tag{50}$$

We can simplify equation (50) further by recalling the Lorentz gauge freedom present in our setup. Under a Lorentz gauge transformation parameterised by the rotor R the Riemann tensor transforms according to

$$\mathsf{R}(B) \mapsto \mathsf{R}'(B) = R\mathsf{R}(\tilde{R}BR)\tilde{R}. \tag{51}$$

In our axisymmetric setup the rotor R is restricted to the form of equation (36). Under this transformation we find that β transforms to

$$\beta' = \cos(2w)\beta - \tfrac{1}{2}(\alpha_1 - \alpha_2)\sin(2w) \tag{52}$$

where the scalar + pseudoscalar combination w is treated as a complex number. We can therefore eliminate the off-diagonal term from $\mathsf{R}(B)$ by setting $\tan(w) = 2\beta/(\alpha_1 - \alpha_2)$. This fixes w and so removes the rotational degrees of freedom. The conclusion is that the Riemann tensor for the vacuum outside an axisymmetric source can be written

$$\begin{aligned}
\mathsf{R}(\sigma_r) &= \alpha_1\sigma_r \\
\mathsf{R}(\sigma_\theta) &= \alpha_2\sigma_\theta \\
\mathsf{R}(\sigma_\phi) &= -(\alpha_1 + \alpha_2)\sigma_\phi.
\end{aligned} \tag{53}$$

This can be compared with equation (39) to obtain a set of 20 equations, which naturally couple into 10 scalar + pseudoscalar combinations. Four of these equations return explicit formulae for the action of L_r and L_θ on G, T, J, S, H, K,

$$\begin{aligned}
L_r(S+IK) &= -(\bar{S}+I\bar{K})(T+IJ) + T(S+IK) - I(K+\bar{K})(G+IJ) \\
L_r(H+IK) &= -(\bar{S}+I\bar{K})(G+IJ) - G(H+IK) + I(K-\bar{K})(T+IJ) \\
L_\theta(G+IJ) &= (\bar{G}+I\bar{J})(H+IK) - H(G+IJ) + I(J+\bar{J})(S+IK) \\
L_\theta(T+IJ) &= (\bar{G}+I\bar{J})(S+IK) + S(T+IJ) - I(J-\bar{J})(H+IK).
\end{aligned} \tag{54}$$

A further four equations return the remaining coframe derivatives of G, T, J, S, H, K, but also include terms from the Riemann tensor

$$\begin{aligned}
L_r(G+IJ) &= (\bar{S}+I\bar{K})(H+IK) - G(G+IJ) - I(K-\bar{K})(S+IK) + \alpha_2 \\
L_r(T+IJ) &= (S+IK)(\bar{S}+I\bar{K}) + T(T+IJ) + I(K+\bar{K})(H+IK) - \alpha_1 \\
L_\theta(S+IK) &= -(\bar{G}+I\bar{J})(T+IJ) + S(S+IK) + I(J-\bar{J})(G+IJ) - \alpha_2 \\
L_\theta(H+IK) &= -(\bar{G}+I\bar{J})(G+IJ) - H(H+IK) - I(J+\bar{J})(T+IJ) + \alpha_1.
\end{aligned} \tag{55}$$

The final equations give

$$\alpha_1 + \alpha_2 = (G+IJ)(T+IJ) + (S+IK)(H+IK), \tag{56}$$

and

$$L_r(\bar{G}+I\bar{J}) - L_\theta(\bar{S}+I\bar{K}) + \bar{G}(\bar{G}+I\bar{J}) + \bar{S}(\bar{S}+I\bar{K}) = \alpha_1 + \alpha_2. \tag{57}$$

So far we are some way short of a fully determined system. The next step is to impose the Bianchi identities.

5 The bracket structure and Bianchi identities

The Lie bracket structure of equation (28) gives rise to a series of higher order constraints. This information is summarised in the Bianchi identities. For vacuum solutions the Bianchi identities take the simple form [10]

$$\gamma^i\Big(L_i(\mathsf{R}(B)) - \mathsf{R}(L_iB) + \omega_i \times \mathsf{R}(B) - \mathsf{R}(\omega_i \times B)\Big) = 0, \tag{58}$$

which holds for any bivector B. The linearity on I and the properties of the Riemann tensor imply that we only obtain new information for $B = \sigma_r$ and σ_ϕ. At this point it is useful to set

$$\begin{aligned} \alpha &= \alpha_1 + \alpha_2 \\ \delta &= \alpha_1 + 2\alpha_2. \end{aligned} \tag{59}$$

The reason for this choice is that $\delta = 0$ corresponds to a type D solution. Applying equation (58) to $B = \sigma_\phi$ yields the pair of equations

$$\begin{aligned} L_r\alpha &= -(3\alpha - \delta)(G + IJ) + \delta(T + IJ) \\ L_\theta\alpha &= (3\alpha - \delta)(S + IK) - \delta(H + IK). \end{aligned} \tag{60}$$

These are entirely consistent with equations (54), (55) and (56) so contain no new information. Applied to $B = \sigma_r$, however, the Bianchi identity does provide two new equations,

$$\begin{aligned} L_r\delta &= -(3\alpha - \delta)(G + IJ) + 2\delta(T + IJ) + (3\alpha - 2\delta)(\bar{G} + I\bar{J}) \\ L_\theta\delta &= (3\alpha - \delta)(S + IK) - 2\delta(H + IK) - (3\alpha - 2\delta)(\bar{S} + I\bar{K}). \end{aligned} \tag{61}$$

We now have expressions for many of the coframe derivatives of the main physical variables. These derivatives must all be consistent with the bracket structure, which reduces to the single identity

$$[L_r, L_\theta] = -\bar{S}L_r - \bar{G}L_\theta. \tag{62}$$

This identity already holds for all of the unbarred fields. The identity can also be applied to δ and yields one further equation,

$$\begin{aligned} &(3\alpha - 2\delta)\Big(L_r(\bar{S} + I\bar{K}) + L_\theta(\bar{G} + I\bar{J}) + \bar{G}(\bar{S} + I\bar{K}) - \bar{S}(\bar{G} + I\bar{J})\Big) = \\ &\quad 3\alpha\Big((S + H + 2IK)(G + IJ - \bar{G} - I\bar{J}) - (G + T + 2IJ)(S + IK - \bar{S} - I\bar{K})\Big). \end{aligned} \tag{63}$$

But this is as far as the equations can be developed without further physical information. There are no further expressions on which to evaluate the bracket identity, and we do not have a complete set of coframe derivatives. That is, the vacuum structure is currently under-determined. This is to be expected as there is no unique vacuum solution outside a rotating star. To obtain a unique solution we must either impose suitable boundary conditions, or make a further restriction on the form of the Riemann tensor. For example, we could set J, $\bar{J}$, K and $\bar{K}$ to zero. The effect of this is to set α and δ to real variables, so that the equations reduce dramatically. The equations then describe the fields outside a static, axisymmetric source. That is, a non-rotating lump of matter which axial symmetry. This setup was first discussed by Weyl in 1917 [15, 1]. But before we embark on solving the general equations for a particular system, we give a comparison of our equations with the NP formalism.

6 The Newman–Penrose formalism and complex structures

In the present scheme the field equations for axisymmetric vacuum fields are summarised by equations (54), (55), (56), (57), (61) and (63), together with the bracket identity (62). There is clearly a close analogy between these equations and the NP formalism. The first point to note is that all equations now consist of scalar + pseudoscalar combinations. As all other algebraic elements have been removed, the only remaining effect of the Clifford pseudoscalar I is to provide a complex structure through the result (5) that $I^2 = -1$. We can therefore systematically replace I by the unit imaginary i,

$$I \mapsto i. \tag{64}$$

This is quite helpful typographically, as it distinguishes field variables from the the (constant) pseudoscalar. We therefore employ this device at various points. It must be remembered, however, that when we construct the full solution in terms of the ω_i, the imaginary i must be replaced by the pseudoscalar.

Clearly, part of the complex structure in the NP formalism has its geometric origin in the properties of the spacetime pseudoscalar. This unites the complex structure with spacetime duality. But the starting point for the NP formalism is a complex null tetrad. We define the four (Minkowski) null vectors

$$\begin{aligned} l &= \frac{1}{\sqrt{2}}(\gamma_t + \gamma_\phi) & m &= \frac{1}{\sqrt{2}}(\gamma_r + i\gamma_\theta) \\ n &= \frac{1}{\sqrt{2}}(\gamma_t - \gamma_\phi) & \bar{m} &= \frac{1}{\sqrt{2}}(\gamma_r - i\gamma_\theta). \end{aligned} \tag{65}$$

If these vectors replace the $\{\gamma_i\}$ in equation (19) the result is a complex null tetrad. The differential operators defined by the null tetrad are, following the conventions of Kramer *et al.* [1],

$$\begin{aligned} D &= \frac{1}{\sqrt{2}}(L_t + L_\phi) & \delta &= \frac{1}{\sqrt{2}}(L_r + iL_\theta) \\ \Delta &= \frac{1}{\sqrt{2}}(L_t - L_\phi) & \delta^* &= \frac{1}{\sqrt{2}}(L_r - iL_\theta). \end{aligned} \tag{66}$$

One can form bracket identities on these derivatives to obtain each of the spin coefficients in terms of the fields in the ω_i. For example, we find that

$$[\Delta, D] = (\gamma + \gamma^*)D + (\epsilon + \epsilon^*)\Delta - (\tau^* + \pi)\delta - (\tau + \pi^*)\delta^* = [L_t, L_\phi] = 0, \tag{67}$$

where γ, ϵ, τ and π are spin coefficients. Continuing in this manner we obtain expressions for each of the spin coefficients. These include the relations

$$\tau = \frac{1}{2\sqrt{2}}\big(T - G + i(S - H)\big) \tag{68}$$

and

$$\nu = -\frac{1}{2\sqrt{2}}\big(G + T + 2K + i(2J - S - H)\big). \tag{69}$$

There are two problems with the NP formalism. The first is that the key variables (in the spin coefficients) are inappropriate combinations of the natural variables in the ω_i. The second is that there are two distinct complex structures at work. One is geometric, and has its origin in the pseudoscalar; the other is purely formal and is inserted by hand in the null tetrad. If both of these are represented by the same unit imaginary i then the true geometric structure is lost. This is a fundamental flaw, which compromises our ability to solve the equations.

As an aside, it is possible to formulate the null tetrad (65) in terms of a real Clifford algebra. In this case the entire complex structure is provided by the pseudoscalar. A consequence of this is that the null tetrad consists of vector and trivector combinations [16]. This is appropriate for some encodings of supersymmetry, but is not suitable for analysing the gravitational equations, where the null tetrad must consist of complex vectors.

7 Type D vacuum solutions

In order to take our method on to a solution we need to impose some extra constraints on the vacuum fields. Ideally one would like to impose boundary conditions on some fixed 2-surface representing the edge of a rotating source, and then propagate the fields out into the exterior region. Here we follow a slightly different approach and restrict the algebraic form of the Riemann tensor to type D. The relationship between this restriction and the existence of a horizon is described in section (13).

Viewed as a complex linear function the eigenvalues of the Riemann tensor (53) are α_1, α_2 and $-(\alpha_1+\alpha_2)$. This reduces to a type D tensor if two of the eigenvalues are equal. One way of achieving this is to set $\alpha_1 = \alpha_2$. But for this case the equation structure collapses to a simple system which is not asymptotically flat. The remaining two possibilities are gauge equivalent, so we can restrict to type D by setting

$$\alpha_2 = -\alpha_1 - \alpha_2. \tag{70}$$

It follows immediately that $\delta = 0$. The Riemann tensor now reduces to

$$\begin{aligned}\mathsf{R}(\sigma_r) &= 2\alpha\sigma_r\\ \mathsf{R}(\sigma_\theta) &= -\alpha\sigma_\theta\\ \mathsf{R}(\sigma_\phi) &= -\alpha\sigma_\phi,\end{aligned} \tag{71}$$

where

$$\alpha = (G + IJ)(T + IJ) + (S + IK)(H + IK). \tag{72}$$

The Riemann tensor can be encoded neatly in the single expression

$$\mathsf{R}(B) = \tfrac{1}{2}\alpha(B + 3\sigma_r B \sigma_r), \tag{73}$$

an expression which is unique to the Clifford algebra formulation.

Returning to equation (61) and setting $\delta = 0$ we see that we must now have

$$\begin{aligned} \bar{G} + I\bar{J} &= G + IJ \\ \bar{S} + I\bar{K} &= S + IK. \end{aligned} \tag{74}$$

This simplification for type D fields explains our notation using barred and unbarred variables. With this simplification we find that the derivative relations (54) and (55) collapse to give

$$\begin{aligned} L_r(G+iJ) &= -(G+iJ)^2 - T(G+iJ) \\ L_r(T+iJ) &= (S+iK)^2 - \big(2(G+iJ)-T\big)(T+iJ) - 2S(H+iK) \\ L_r(S+iK) &= -iJ(S+iK) - 2iK(G+iJ) \\ L_r(H+iK) &= -(G+iJ)(S+iK) - G(H+iK) \end{aligned} \tag{75}$$

and

$$\begin{aligned} L_\theta(S+iK) &= (S+iK)^2 + H(S+iK) \\ L_\theta(H+iK) &= -(G+iJ)^2 + \big(2(S+iK)-H\big)(H+iK) + 2G(T+iJ) \\ L_\theta(G+iJ) &= iK(G+iJ) + 2iJ(S+iK) \\ L_\theta(T+iJ) &= (G+iJ)(S+iK) + S(T+iJ). \end{aligned} \tag{76}$$

We have here adopted the convention of representing the pseudoscalar I with the unit imaginary i. This is a useful device when analysing this type of equation structure with a symbolic algebra package. The above equations are all consistent with the bracket structure, which now reduces to the single identity

$$[L_r, L_\theta] = -SL_r - GL_\theta. \tag{77}$$

This set of equations is now complete — we have explicit forms for the intrinsic derivatives of all of our variables, these are all consistent with the bracket structure, and the full Bianchi identities are all satisfied. Obtaining a set of 'intrinsic' equations such as these is a key step in our method.

The equations (75) and (76) exhibit a remarkable symmetry. The L_r and L_θ derivatives can be obtained from each other through the interchange

$$\begin{aligned} &G \leftrightarrow S \qquad && J \leftrightarrow K \\ &T \leftrightarrow H \qquad && L_r \leftrightarrow -L_\theta. \end{aligned} \tag{78}$$

The origin of this symmetry is explained in Section (12), where it is related to the conjugacy transformation discussed by Chandrasekhar [4].

8 The Schwarzschild solution

Before proceeding to the Kerr solution, it is helpful to look at how the Schwarzschild solution solution fits into our scheme. The restriction to spherical symmetry sets all of a_2, b_2, c_2 and d_2 to zero, and we also require that $d_1 = c_1$. The remaining functions, a_1, b_1 and c_1 are all functions of r only. The bracket relations now tell us that the only remaining functions in ω_i are T, $G = \bar{G}$ and H, with H given by

$$H = \frac{c_1 \cos\theta}{r \sin\theta}. \tag{79}$$

We are therefore left with the pair of equations

$$\begin{aligned} L_r G &= -G^2 - GT \\ L_r T &= -2GT + T^2, \end{aligned} \tag{80}$$

where $L_r = b_1 \partial_r$. Dividing these, and solving the resulting homogeneous differential equation, we obtain

$$GT = \lambda_0 (G^2 - 2GT)^{3/2}, \tag{81}$$

where λ_0 is an arbitrary constant. A significant point is that this relation is derived without fixing the diffeomorphism gauge. The relationship is an intrinsic feature of the solution.

The Riemann invariant α is now simply GT, and satisfies

$$L_r \alpha = -3G\alpha. \tag{82}$$

The equation for $L_\theta H$ reduces to the algebraic expression

$$\frac{c_1^2}{r^2} = G^2 - 2GT. \tag{83}$$

If follows that

$$\alpha = GT = \lambda_0 \frac{{c_1}^3}{r^3}. \tag{84}$$

We retain the diffeomorphism freedom to perform an r-dependent transformation, which we can employ to fix the functional dependence of one of our variables. The obvious way to employ this freedom now is to set $c_1 = 1$. This ensures that r defines the proper circumference of a sphere in the standard Euclidean manner. The gauge choice also ensures that the tidal force, controlled by α, falls off as r^{-3}, which agrees with the Newtonian result. It is then clear, from comparison with the Newtonian result, that λ_0 can be identified with the mass, so

$$\alpha = \frac{M}{r^3}. \tag{85}$$

This ensures that the status of r is lifted from being an arbitrary coordinate to a physical field controlling the magnitude of the tidal force.

Equation (82) now tell us that G is given by

$$G = \frac{b_1}{r} \tag{86}$$

and equation (80) becomes

$$b_1 \frac{\partial}{\partial r}\left(\frac{b_1}{r}\right) + \frac{{b_1}^2}{r^2} = \frac{M}{r^3}. \tag{87}$$

This integrates to give

$${b_1}^2 = c_0 - 2M/r \tag{88}$$

where c_0 is the arbitrary constant of integration. A combination of a further constant rescaling of r and a global scale change can be used to set $c_0 = 1$, so that g_r tends to a flat space polar coordinate vector at large r.

Finally, we recover a_1 from the $[L_t, L_r]$ bracket relation, which yields

$$L_r a_1 = T a_1. \tag{89}$$

Since $L_r b_1 = -T b_1$, it follows that $a_1 b_1$ must be constant. This constant can be gauged to 1 by a constant rescaling of the time coordinate t, so we have

$$a_1 = (1 - 2M/r)^{-1/2}. \tag{90}$$

This solves for all of the terms in $\{g^\mu\}$, and calculating the associated metric recovers the Schwarzschild line element.

This derivation illustrates a number of the key features which reappear in the derivation of the Kerr solution. The intrinsic differential structure can be taken almost completely through to solution without ever introducing any coordinatisation. Coordinates are only fixed in the final step, when one solves for the metric coefficients. For the Schwarzschild case this results in equation (87), which is solved directly be integration and could have been computed numerically had a simple analytic solution not been available. All of the remaining terms are found algebraically, or by integration alone. Another feature is the employment of any residual gauge freedom to set the asymptotic conditions so that the $\{g_\mu\}$ vectors map to a flat space coordinate frame as $r \mapsto \infty$. Finally, the form of the solution shows that the fields are only valid for $r > 2M$. Our initial ansatz (17) is not sufficiently general to describe a global solution when a horizon is present. This problem is only resolved by introducing further terms to allow for a t–r coupling [10].

9 Simplifying the Kerr equations

We are now in a position to complete the derivation of the Kerr solution. Our approach is to construct a solution which is asymptotically flat at large r, for all angles. This condition turns out to be sufficient to construct a unique solution, which is then easily seen to be the Kerr solution.

The first step in solving the equation structure of (75) and (76) is the identification of a number of integrating factors. If a pair of variables A and B satisfy the equation

$$L_\theta A - L_r B = GB + SA \tag{91}$$

then an integrating factor x exists defined (up to an arbitrary magnitude) by

$$L_r x = Ax, \qquad L_\theta x = Bx. \tag{92}$$

Equation (91) ensures that the definition of the integrating factor is consistent with the bracket structure (77). This means that, when a coordinatisation is chosen, the equations for the partial derivatives of x are consistent, and x can then be computed by straightforward integration.

The first integrating factor is constructed from α. The factor of 3 in equation (60) prompts us to define

$$Z = Z_0 \alpha^{-1/3}, \tag{93}$$

where Z_0 is an arbitrary complex constant. From equation (60), Z satisfies

$$\begin{aligned} L_r Z &= (G + iJ) Z \\ L_\theta Z &= -(S + iK) Z. \end{aligned} \tag{94}$$

We can therefore use Z to simplify a number of our equations. Separating into modulus X and argument Y,

$$Z = X e^{iY}, \tag{95}$$

we find that

$$\begin{aligned} L_r X &= G X \\ L_\theta X &= -S X, \end{aligned} \tag{96}$$

which provides an integrating factor for G and S. This is particularly important, as a comparison with the bracket relation (77) now tells us that

$$[X L_r, X L_\theta] = 0. \tag{97}$$

Recovering a pair of commuting derivatives like this tells us that we should fix our displacement gauge freedom by setting $b_2 = c_2 = 0$. With this done, we can write

$$\begin{aligned} X L_r &= b(r) \partial_r \\ X L_\theta &= c(\theta) \partial_\theta, \end{aligned} \tag{98}$$

where $b(r)$ and $c(\theta)$ are arbitrary functions which we can choose with further gauge fixing.

A search through the remaining derivatives reveals that the pair T and $-H$ satisfy an equation of the form of (91). We can therefore introduce a further integrating factor, F, with the properties

$$L_r F = T F, \qquad L_\theta F = -H F. \tag{99}$$

Both Z and F are unchanged by the conjugacy operation of equation (78).

Now that we have suitable integrating factors at our disposal, we can considerably simplify our equations for $G + iJ$ and $S + iK$ to read

$$\begin{aligned} L_r\big(FZ(G + iJ)\big) &= 0 \\ L_\theta\big(FZ(S + iK)\big) &= 0 \\ L_r\big(XZ(S + iK)\big) &= 2XZ(SG + JK) \\ L_\theta\big(XZ(G + iJ)\big) &= -2XZ(SG + JK) \end{aligned} \tag{100}$$

These equations retain the symmetry described by equation (78). Equations (100) now focus attention on the quantity $SG + JK$. Computing the derivatives of this, we find that

$$\begin{aligned} L_r(SG + JK) &= -(G + T)(SG + JK) \\ L_\theta(SG + JK) &= (H + S)(SG + JK). \end{aligned} \tag{101}$$

It follows that

$$L_r\big(XF(SG + JK)\big) = L_\theta\big(XF(SG + JK)\big) = 0, \tag{102}$$

and hence that $XF(SG+JK)$ is a constant. It turns out that the equations can be solved if this constant is non-zero, but the solutions require infinite disks as sources and are not asymptotically flat in all directions. These solutions are presented elsewhere. Since we are interested here in asymptotically flat vacuum solutions we must set this constant to zero, which implies that

$$SG + JK = 0. \tag{103}$$

Equations (100) now yield

$$L_r\big(XFZ^2(G + iJ)(S + iK)\big) = L_\theta\big(XFZ^2(G + iJ)(S + iK)\big) = 0. \tag{104}$$

It follows that

$$XFZ^2(G + iJ)(S + iK) = C_1, \tag{105}$$

where C_1 is an arbitrary complex constant.

Remarkably, we are now in a position to separate the remaining equations into radial and angular derivatives. With L_r and L_θ defined by equation (98), we see that we can set

$$\begin{aligned} FZ(G + iJ) &= W(\theta) \\ FZ(S + iK) &= U(r). \end{aligned} \tag{106}$$

We therefore have

$$XW(\theta)U(r) = C_1 F, \tag{107}$$

and

$$\begin{aligned} XZ(G + iJ) &= C_1/U(r) \\ XZ(S + iK) &= C_1 W(\theta). \end{aligned} \tag{108}$$

This provides a structure consistent with our derivatives of $G + iJ$ and $S + iK$.

We next form

$$\frac{W(\theta)}{U(r)} = \frac{G + iJ}{S + iK} = i\frac{SJ - GK}{S^2 + K^2}, \tag{109}$$

which is a pure imaginary quantity. This implies that W and U are $\pi/2$ out of phase. Since these are separately functions of r and θ, their phases must be constant. Returning to the derivatives of Z we see that

$$XL_rZ = b(r)\partial_r Z = XZ(G + iJ) = C_1/U(r) \tag{110}$$

and

$$XL_\theta Z = c(\theta)\partial_\theta Z = XZ(S + iK) = C_1/W(\theta). \tag{111}$$

It follows that Z must be the sum of a function of r and a function of θ. Furthermore, these functions must also have constant phases $\pi/2$ apart. Since the overall phase of Z is arbitrary (Z was defined up to an arbitrary complex scale factor), we can write

$$Z = R(r) + i\Psi(\theta) \tag{112}$$

where $R(r)$ and $\Psi(\theta)$ are real functions. With Z reduced to this simple form we can solve the remaining equations by finite power series.

10 Power series solution

With Z given by equation (112), we now have

$$XZ(G + iJ) = XL_r Z = XL_r R = R' \tag{113}$$

and

$$XZ(S + iK) = -XL_\theta Z = -iXL_\theta \Psi = -i\dot{\Psi}. \tag{114}$$

Here we have introduced the abbreviations

$$\begin{aligned} R' &= XL_r R \\ \dot{\Psi} &= XL_\theta \Psi. \end{aligned} \tag{115}$$

These help to encode the fact that XL_rR is a function of r only, and $XL_\theta\Psi$ is a function of θ only. The functions implied in the R' and $\dot{\Psi}$ notation are not necessarily pure partial derivatives, though they can be chosen as such with suitable gauge choices.

We now have

$$\frac{F}{X} = \frac{i}{C_1 R' \dot{\Psi}}, \tag{116}$$

which implies that C_1 is an imaginary constant. Since F is only defined up to an arbitrary scaling, we can choose this constant as i and write

$$\frac{F}{X} = \frac{1}{R'\dot{\Psi}}. \tag{117}$$

Differentiating this relation now yields

$$X(T - G) = X\frac{X}{F}L_r\frac{F}{X} = -\frac{XL_rR'}{R'} = -\frac{R''}{R'} \tag{118}$$

and

$$X(H - S) = -X\frac{X}{F}L_\theta\frac{F}{X} = \frac{XL_\theta\dot{\Psi}}{\dot{\Psi}} = \frac{\ddot{\Psi}}{\dot{\Psi}}, \tag{119}$$

where we have introduced the obvious notation R'' and $\ddot{\Psi}$ for the second derivatives.

The final relation we need to satisfy is to construct α from R and Ψ and equate this with $(Z_0/Z)^3$. On writing

$$\alpha = (G + iJ)^2 + (T - G)(G + iJ) + (S + iK)^2 + (H - S)(S + iK) \tag{120}$$

we see that

$$
\begin{aligned}
X^2Z^2\alpha &= R'^2 - R''(R+i\Psi) - \dot{\Psi}^2 - i\ddot{\Psi}(R+i\Psi) \\
&= R'^2 - RR'' - \dot{\Psi}^2 + \Psi\ddot{\Psi} - i(R''\Psi + \ddot{\Psi}R).
\end{aligned} \tag{121}
$$

This must be equated with

$$
X^2Z^2\alpha = {Z_0}^3X^2/Z = {Z_0}^3Z^* = {Z_0}^3(R-i\Psi) \tag{122}
$$

with ${Z_0}^3$ an arbitrary complex constant. These are the only differential relations we need to satisfy in order to obtain a valid solution.

We could proceed now by fixing the remaining displacement gauge freedom and specifying forms for R and Ψ, or $d(r)$ and $c(\theta)$. While this approach certainly works, we can in fact do better by continuing to work in our abstract, intrinsic fashion. The key is to notice that R'^2 and $\dot{\Psi}^2$ must be power series in R and Ψ respectively in order for the right-hand side of equation (121) to be linear in R and Ψ. We therefore set

$$
\begin{aligned}
R'^2 &= k_2R^2 + k_1R + k_0 \\
\dot{\Psi}^2 &= l_2\Psi^2 + l_1\Psi + l_0,
\end{aligned} \tag{123}
$$

where the k_i and l_i are a set of six constants (any higher order terms vanish). These equations are differentiated implicitly to yield

$$
\begin{aligned}
R'' &= k_2R + k_1/2 \\
\ddot{\Psi} &= l_2\Psi + l_1/2.
\end{aligned} \tag{124}
$$

We now substitute these series into equation (121). In order to remove the constant and $R\Psi$ terms we must set

$$
l_0 = k_0, \qquad l_2 = -k_2. \tag{125}
$$

The remaining terms factorise to yield

$$
X^2Z^2\alpha = \tfrac{1}{2}(k_1 - il_1)(R - i\Psi), \tag{126}
$$

which now satisfies equation (122). We have therefore solved all of the equations 'intrinsically'. This process has revealed the presence of an arbitrary complex constant controlling the Weyl tensor, which must be understood physically. The remaining tasks are to pick a suitable coordinatisation and to integrate the bracket relations to reconstruct the g^μ vectors and the metric.

11 Integrating the bracket structure

In arriving at an intrinsically-defined power series solution we have completed much of the work required to generate a solution. In particular, we can now integrate the bracket relations (28) to solve directly for the coframe vectors, and hence for the metric. In so doing we have to include a number of constants of integration. Some of these are gauge artifacts and others can be removed by imposing either elementary or asymptotic flatness. The end result of this process, as we show below, is that the constant l_1 must be set to zero to achieve a solution which is well-defined everywhere outside the horizon, and asymptotically flat. A choice of position gauge can then be employed to set

$$\alpha = -\frac{M}{(r - iL\cos\theta)^3}. \tag{127}$$

This could then be used to find the coframe vectors directly, but it is more instructive to see how the bracket structure integrates intrinsically.

The bracket relations (28) produce a series of relations for a_1, a_2, b_1 and b_2. By applying the known integrating factors these become

$$\begin{aligned} L_r\left(\frac{1}{F}a_1\right) &= \frac{2K}{F}\frac{a_2}{r\sin\theta} & L_\theta(Xa_1) &= 0 \\ L_r\left(X\frac{a_2}{r\sin\theta}\right) &= 0 & L_\theta\left(\frac{1}{F}\frac{a_2}{r\sin\theta}\right) &= \frac{2J}{F}a_1 \\ L_r\left(X\frac{d_1}{r\sin\theta}\right) &= 0 & L_\theta\left(\frac{1}{F}\frac{d_1}{r\sin\theta}\right) &= \frac{2J}{F}d_2 \\ L_r\left(\frac{1}{F}d_2\right) &= \frac{2K}{F}\frac{d_1}{r\sin\theta} & L_\theta(Xd_2) &= 0. \end{aligned} \tag{128}$$

To simplify these equations (128) we first form

$$\frac{2J}{XF} = \frac{2J}{X^2}R'\dot{\Psi} = -\frac{2\Psi\dot{\Psi}}{X^5}R'^2 = L_\theta\Big(\frac{R'^2}{X^2}\Big) \tag{129}$$

and

$$\frac{2K}{XF} = \frac{2K}{X^2}R'\dot{\Psi} = -\frac{2RR'}{X^5}\dot{\Psi}^2 = L_r\Big(\frac{\dot{\Psi}^2}{X^2}\Big). \tag{130}$$

The set of equations (128) therefore integrate simply to yield

$$\begin{aligned} \frac{a_1R'}{X} - \frac{a_2\dot{\Psi}}{Xr\sin\theta} &= \delta_1 \\ \frac{d_2R'}{X} - \frac{d_1\dot{\Psi}}{Xr\sin\theta} &= \epsilon_1 \end{aligned} \tag{131}$$

where δ_1 and ϵ_1 are two arbitrary constants. But we know that Xa_1 and Xd_2 are independent of θ, and $Xd_1/(r\sin\theta)$ and $Xa_2/(r\sin\theta)$ are independent of r. We can therefore separate the above relations and write

$$\begin{aligned} a_1 &= \frac{\delta_0 + \delta_1R^2}{XR'} & \frac{a_2}{r\sin\theta} &= \frac{\delta_0 - \delta_1\Psi^2}{X\dot{\Psi}} \\ d_2 &= \frac{\epsilon_0 + \epsilon_1R^2}{XR'} & \frac{d_1}{r\sin\theta} &= \frac{\epsilon_0 - \epsilon_1\Psi^2}{X\dot{\Psi}}, \end{aligned} \tag{132}$$

where δ_0 and ϵ_0 are two further constants.

The next problem to address is the behaviour of the solution on the z-axis. The problem here is the ambiguity in the coordinate system at the two poles. The situation can be easily sorted out by converting back to Cartesian coordinates and demanding that the equivalent vectors $\{g^t, g^x, g^y, g^z\}$ are well-defined on the axis. We quickly see that on the axis we must have $a_2 = 0$ and $d_1 = c_1$. Similar considerations hold for the ω_i bivectors. For these we see that $\dot{\Psi}$ must vanish on the axis and so must contain a factor of $\sin\theta$. It follows that

$$l_2\Psi(0)^2 + l_1\Psi(0) + l_0 = l_2\Psi(\pi)^2 + l_1\Psi(\pi) + l_0 = 0. \tag{133}$$

A further requirement from the ω_i bivectors is that we must have

$$\epsilon_0 - \epsilon_1\Psi^2 - \cos\theta\ddot{\Psi} = 0, \qquad \theta = 0, \pi \tag{134}$$

which tells us that $\ddot{\Psi}$, and hence Ψ, must change sign between 0 and π. Since Ψ^2 must have the same value at 0 and π to ensure that a_2 vanishes on the axis, we see that we must have

$$\Psi(0) = -\Psi(\pi). \tag{135}$$

We cannot have $\Psi = 0$ at the poles, so equations (133) and (135) enforce $l_1 = 0$. Solutions with a non-zero value of l_1 can be constructed, but these are only valid in the upper or lower half planes separately. Matching these together introduces an infinite disk of matter as a source, and the solutions are not true vacuum. These will be discussed elsewhere.

With $l_1 = 0$, we now have

$$\ddot{\Psi} = -k_2\Psi. \tag{136}$$

A sensible gauge choice, which produces a well-defined solution for all θ, is to set $c(\theta) = 1$ in equation (98), and scale the solution so that $k_2 = 1$. It then follows that Ψ is a linear combination of $\sin\theta$ and $\cos\theta$. Since $\dot{\Psi}$ vanishes at the poles, we must have

$$\Psi(\theta) = -L\cos\theta, \qquad l_2 = -1, \qquad l_0 = L^2. \tag{137}$$

The Riemann tensor is now controlled by the complex function $k_1(R - iL\cos\theta)^{-3}/2$, which determines the magnitude of the tidal forces experienced in different directions. A large distance from the source we expect this to tend to the Schwarzschild value of $-M/r^3$, so a sensible choice for the radial gauge is

$$R = r, \qquad k_1 = -2M. \tag{138}$$

From equation (98) it follows that

$$b(r) = XL_r r = R', \tag{139}$$

and from equation (123) we have

$$R'^2 = b(r)^2 = r^2 - 2Mr + L^2 = \Delta. \tag{140}$$

Here we have introduced the standard symbol Δ for the radial function $b(r)^2$.

With the preceding gauge choices we now have

$$
\begin{aligned}
G + iJ &= \frac{\Delta^{1/2}}{\rho(r - iL\cos\theta)} & S + iK &= \frac{-iL\sin\theta}{\rho(r - iL\cos\theta)} \\
T - G &= -\frac{r - M}{\rho\Delta^{1/2}} & H - S &= \frac{\cos\theta}{\rho\sin\theta},
\end{aligned}
\tag{141}
$$

where

$$
\rho^2 = X^2 = r^2 + L^2\cos^2\theta. \tag{142}
$$

The equation for T shows that a horizon exists at $\Delta = 0$. We also now have

$$
b_1 = \frac{\Delta^{1/2}}{\rho}, \qquad c_1 = \frac{r}{\rho}. \tag{143}
$$

Similarly, we now have

$$
\begin{aligned}
a_1 &= \frac{1}{\rho\Delta^{1/2}}(\delta_0 + \delta_1 r^2) & a_2 &= \frac{r}{L\rho}(\delta_0 - \delta_1 L^2\cos^2\theta) \\
d_1 &= \frac{r}{L\rho}(\epsilon_0 - \epsilon_1 L^2\cos^2\theta) & d_2 &= \frac{1}{\rho\Delta^{1/2}}(\epsilon_0 + \epsilon_1 r^2).
\end{aligned}
\tag{144}
$$

In order that the g^μ vectors approach a flat space coordinate frame at infinity we set $\delta_1 = 1$ and $\epsilon_1 = 0$. The former corresponds to a constant rescaling in r and the latter is achieved with a constant rotation, both of which are gauge transformations. The on-axis conditions $d_1 = c_1$ and $a_2 = 0$ set $\epsilon_0 = L$ and $\delta_0 = L^2$ respectively. We then finally arrive at the vectors

$$
\begin{aligned}
g^t &= \frac{r^2 + L^2}{\rho\Delta^{1/2}}\gamma^t + \frac{L\sin\theta}{\rho}\gamma^\phi \\
g^r &= \frac{\Delta^{1/2}}{\rho}\gamma^r \\
g^\theta &= \frac{1}{\rho}\gamma^\theta \\
g^\phi &= \frac{1}{\rho\sin\theta}\gamma^\phi + \frac{L}{\rho\Delta^{1/2}}\gamma^t.
\end{aligned}
\tag{145}
$$

The reciprocal set is given by

$$
\begin{aligned}
g_t &= \frac{\Delta^{1/2}}{\rho}\gamma_t - \frac{L\sin\theta}{\rho}\gamma_\phi \\
g_r &= \frac{\rho}{\Delta^{1/2}}\gamma_r \\
g_\theta &= \rho\gamma_\theta \\
g_\phi &= \frac{(r^2 + L^2)\sin\theta}{\rho}\gamma_\phi - \frac{L\Delta^{1/2}\sin^2\theta}{\rho}\gamma_t.
\end{aligned}
\tag{146}
$$

The metric derived from these vectors gives the Kerr solution in Boyer–Lindquist coordinates.

12 Conjugate solutions

At various points in the text we have pointed out the symmetry in the equations described by equation (78). We are now in a position to understand how this comes about, and relate the symmetry to the conjugation operation discussed by Chandrasekhar [4]. We first need the general version of equation (128) applicable when the barred and unbarred variables are not necessarily equal. These are:

$$
\begin{aligned}
L_r \frac{d_1}{r\sin\theta} &= -G\frac{d_1}{r\sin\theta} - (K-\bar{K})d_2 & L_\theta a_1 &= S a_1 - (J-\bar{J})\frac{a_2}{r\sin\theta} \\
L_r \frac{a_2}{r\sin\theta} &= -G\frac{a_2}{r\sin\theta} - (K-\bar{K})a_1 & L_\theta d_2 &= S d_2 - (J-\bar{J})\frac{d_1}{r\sin\theta} \\
L_\theta \frac{d_1}{r\sin\theta} &= -H\frac{d_1}{r\sin\theta} + (J+\bar{J})d_2 & L_r a_1 &= T a_1 + (K+\bar{K})\frac{a_2}{r\sin\theta} \\
L_\theta \frac{a_2}{r\sin\theta} &= -H\frac{a_2}{r\sin\theta} + (J+\bar{J})a_1 & L_r d_2 &= T d_2 + (K+\bar{K})\frac{d_1}{r\sin\theta}.
\end{aligned} \tag{147}
$$

The full conjugacy symmetry is therefore described by

$$
\begin{aligned}
a_1 &\leftrightarrow \frac{d_1}{r\sin\theta} & d_2 &\leftrightarrow -\frac{a_2}{r\sin\theta} \\
L_r &\leftrightarrow -L_\theta & T &\leftrightarrow H \\
G &\leftrightarrow S & \bar{G} &\leftrightarrow \bar{S} \\
J &\leftrightarrow K & \bar{J} &\leftrightarrow \bar{K}.
\end{aligned} \tag{148}
$$

From the form of the Riemann tensor (39) we can see that this interchange simply swaps round the various coefficients in $\mathsf{R}(B)$. For a general, non-vacuum configuration this interchange will affect the Einstein tensor and therefore alter the matter distribution. For a vacuum solution, however, the Riemann tensor reduces to the form of equation (53). This is unchanged under the above transformation, which therefore does yield a new vacuum solution.

In terms of the metric, we already know how the g_t and g_ϕ terms transform. The $L_r \leftrightarrow -L_\theta$ interchange is achieved by

$$
b_1 \leftrightarrow -b_2/r, \qquad c_2 \leftrightarrow -c_1/r. \tag{149}
$$

The effect of these interchanges on the line element (21) is to leave the dr^2, $dr\,d\theta$ and $d\theta^2$ coefficients unchanged, and to transform the remaining terms in the following manner:

$$
Adt^2 + 2Bdt\,d\phi - Cd\phi^2 \leftrightarrow Cdt^2 + 2Bdt\,d\phi - Ad\phi^2. \tag{150}
$$

This is precisely the conjugation operation described by Chandrasekhar [4]. This operation produces a new vacuum solution because the procedure can be viewed mathematically as the complex coordinate transformation $t \leftrightarrow i\phi$. Our approach reveals more clearly the effect of the transformation on the various physical terms in the ω_i bivectors and the Riemann tensor. In particular, the interchange of G and S is likely to be problematic, as S is required to vanish on the z-axis, whereas no such restriction exists for G. This is certainly true of the Kerr solution, for which conjugacy symmetry does not yield a globally well-defined solution.

13 The Carter–Robinson Uniqueness Theorem

One of the most remarkable features of the Kerr solution is that it represents the unique vacuum solution to the Einstein equations which is stationary, axisymmetric, asymptotically flat and everywhere nonsingular outside an event horizon. This result was finally proved by Robinson [17], building on some initial work of Carter [18]. The proof proceeds in an analogous manner to standard uniqueness proofs by constructing an integral over the space outside the horizon. The boundary conditions are then employed carefully to show that a term representing the difference of two solutions must vanish [4]. The proof is highly mathematical, so it is instructive to see how the present formalism highlights what is going on.

So far, we have shown that the Kerr solution is the unique type D axisymmetric vacuum solution which is asymptotically flat and nonsingular outside the horizon. This proof is constructive — we constructed the most general solution with these properties and it turned out to be the Kerr solution. This is useful, as it shows that the role of the horizon is to force the Riemann tensor to be type D. To see how this comes about, we must return to the general set of equations before the restriction to a type D vacuum was made. These are embodied in equations (54), (55), (56), (57), (61) and (63), together with the bracket identity (62).

Our first task is to identify the horizon. The definition of the horizon is the surface over which the bivector defined by the Killing vectors g_t and g_ϕ is null. The equation for the horizon is therefore

$$(g_t \wedge g_\phi)^2 = 0. \tag{151}$$

The volume element defined by the $\{g_\mu\}$ frame does not vanish anywhere, so an equivalent condition is

$$(g^r \wedge g^\theta)^2 = 0. \tag{152}$$

Since we are free to set $c_2 = b_2 = 0$ with a choice of displacement gauge, the horizon in the present setup is characterised by ${g^r}^2 = 0$, or $b_1 = 0$. It follows that, at the horizon, L_r acting on any finite, continuous quantity must vanish.

From the $[L_r, L_\theta]$ bracket relation (28) we see that

$$\bar{G} = -\frac{r}{c_1} L_r\Big(\frac{c_1}{r}\Big), \tag{153}$$

Since c_1 must be finite at the horizon, it follows that $\bar{G}$ vanishes there. We also have

$$\bar{S} = \frac{1}{b_1} L_\theta b_1 = \frac{c_1}{r b_1} \partial_\theta b_1. \tag{154}$$

But $\bar{S}$ cannot be singular at the horizon, as this would imply that the Riemann tensor itself was singular. It follows that b_1 must take the form

$$b_1 = (r - r_0)^\eta \tilde{b}_1(r, \theta) \tag{155}$$

where r_0 defines the horizon, which is now a surface of constant r, η is some arbitrary exponent, and $\tilde{b}_1$ is finite at the horizon.

Next, we consider the force necessary to remain at a constant radius. This is simplest to analyse on the z axis. An observer remaining at rest on the z-axis must apply a force of magnitude T in the outward direction. This must be singular at the horizon, otherwise it would be possible to escape. This argument holds at all angles, and the horizon is therefore also defined (in this gauge) by $T \mapsto \infty$.

The invariants in the Riemann tensor, α and δ must be finite at the horizon, so G and J must be zero there to ensure that the product $T(G+iJ)$ in α is finite. We also know that $L_r\alpha$ and $L_r\delta$ must vanish at the horizon. Since T is singular there, equations (61) force $\delta = 0$ at the horizon. It further follows from the equation for $L_r\delta$ that $\bar{G} + i\bar{J}$ must vanish at the horizon (and hence are equal to their unbarred counterparts). Finally, since the horizon is a surface of constant r, and δ vanishes there, δ must take the form

$$\delta = (r - r_0)^{\eta'} \tilde{\delta}(r, \theta), \tag{156}$$

where η' is a positive exponent and $\tilde{\delta}$ is finite at the horizon. It follows that $L_\theta\delta$ must also vanish at the horizon, which in turn sets $S + iK = \bar{S} + i\bar{K}$ there.

The above argument shows that, at the horizon, the Riemann tensor is of type D and all the barred variables are equal to their unbarred counterparts. These are precisely the conditions met by the Kerr solution. In the light of this result, it is perhaps less surprising that the vacuum outside a horizon is described by the Kerr solution. Of course, this is not a complete proof of uniqueness, but it does serve to make the result more physically natural. One can take the above chain of reasoning further to show that the higher derivatives of δ also vanish at the horizon, from which it can be argued that $\delta = 0$ throughout the exterior vacuum. As shown earlier, this alone is sufficient to restrict to the Kerr solution.

The main weakness in the preceding argument (which is shared with all proofs of the uniqueness of the Kerr solution) is the employment of a 'bad' gauge for the solution. The g^μ vectors and ω_i bivectors are only defined outside the horizon, and various terms go singular on the horizon. To avoid this problem one must work with a more general ansatz for the g^μ vectors which allows some coupling between t and r. Only then can one write down globally valid solutions. It would certainly be useful to have a version of the above argument which does not rely on a bad choice of gauge, and this is a subject for future research.

14 Conclusions

We have introduced a new framework for the study of axisymmetric gravitational fields. The technique draws on results from the gauge treatment of gravity, and builds on earlier work on spherical [10] and cylindrical [19] systems. In this paper we have concentrated on the vacuum equations. Further publications will detail how the technique is applied to matter configurations. We believe that this approach can help make progress in the long-standing problem of finding the fields inside and outside a rotating star. It is also expected that this framework should be helpful in analysing the disk solutions proposed by Pichon and Lynden-Bell [20], and Neugebauer and Meinel [21]. The equations with matter included will be presented elsewhere. A significant feature of these is that a natural set of relations for the fluid velocity field emerge which can be carried into the vacuum sector [22]. This velocity field in turn provides a physical basis for the Ernst equation.

The vacuum equations developed here are under-determined. They are only capable of unique solution when further physical restrictions are applied. The restriction to type D vacua is sufficient to yield a fully-determined set of consistent, ‘intrinsic’ equations. The development of such a set is similar to the process of obtaining an involutive set of equations in the formal theory of partial differential equations [2]. The point of the latter process is that the equations can then be solved by a power series. In this respect, the power series solution of the equations developed here is remarkable. It hints at a power series solution method which does not rely on any definite coordinatisation. This idea warrants serious further investigation.

We have shown that the twin restrictions of a type D vacuum and asymptotic flatness are sufficient to force us to the Kerr solution. The uniqueness theorem for the Kerr solution can therefore be interpreted as showing that the presence of a horizon forces a restriction on the algebraic form of the Riemann tensor. This provides a useful physical understanding of the uniqueness of the Kerr solution. Some indication of how this restriction comes about can be seen in the more general vacuum equations. But work remains to recast the formalism in a gauge which is not singular at the horizon.

A future goal of this approach is to streamline and refine the techniques to the extent where time dependence can be included. It would then be possible to study the formation of rotating black holes, and their spin-up under accretion. Our approach opens up many new possibilities for the study of such systems.

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