Geometric Algebra, Spacetime Physics and Gravitation

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Abstract

Clifford's '*geometric algebra*' is presented as the natural language for expressing geometrical ideas in mathematical physics. Its spacetime version '*spacetime algebra*' is introduced and is shown to provide a powerful, invariant description of relativistic physics. Applications to electromagnetism and gravitation are discussed.

1 Introduction

For some time we have been convinced that *geometric algebra* is the best available language for mathematical physics. Geometric algebra offers a description of physical laws that is independent of any coordinate frame and so separates the physics from coordinate artefacts. It also offers a powerful new tool: a single, associative product of vectors. These *algebraic* properties confer great computational power, but the vectors always retain their *geometric* interpretation as directed line segments.

This paper contains an introduction to the essential ideas of geometric algebra and, in particular, to its spacetime version – the spacetime algebra. We then outline how these ideas can be applied to the central theme of Donald Lynden-Bell's interests: gravitation theory. We start from a very simple idea: that *physical quantities can be represented by fields whose values are functions of position in an ordinary, flat Minkowski vector space.* This approach provides an excellent description of physical phenomena in the absence of gravitational fields. But the mapping between physical events and spacetime vectors is not unique — the physical relationships between fields at a point are unaffected if all the fields are displaced or rotated in the same way. By ensuring that this symmetry also holds for differential relationships we arrive naturally at a gauge theory that describes gravitation. The predictions of this theory agree with those of general relativity for a wide range of phenomena, although the underlying, flat Minkowski space is retained throughout. Differences between the theories emerge over issues such as the global structure of solutions and the interaction with quantum spin.

2 Geometric (Clifford) algebra

2.1 An outline of geometric algebra

Geometric algebra arose from Clifford's attempts to generalise Hamilton's quaternion algebra into a language for vectors in arbitrary dimensions ([1]). Clifford discovered that both complex numbers and quaternions are special cases of an algebraic framework in which vectors have a single, associative product which is distributive over addition. This associative product generates a 'geometric (Clifford) algebra', which is a *graded*, *linear* space, where the grade of an element determines its physical interpretation. For example, the grade 0 elements of space are *real scalars*, and the grade 1 elements are *vectors*, and are to be interpreted as directed line segments. For vectors a and b the 'geometric product' is written simply as ab. A key feature of the geometric product is that the square of any vector is a scalar (the square of its length). By expanding the square of the sum $a+b$,

$$(a+b)^2 = (a+b)(a+b) = a^2 + (ab+ba) + b^2 = a^2 + 2a\cdot b + b^2, \tag{1}$$

we see that the familiar *inner product* can be written as

$$a\cdot b \equiv \tfrac{1}{2}(ab+ba). \tag{2}$$

The remaining antisymmetric part of the the geometric product represents the directed area swept out by displacing a along b. This is the 'outer' product introduced by [2]. The outer product of two vectors is written with a wedge:

$$a\wedge b \equiv \tfrac{1}{2}(ab-ba). \tag{3}$$

The outer product $a\wedge b$ is a directed area, or 'bivector', and has grade 2.

The geometric product combines the inner and outer products:

$$ab = a\cdot b + a\wedge b. \tag{4}$$

By forming further geometric products of vectors the entire geometric algebra is generated. General elements are called 'multivectors' and these decompose into sums of elements of different grades.

2.2 Example: the Pauli algebra

The algebra of ordinary 3-dimensional space is a good illustration of a geometric algebra. The Pauli algebra of space is generated by 3 orthonormal vectors $\{\sigma_1, \sigma_2, \sigma_3\}$, with

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1\ . \tag{5}$$

Products of pairs of these vectors produce 3 independent bivectors $\sigma_1\sigma_2$, $\sigma_2\sigma_3$, and $\sigma_3\sigma_1$, with

$$\sigma_1\sigma_2 = \sigma_1\wedge\sigma_2 = -\sigma_2\sigma_1\ . \tag{6}$$

The square of any of these bivectors is minus one:

$$(\sigma_1\sigma_2)^2 = \sigma_1\sigma_2\sigma_1\sigma_2 = -\sigma_1\sigma_1\sigma_2\sigma_2 = -1\ . \tag{7}$$

The product of all three vectors is a grade-3 'trivector', which we denote by the special symbol $i \equiv \sigma_1\sigma_2\sigma_3$, because it commutes with all vectors and squares to give minus one:

$$i^2 = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3 = (\sigma_1\sigma_2)^2\sigma_3^2 = -1 \ . \tag{8}$$

The other important relations are

$$\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_3\sigma_3 = i\sigma_3 \ ; \quad \sigma_2\sigma_3 = i\sigma_1 \ ; \quad \sigma_3\sigma_1 = i\sigma_2 \ . \tag{9}$$

The full algebra is then 8-dimensional:

$$\begin{array}{cccc} 1 & \{\sigma_1, \sigma_2, \sigma_3\} & \{i\sigma_1, i\sigma_2, i\sigma_3\} & i \\ \text{1 Scalar} & \text{3 vectors} & \text{3 bivectors} & \text{1 trivector} \end{array} \tag{10}$$

The pattern $1+3+3+1 = 8 = 2^3$ is an example of the general result that the geometric algebra for n-dimensional space is 2^n-dimensional. From the relations (9) the algebra of 3-space is isomorphic to the algebra of the Pauli 'spin matrices' used in quantum mechanics. We emphasise, however, that here the $\{\sigma_i\}$ represent a set of orthonormal vectors in space, not the components of an abstract 'vector' in 'spin space'.

2.3 Reflections and rotations

The innovative feature of Clifford's product (4) lies in its mixing of two different types of object: scalars and bivectors. This is not problematic — the addition implied by (4) is precisely that which is implied when a real number is added to an imaginary number to form a complex number. But why should we want to add these two geometrically distinct objects? The answer is provided by considering reflections and rotations. Suppose that the vector a is reflected in the (hyper)plane perpendicular to the unit vector n. The result is the new vector

$$a - 2a\cdot nn = a - (an + na)n = -nan. \tag{11}$$

The utility of the geometric algebra form of the resultant vector, $-nan$, becomes clear when a second reflection is performed. If this second reflection is in the hyperplane perpendicular to the unit vector m, then the combined effect is

$$a \mapsto mnanm. \tag{12}$$

But the combined effect of two reflections is a rotation so, defining the geometric product mn as the scalar plus bivector quantity R, we see that rotations are represented by

$$a \mapsto Ra\tilde{R}. \tag{13}$$

The object R is called a *rotor*. Rotors can be written as an even (geometric) product of unit vectors and satisfy the relation $R\tilde{R} = 1$. The quantity $\tilde{R} = nm$ is called the '*reverse*' of R and is obtained by reversing the order of all geometric products. The representation of rotations afforded by (13) has many advantages over tensor techniques. By defining $\cos\theta \equiv m{\cdot}n$ we can write

$$R = mn = \exp\{\frac{m\wedge n}{|m\wedge n|}\theta/2\}, \tag{14}$$

which relates the rotor R directly to the plane in which the rotation takes place. Equation (14) generalises to arbitrary dimensions the representation of planar rotations afforded by complex numbers. This generalisation provides a good example of how the full geometric product, and the implied sum of objects of different grades, can enter physics at a very basic level. The fact that equation (14) encapsulates a simple geometric relation should dispel the mistaken notion that Clifford algebras are intrinsically 'quantum' in origin. The derivation of (13) has implied nothing about the size of the space being employed, so the formula applies in arbitrary dimensions. The same formula (13) also applies for boosts, except that now $R = \exp(\alpha B/2)$ with B a spatial bivector ($B^2 = 1$).

2.4 The Spacetime Algebra

Of central importance to relativistic physics is the geometric algebra of spacetime, the *spacetime algebra* ([3]). To describe the spacetime algebra (STA) it is helpful to introduce a set of four orthonormal basis vectors $\{\gamma_\mu\}$, $\mu = 0 \ldots 3$, satisfying

$$\gamma_\mu{\cdot}\gamma_\nu = \eta_{\mu\nu} = \mathrm{diag}(+ - - -). \tag{15}$$

The vectors $\{\gamma_\mu\}$ satisfy the same algebraic relations as Dirac's γ-matrices, but they now form a set of four independent basis vectors for spacetime, not four components of a single vector in an internal 'spin-space'.

A frame of timelike bivectors $\{\sigma_k\}$, $k = 1 \ldots 3$ is defined by

$$\sigma_k \equiv \gamma_k\gamma_0, \tag{16}$$

and forms an orthonormal frame of vectors in the space relative to the γ_0 direction. The algebraic properties of the $\{\sigma_k\}$ are the same as those of the Pauli spin matrices, but in the STA they again represent an orthonormal frame of vectors in space and not three components of a vector in spin-space. The highest-grade element (or 'pseudoscalar') is denoted by i and is defined as:

$$i \equiv \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3. \tag{17}$$

The pseudoscalar i must not be confused with the unit scalar imaginary — we are in a space of even dimension so i *anti*commutes with odd-grade elements, and commutes only with even-grade elements. With these definitions, a basis for the 16-dimensional STA is provided by

$$\begin{array}{ccccc} 1 & \{\gamma_\mu\} & \{\sigma_k, i\sigma_k\} & \{i\gamma_\mu\} & i \\ \text{1 scalar} & \text{4 vectors} & \text{6 bivectors} & \text{4 trivectors} & \text{1 pseudoscalar,} \\ \text{grade 0} & \text{grade 1} & \text{grade 2} & \text{grade 3} & \text{grade 4} \end{array} \tag{18}$$

Geometric significance is attached to the preceding relations as follows. An inertial system is completely characterised by a future-pointing timelike (unit) vector. If this is chosen to be the γ_0 direction, then the γ_0-vector determines a map between spacetime vectors $a = a^\mu\gamma_\mu$ and the even subalgebra of the full STA via

$$a\gamma_0 = a_0 + \boldsymbol{a}, \tag{19}$$

where

$$a_0 = a\cdot\gamma_0 \quad \text{and} \quad \boldsymbol{a} = a\wedge\gamma_0. \tag{20}$$

The 'relative vector' $\boldsymbol{a}$ can be decomposed in the $\{\sigma_k\}$ frame and represents a spatial vector as seen by an observer in the γ_0 frame. Equation (19) shows that the algebraic properties of vectors in relative space are determined entirely by the properties of the relativistic STA.

The split of the six spacetime bivectors into relative vectors and relative bivectors is a frame-dependent operation; different observers determine different relative spaces. This fact is well illustrated using the Faraday bivector F. In the γ_0-system, F can be separated into parts which anticommute and commute with γ_0,

$$F = \boldsymbol{E} + i\boldsymbol{B}, \tag{21}$$

where

$$\boldsymbol{E} \equiv \tfrac{1}{2}(F - \gamma_0 F \gamma_0) \quad \text{and} \quad i\boldsymbol{B} \equiv \tfrac{1}{2}(F + \gamma_0 F \gamma_0). \tag{22}$$

Both $\boldsymbol{E}$ and $\boldsymbol{B}$ are spatial vectors in the γ_0-frame, and $i\boldsymbol{B}$ is a spatial bivector. Equation (21) decomposes F into separate electric and magnetic fields, and the explicit appearance of γ_0 in the formulae for $\boldsymbol{E}$ and $\boldsymbol{B}$ reveals how this split is observer-dependent.

We employ natural units ($G = c = \hbar = 1$) and summation convention throughout. Spacetime vectors are usually denoted in lower case Latin, $a = a^\mu \gamma_\mu$, or Greek for basis frame vectors. Given a frame of vectors $\{e_\mu\}$, say, the reciprocal frame is denoted by $\{e^\mu\}$ and satisfies $e^\mu \cdot e_\nu = \delta^\mu_\nu$. The key differential operator is the vector derivative, ∇, defined by

$$\nabla \equiv \gamma^\mu \frac{\partial}{\partial x^\mu} \tag{23}$$

where the $\{x^\mu\}$ are a set of Cartesian coordinates. This definition ensures that ∇ inherits the algebraic properties of a vector, as well as a calculus from the directional derivatives.

3 Electromagnetic field of a point charge

As a simple example of the power of the STA in relativistic physics, we give a compact formula for the fields of a radiating charge. We suppose that a charge q moves along a world-line defined by $x_0(\tau)$, where τ is the proper time for the path. An observer at spacetime position x receives an electromagnetic influence from the charge when it lies on that observer's past light-cone. The vector

$$X \equiv x - x_0(\tau) \tag{24}$$

is the separation vector down the light-cone, joining the observer to the intersection point with the charge's worldline. The condition $X^2 = 0$ defines a map from the spacetime position x to a value of the particle's proper time τ. In this sense we can write $\tau = \tau(x)$ and treat τ as a scalar field.

In the STA the Liénard-Wiechert potential for a charge with an arbitrary velocity $v \equiv dx_0/d\tau$ takes the form

$$A = \frac{q}{4\pi\epsilon_0} \frac{v}{|X \cdot v|}. \tag{25}$$

We now wish to differentiate the potential to find the Faraday bivector. Since the gradient operator is a vector we must take account of its commutation properties. While it is obvious that $\nabla x = 4$, we must also deal with expressions such as $\nabla(ax)$, where a is a constant vector and the ∇ operates only on x. The result is found by anticommuting the x past the a to obtain $ax = 2x\cdot a - xa$, and then differentiating this, giving $\nabla(ax) = -2a$. We can also exploit the fact that the chain rule applies in the STA as in ordinary calculus so that, for example,

$$\nabla x_0 = \nabla\tau\, v. \tag{26}$$

(Here we adopt the convention that, in the absence of brackets, ∇ operates only on the object immediately to its right.)

We can now quickly derive the results needed to assemble the Faraday bivector. First, since

$$0 = \nabla X^2 = (4 - \nabla\tau\, v)X + (-2X - \nabla\tau\, Xv), \tag{27}$$

it follows that

$$\nabla\tau = \frac{X}{X\cdot v}. \tag{28}$$

(Incidentally, finding an explicit expression for $\nabla\tau$ confirms that the particle's proper time can be treated as a scalar field.) Second, we need an expression for the result of $\nabla|X\cdot v|$. Using the results already established we find that

$$\nabla|X\cdot v| = \frac{X\dot{v}X - X + vXv}{2|X\cdot v|}. \tag{29}$$

Combining the above results we now see that

$$\begin{aligned} \nabla A &= \frac{q}{4\pi\epsilon_0}\left\{\frac{\nabla v}{|X\cdot v|} - \frac{1}{|X\cdot v|^2}\nabla|X\cdot v|v\right\} \\ &= \frac{q}{8\pi\epsilon_0|X\cdot v|^3}\left(X\dot{v}vX + Xv - vX\right). \end{aligned} \tag{30}$$

It follows that $\nabla\cdot A = 0$, which confirms that the A field (25) is in the Lorentz gauge. The Faraday bivector can now be written as

$$F = \frac{q}{4\pi\epsilon_0}\frac{X\wedge v + \frac{1}{2}X\Omega_v X}{|X\cdot v|^3}, \tag{31}$$

where $\Omega_v \equiv \dot{v}v$ is the 'acceleration bivector' of the particle. This form of the Faraday bivector is very revealing. It displays a clean split into a velocity term proportional to $1/(\text{distance})^2$ and a long-range radiation term proportional to $1/(\text{distance})$. The first term is exactly the Coulomb field in the rest frame of the charge, and the radiation term,

$$F_{\text{rad}} = \frac{q}{4\pi\epsilon_0} \frac{\frac{1}{2}X\Omega_v X}{|X\cdot v|^3}, \tag{32}$$

is proportional to the rest-frame acceleration projected down the null-vector X.

4 Spacetime algebra and gravitation

The STA provides an ideal language for formulating relativistic physics because the geometric structure of Minkowski spacetime is written into the algebra at its most basic level. Events in spacetime are labelled by vectors x in the STA and physical quantities are modelled by multivector fields $\psi(x)$. We label events with *vectors*, rather than points in a more abstract manifold, so that the algebraic power of the STA is retained when we manipulate them. But the mapping between physical quantities and fields is partly arbitrary. We could, for example, make a new mapping in which physical quantities are represented by new fields $\psi'(x)$ obtained from the old fields $\psi(x)$ by $\psi'(x) \equiv \psi(x+a)$, where a is a constant vector. All physical predictions would be unaffected by such a change. Similarly we could apply a constant rotation to all the the multivector fields $\psi(x)$ that correspond to physical observables and still obtain the same physical predictions.

If the geometric structure of flat spacetime is so deeply embedded in the STA, how can we use it to describe gravitational phenomena? After all, general relativity (GR) starts from the viewpoint that spacetime cannot be flat if gravitational fields are present. As a result GR is developed as a metric theory, with field equations relating the matter fields to second-order derivatives of the metric. Being a second-order theory in a curved space, GR sits uneasily with other modern physical theories, such as QED or QCD, which are all gauge theories constructed in a flat background spacetime. There have been many attempts to cast GR as a gauge theory — the first work which recovered features of GR from a gauging argument was that of [4], who elaborated on an earlier, unsuccessful attempt by [5]. Kibble used the 10-component Poincaré group of passive infinitesimal coordinate transformations as the global symmetry group. By gauging this group, Kibble arrived at a set of gravitational field equations. These were not the Einstein equations, but those of a slightly more general theory, known as a ‘spin-torsion’ theory. Kibble’s use of passive transformations was criticised by [6], who reproduced Kibble's derivation from the standpoint of active transformations of the matter fields, and also arrived at a spin-torsion theory. However, all these attempts regard the gauge fields as determining the curvature and torsion of a Riemann-Cartan spacetime, which then has a dynamical role in the theory.

With the STA at our disposal, we can immediately see how to construct a more satisfactory gauge theory of gravity — we simply demand that the global STA displacement and rotation symmetries be promoted to local symmetries and introduce appropriate gauge fields to ensure this. The principles underlying our theory are therefore summarised as follows:

- The physical content of a field equation in the STA must be invariant under arbitrary local displacements of the fields. (This is called position-gauge invariance.)
- The physical content of a field equation in the STA must be invariant under arbitrary local rotations of the fields. (This is called rotation-gauge invariance.)

In this theory predictions for all physically-measurable quantities, including distances and angles, must be derived from gauge-invariant relations between the field quantities.

Whilst the mapping of fields onto spacetime positions is arbitrary, the fields themselves must be well-defined in the STA. Fields cannot be allowed to be singular except at a few special points. Furthermore, any re-mapping of the fields in the STA must be one-to-one, or we would cut out some region of physical significance. In fact GR does allow analogous operations in which regions of spacetime are removed. These are achieved through the use of singular coordinate transformations and are the origin of a number of differences between our gauge theory and GR.

The result of this approach is a theory containing two separate gauge fields, one for arbitrary displacements in a flat Minkowski spacetime, and one for Lorentz rotations at a point. In [7] the theory is developed as a gauge theory based on the Dirac equation. This approach ensures consistency between our gauge theory of gravity and relativistic quantum theory. The field equations are a set of first-order differential relations between the gauge fields and matter fields. If the matter has no spin, any solution to the field equations can be used to generate a metric which solves the Einstein equations. The converse is not always true, however; metrics exist in GR which cannot be reached from the gauge-theoretical starting point.

Our theory is therefore slightly more restrictive than full GR — although entirely consistent with experiment.

4.1 The gravitational gauge fields and the field equations

The first gravitational gauge field is a position-dependent linear function $\overline{h}(a)$, or $\overline{h}(a, x)$. This is linear in its vector argument a and is a general non-linear function of the position vector x. (The overbar on $\overline{h}(a)$ denotes that this is a linear function of a.) If arbitrary orthogonal and coordinate frames are introduced, the linear function $\overline{h}(a)$ can be used to define a vierbein. The origin of this gauge field is related to displacements, however, and has nothing to do with frames or coordinate systems.

The second gauge field is denoted $\omega(a) = \omega(a, x)$, and is a bivector-valued linear function of its argument a. The position-dependence of $\omega(a)$ is non-linear in general. On defining the derivative operators

$$L_a \equiv a\cdot\overline{h}(\nabla), \tag{33}$$

the $\overline{h}(a)$ and $\omega(a)$ gauge fields are related by

$$[L_a, L_b] = L_c, \tag{34}$$

where, in the absence of spin,

$$c = L_a b + \omega(a)\cdot b - L_b a - \omega(b)\cdot a. \tag{35}$$

Spacetime rotations (*i.e.* boosts and rotations) are generated by a rotor R. In terms of this rotor, a spacetime rotation of a vector a is performed in the standard manner by $a \mapsto Ra\tilde{R}$. Under a local rotation, the gauge fields are defined to transform as:

$$\overline{h}(a) \mapsto \overline{h}'(a) \equiv R\overline{h}(a)\tilde{R}, \tag{36}$$

and

$$\omega(a) \mapsto \omega'(a) \equiv R\omega(\tilde{R}aR)\tilde{R} - 2L_{\tilde{R}aR}R\,\tilde{R}. \tag{37}$$

These transformation laws ensure that the bracket structure (34) remains intact.

Under the displacement $x \mapsto x' \equiv f(x)$ the $\overline{h}(a)$ function is defined to transform as

$$\overline{h}(a, x) \mapsto \overline{h}'(a, x) \equiv \overline{h}(\overline{f}^{-1}(a), f(x)) \tag{38}$$

where $\underline{f}(a) \equiv a\cdot\nabla f(x)$, and the underbar/overbar notation denotes a linear function and its adjoint. This transformation law ensures that a vector field such as $a(x) \equiv \overline{h}(\nabla\phi(x))$, where $\phi(x)$ is a scalar field, transforms covariantly under displacements. That is, if we replace $\phi(x)$ by $\phi'(x) = \phi(x')$ and $\overline{h}$ by $\overline{h}'$ then $a(x)$ transforms simply to $a'(x) = a(x')$. Similarly, the $\omega(a)$ field transforms by changing its position dependence,

$$\omega(a,x) \mapsto \omega'(a,x) \equiv \omega(a, f(x)). \tag{39}$$

Much of the gauge-invariant information is contained in the Riemann tensor, $\mathcal{R}(a\wedge b)$, which is defined by

$$\mathcal{R}(a\wedge b) = L_a\omega(b) - L_b\omega(a) + \omega(a)\times\omega(b) - \omega(c), \tag{40}$$

with c determined by equation (35). The Ricci and Einstein tensors and the Ricci scalar are defined from this by

$$\text{Ricci Tensor:}\quad \mathcal{R}(b) = \gamma_\mu\cdot\mathcal{R}(\gamma^\mu\wedge b) \tag{41}$$

$$\text{Ricci Scalar:}\quad \mathcal{R} = \gamma_\mu\cdot\mathcal{R}(\gamma^\mu) \tag{42}$$

$$\text{Einstein Tensor:}\quad \mathcal{G}(a) = \mathcal{R}(a) - \tfrac{1}{2}a\mathcal{R}. \tag{43}$$

The field equations consist of the bracket relations (34), which can be inverted to solve for $\omega(a)$ in terms of $\overline{h}(a)$, together with the Einstein equation

$$\mathcal{G}(a) = 8\pi\mathcal{T}(a), \tag{44}$$

where $\mathcal{T}(a)$ is the matter stress-energy tensor.

A significant advantage of the STA approach is that all relations are coordinate-free. The spacetime vector x can, of course, be written in terms of its components $\{x^\mu\}$ in the $\{\gamma_\mu\}$ frame, $x = x^\mu\gamma_\mu$, where $x^\mu = x\cdot\gamma^\mu$. But the vector x itself remains *invariant*, whatever coordinate system is chosen. Physical relationships in our theory all have the generic form $a(x) = b(x)$, where $a(x)$ and $b(x)$ are spacetime fields. The physical content of an equality such as this is clearly unaffected by arbitrary displacements and rotations, since the equations $a(x') = b(x')$ and $Ra(x)\tilde{R} = Rb(x)\tilde{R}$ are both also expressions of the same physical equality. Freedom from coordinates also means that the absence of gravitational forces can always be expressed by $\omega(a) \mapsto 0$. Unlike GR, this affords a clean separation between gravitational forces and 'fictitious' forces arising from the choice of coordinate system.

5 Selected aspects of black-hole physics

Rather than attempt a full formal derivation of the properties of black holes in our gauge theory of gravity, we will simply highlight a few aspects. The simplest solution to the gravitational field equations is for a radially-symmetric point source. This solution can be written in the particularly simple form

$$\overline{h}(a) = a - \sqrt{2M/r}\, a \cdot e_r \gamma_0, \tag{45}$$

where $r \equiv |x \wedge \gamma_0|$ and $e_r \equiv \partial_r x$. In this gauge (which we call the 'Newtonian' gauge) the time coordinate $t = x \cdot \gamma_0$ agrees with the time measured by freely-falling observers. The solution (45) thus only differs from the identity (no gravitational fields) by a term governed by the Newtonian free-fall velocity. The properties of black holes are remarkably easy to understand in the Newtonian gauge since Newtonian reasoning need barely be modified. For example, the horizon is located at $r = 2M$ precisely because that is where the free-fall velocity becomes equal to the speed of light.

Of course, one could choose to study the solution (45) in different gauges, but it is possible to show that the solutions always fall into one of two distinct gauge classes. These classes are related by the discrete symmetry of time-reversal. Furthermore, no globally valid solution exists which is symmetric under time-reversal (this is a result of the existence of a horizon in the solution (45)). This is quite different from GR, which does admit eternal, time-reverse symmetric black hole solutions in the form of Kruskal's maximal extension of the Schwarzschild solution.

An interesting way to probe the properties of the solution (45) is to study the field lines from a point charge held at rest outside the horizon. The effect of the gravitational fields on the Maxwell equations can be interpreted as defining the dielectric properties of the vacuum ([7]). The $\boldsymbol{D}$ field then satisfies the familiar 3D equation $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho$, where $\boldsymbol{\nabla}$ is the 3D vector derivative. It follows that the streamlines defined by $\boldsymbol{D}$ must spread out from the source and ultimately end on opposite charges. These field lines for a charge held at rest outside the horizon of a spherically-symmetric black hole are shown in Figure 1. It can be shown that the information contained in these plots is genuinely physical (*i.e.* gauge independent).

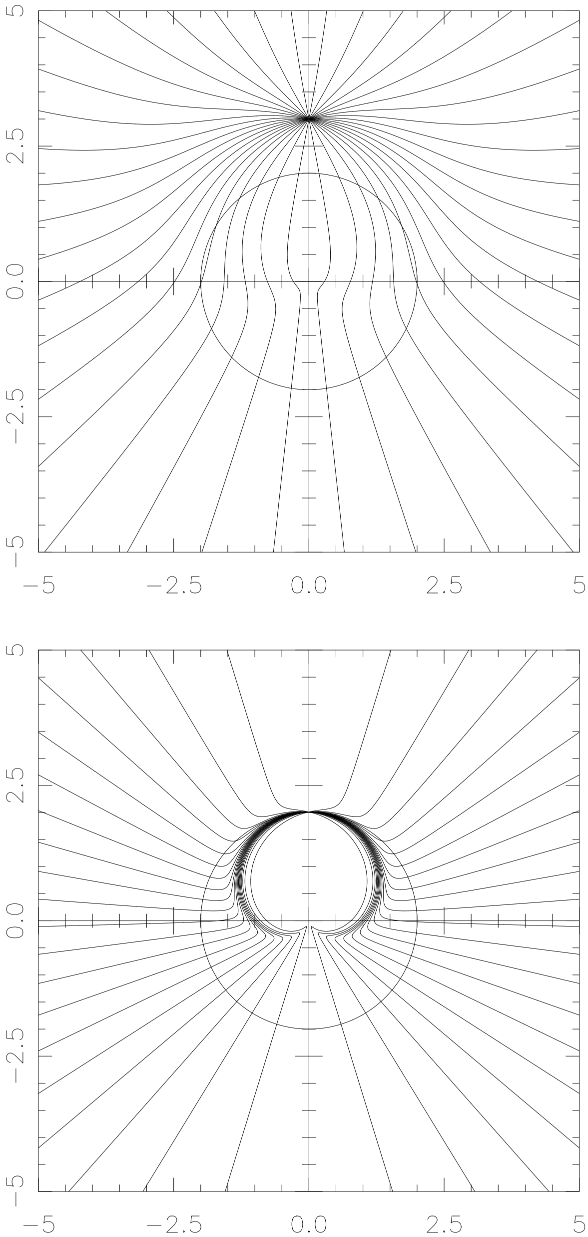

Figure 1: Field lines of the $\boldsymbol{D}$ field. The horizon is at $r = 2$ and the charge is placed on the z-axis. The charge is at $z = 3$ and $z = 2.01$ for the top and bottom diagrams respectively. The field lines are seeded so as to reflect the magnitude of $\boldsymbol{D}$. They are attracted towards the origin but never actually meet it. Note the appearance of a 'cardioid of avoidance' as the charge approaches very close to the horizon.

6 Summary of gauge theory gravity

We end by summarising some of the main features of our gauge theory of gravity. Full details are contained in [7]. The first point is its relation to GR. Whilst the gauge theory approach says nothing about metrics and curvature, there is an equivalent 'induced metric'

$$g(a) = \overline{h}^{-1}\underline{h}^{-1}(a). \tag{46}$$

If a set of coordinates $\{x^\mu\}$ are introduced, with the associated coordinate frame defined by $e_\mu \equiv \partial_{x^\mu} x$, a metric tensor is then defined by

$$g_{\mu\nu} = e_\mu \cdot g(e_\nu). \tag{47}$$

In the absence of spin, this metric tensor solves the Einstein equations for the given matter distribution. Differences between GR and our gauge approach arise from the status of $g_{\mu\nu}$. In GR $g_{\mu\nu}$ is the fundamental entity, and the only restrictions applied to it are derived from the Einstein equations. In our gauge theory $g_{\mu\nu}$ is derived from more primitive quantities, and this imposes further limitations on $g_{\mu\nu}$. These additional limitations typically have the effect of disallowing metrics which fail to cover all of spacetime (such as the Schwarzschild metric). Furthermore, the only singularities in the gauge fields that are permitted are at δ-function sources — the coordinate-independence of the STA formalism means that issues such as the existence of 'coordinate singularities' are avoided. Many weird topologies are also ruled out, including the multiple-coverings frequently invoked in modern treatments of black holes. Some interesting 'topological' solutions remain valid, however, including the cosmic string solutions discussed in cosmology.

A key advantage in formulating gravity as a gauge theory is the removal of any ambiguity as to what constitutes a physically measurable quantity. All physical predictions must be made in a gauge-invariant manner, and the formalism makes it simple to see what is or is not gauge invariant. GR, with its reliance on coordinates, is less clear on such issues.

Since the field equations consist of a set of differential equations in a flat background spacetime, the theory can be recast in terms of integral equations. These enable us to probe the structure of matter singularities more carefully than in GR. For example, it is possible to show that the stress-energy tensor for a non-rotating black hole does contain a δ-function at the origin. This is in stark contrast to GR, which allows 'mass without mass' ([8]). A similar calculation for the Kerr solution shows that the matter is distributed in a disc.

Our final comments concern the interaction with the Dirac theory. A spin interaction is an inevitable consequence of a gauge theory based on the Dirac equation, since the minimal coupling procedure automatically induces a coupling between $\omega(a)$ and the spin of the matter fields. A remarkable feature of this interaction is that self-consistency between the minimal coupling and variational procedures implies that the only permitted form for the gravitational Lagrangian is the Ricci scalar. All possible higher-order terms leed to inconsistencies in the formalism and so are ruled out. Satisfyingly, this forces us to a theory which is first-order in the derivatives of the fields. The only ambiguity in the theory lies in the possible inclusion of a cosmological constant, which cannot be ruled out on the grounds of consistency alone.

The Dirac electron wavefunction couples to the gauge fields in a different manner to classical fields, and so probes the theory at a deeper level than, for example, electromagnetism. A cosmological consequence of this fact is that in a non-spatially flat universe the Dirac wavefunction sees a preferred spatial direction. The only cosmological models which are consistent with homogeneity at the level of Dirac wavefunctions are those that are spatially flat.

A wide class of physical theories have now been successfully formulated in terms of geometric algebra. These include classical mechanics, relativistic dynamics, Dirac theory, electrodynamics and, as discussed here, gravitation theory. In every case, geometric algebra has offered demonstrable advantages over other techniques and has provided novel insights and unifications between disparate branches of physics and mathematics. We believe that all physicists should be exposed to its benefits and insights.

References

[1] W.K. Clifford. Applications of Grassmann's extensive algebra. *Am. J. Math.*, 1:350, 1878.

[2] H. Grassmann. Der ort der Hamilton'schen quaternionen in der Ausdehnungslehre. *Math. Ann.*, 12:375, 1877.

[3] D. Hestenes. *Space–Time Algebra.* Gordon and Breach, New York, 1966.

[4] T.W.B. Kibble. Lorentz invariance and the gravitational field. *J. Math. Phys.*, 2(3):212, 1961.

[5] R. Utiyama. Invariant theoretical interpretation of interaction. *Phys. Rev.*, 101(5):1597, 1956.

[6] F.W. Hehl, P. von der Heyde, G.D. Kerlick, and J.M. Nester. General relativity with spin and torsion: Foundations and prospects. *Rev. Mod. Phys.*, 48:393, 1976.

[7] A.N. Lasenby, C.J.L. Doran, and S.F. Gull. Gravity, gauge theories and geometric algebra. *Phil. Trans. R. Soc. Lond. A*, 356:487–582, 1998.

[8] C.W. Misner, K.S. Thorne, and J.A. Wheeler. *Gravitation*. W.H. Freeman and Company, San Francisco, 1973.