INTEGRAL EQUATIONS AND KERR-SCHILD FIELDS I. TIME-DEPENDENT, SPHERICALLY-SYMMETRIC FIELDS

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Abstract

Kerr-Schild solutions to the vacuum Einstein equations are considered from the viewpoint of a gauge-theoretic formulation of gravity. This formulation employs the spacetime algebra (the Clifford algebra of spacetime), which offers a number of novel insights into the nature of the solutions. Working with gauge fields defined over a background Minkowski spacetime allows the Einstein equations to be recast as integral equations, which can be used to explore the nature of the gravitational singularities. For the Schwarzschild and Vaidya solutions the fields are shown to result from a δ-function point source. For the Reissner-Nordstrom solution the integral equations reveal that inclusion of the gravitational fields removes the divergent self-energy familiar from classical electromagnetism.

1 Introduction

Many of the important solutions to the Einstein field equations can be expressed in Kerr-Schild form (see, for example, the discussion in [1]). In this and a following paper [2] Kerr-Schild solutions are analysed from the viewpoint of the gauge theory approach to gravity introduced in [3, 4]. In this approach the gravitational fields are gauge fields defined over a flat Minkowski spacetime. These fields ensure that all relations between physical quantities are independent of the position and orientation of the matter fields — a scheme which ensures that the background spacetime plays no dynamic role in the physics and has no measurable properties. Kerr-Schild metrics are constructed from a null vector field in the background Minkowski spacetime, so are particularly well-suited to analysis via this gauge-theoretic approach. Furthermore, the aproach developed in [3, 4] enables all manipulations to be carried out in a coordinate-free manner, affording a clear separation between physical effects and gauge artefacts.

In order to fully develop a coordinate-free gauge theory of gravity, the theory developed in [3, 4] is presented in the language of 'spacetime algebra' [5, 6]. The algebraic structure of the spacetime algebra (STA) is that of the Dirac γ-matrices. Using this algebraic structure one can develop a mathematical language which is adept at describing many aspects of relativistic physics. This language includes a calculus which goes some way beyond what is available in alternative languages. The theory developed in [3, 4] takes on its most natural and compelling form when expressed in the STA, and for this reason the applications discussed in this and the following paper are also formulated in the

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STA. A brief introduction to the STA is included here, giving only the necessary conventions and notations. Further details can be found in [4, 6, 7] and references contained therein. Reference [4] includes an appendix describing how STA-based expressions can be converted into conventional tensor calculus.

After introducing the STA, this paper proceeds to a derivation of the gravitational field equations. It is shown that for all fields of Kerr-Schild type the Einstein tensor is a total divergence in the background Minkowski spacetime. In this paper, various consequences of this result are explored for spherically-symmetric fields. Gauss' theorem is used to convert volume integrals of the Einstein tensor to surface integrals, so probing the nature of the matter singularities generating the gravitational fields. Here three spherically-symmetric solutions are considered: the Schwarzschild, Reissner-Nordstrom and Vaidya solutions. In all cases the integrals provide sensible results for the total energy contained in the fields, with the mass contribution to the energy residing in a point-source δ-function. For the Reissner-Nordstrom solution the inclusion of gravitational fields removes the infinite electromagnetic self-energy for a point charge familiar from classical electromagnetism [8]. This is achieved via a simple regularisation procedure, which ensures that the total electromagnetic self-energy is zero.

2 Spacetime Algebra

The basic algebraic structure behind the STA will be familiar to most physicists in the guise of the algebra of the Dirac γ-matrices. The geometric interpretation the STA attaches to this algebra may be less familiar, though it is remarkably well-suited to most problems in relativistic physics [5, 7]. The STA is generated by four vectors $\{\gamma_\mu\}$, $\mu = 0 \ldots 3$, equipped with an associative (Clifford) product denoted by juxtaposition. The symmetric and antisymmetric parts of this product define the inner and outer products, and are denoted with a dot and a wedge respectively:

$$\gamma_\mu \cdot \gamma_\nu \equiv \tfrac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \eta_{\mu\nu} = \mathrm{diag}(+ - - -) \tag{1}$$

$$\gamma_\mu \wedge \gamma_\nu \equiv \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \tag{2}$$

The outer product of two vectors defines a bivector — a directed plane segment representing the plane defined by the two vectors. A full basis for the STA is provided by the set

$$\begin{array}{ccccc} 1 & \{\gamma_\mu\} & \{\boldsymbol{\sigma}_k, i\boldsymbol{\sigma}_k\} & \{i\gamma_\mu\} & i \\ \text{1 scalar} & \text{4 vectors} & \text{6 bivectors} & \text{4 trivectors} & \text{1 pseudoscalar,} \\ \text{grade 0} & \text{grade 1} & \text{grade 2} & \text{grade 3} & \text{grade 4} \end{array} \tag{3}$$

where $\boldsymbol{\sigma}_k \equiv \gamma_k\gamma_0$, $k = 1 \ldots 3$, and $i \equiv \gamma_0\gamma_1\gamma_2\gamma_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$. The pseudoscalar i squares to -1, anticommutes with all odd-grade elements and commutes with even grade elements. Both the $\{\boldsymbol{\sigma}_k\}$ and $\{\gamma_\mu\}$ are algebraic entities with clear geometric significance. They should not be thought of as matrices acting on an internal spin space. (The same symbols as employed in quantum theory are used here simply because the algebraic relations are the same.)

An arbitrary real superposition of the basis elements (3) is called a 'multivector' and these inherit the associative Clifford product of the $\{\gamma_\mu\}$ generators. For a grade-r multivector A_r and a grade-s multivector B_s we define the inner and outer products respectively by:

$$A_r \cdot B_s \equiv \langle A_r B_s \rangle_{|r-s|}, \qquad A_r \wedge B_s \equiv \langle A_r B_s \rangle_{r+s}, \tag{4}$$

where $\langle M \rangle_r$ denotes the projection onto the grade-r components of M. We also employ the commutator product,

$$A \times B \equiv \tfrac{1}{2}(AB - BA). \tag{5}$$

Vectors are usually denoted in lower case Latin, $a = a^\mu \gamma_\mu$, or Greek for basis frame vectors. In the absence of brackets the inner, outer and commutator products take precedence over geometric products.

An inertial system is picked out by a future-pointing timelike (unit) vector. If this is chosen to be the γ_0 direction then the γ_0-vector determines a map between spacetime vectors $a = a^\mu \gamma_\mu$ and the even subalgebra of the full STA via

$$a\gamma_0 = a_0 + \boldsymbol{a}, \tag{6}$$

where

$$a_0 = a \cdot \gamma_0, \qquad \text{and} \qquad \boldsymbol{a} = a \wedge \gamma_0. \tag{7}$$

The 'relative vector' $\boldsymbol{a}$ can be decomposed in the $\{\boldsymbol{\sigma}_k\}$ frame and represents a spatial vector as seen by an observer in the γ_0-frame. Relative (or spatial) vectors in the γ_0-system are written in bold type to record the fact that in the STA they are actually bivectors. This distinguishes them from spacetime vectors, which are left in normal type. The $\{\boldsymbol{\sigma}_k\}$ generate the (Pauli) algebra of three-dimensional space, and we occasionally require that the dot and wedge symbols define the three-dimensional inner and outer products. The convention we adopt is that if both arguments of a dot or wedge product are written in bold, then the product takes its three-dimensional meaning. For example, $\boldsymbol{a} \wedge \boldsymbol{b}$ is a relative bivector, and so also a spacetime bivector, and not a spacetime four-vector.

The vector derivative, ∇, is defined by

$$\nabla \equiv \gamma^\mu \frac{\partial}{\partial x^\mu} \tag{8}$$

where the $\{x^\mu\}$ are a set of Cartesian coordinates and the $\{\gamma^\mu\}$ are the reciprocal frame to the associated coordinate frame. The spacetime split of the vector derivative ∇ goes through slightly differently, since it is desirable to have the $\boldsymbol{\nabla}$ symbol agreeing with its conventional 3D meaning. To achieve this we define

$$\gamma_0 \nabla = \partial_t + \boldsymbol{\nabla}, \tag{9}$$

so that $\boldsymbol{\nabla} = \boldsymbol{\sigma}_i \partial_{x^i}$. The ∇ operator has the algebraic properties of a vector, and often acts on objects which it is not adjacent to. The 'overdot' notation is a convenient way to encode this:

$$\dot{\nabla} A \dot{B} \equiv \gamma^\mu A \frac{\partial B}{\partial x^\mu}. \tag{10}$$

The ∇ operator acts on the object to its immediate right unless brackets or overdots are present. If brackets are present then ∇ operates on everything in the bracket, so that, for example, $\nabla(AB) = \nabla AB + \dot{\nabla} A\dot{B}$. The same rules apply to $\boldsymbol{\nabla}$.

Linear functions mapping vectors to vectors are usually denoted with an underbar, $\underline{h}(a)$ (where a is the vector argument), with the adjoint denoted with an overbar, $\bar{h}(a)$. Linear functions extend to act on multivectors via the rule

$$\underline{h}(a\wedge b\cdots\wedge c) \equiv \underline{h}(a)\wedge\underline{h}(b)\wedge\cdots\wedge\underline{h}(c), \tag{11}$$

which defines a grade-preserving linear operation. The pseudoscalar is unique up to a scale factor, and the determinant is defined by

$$\underline{h}(i) = \det(\underline{h})i. \tag{12}$$

The underbars are not strictly necessary and are often dropped for symmetric functions, but many expressions take on a more symmetric appearance if they are employed. For example, a function and its adjoint are related by [6]

$$\begin{aligned} A_r\cdot\bar{h}(B_s) &= \bar{h}[\underline{h}(A_r)\cdot B_s] \qquad r \leq s \\ \underline{h}(A_r)\cdot B_s &= \underline{h}[A_r\cdot\bar{h}(B_s)] \qquad r \geq s. \end{aligned} \tag{13}$$

A number of manipulations in linear algebra are simplified by using the vector derivative in place of frame contractions. For example, the trace of $\underline{h}(a)$ can be written as

$$\mathrm{Tr}(\underline{h}) = \gamma^\mu\cdot\underline{h}(\gamma_\mu) = \partial_a\cdot\underline{h}(a), \tag{14}$$

where ∂_a is the vector derivative with respect to a. The following results are also useful:

$$\partial_a\, a\cdot A_r = rA_r \tag{15}$$

$$\partial_a\, a\wedge A_r = (n-r)A_r \tag{16}$$

$$\partial_a A_r a = (-1)^r(n-2r)A_r, \tag{17}$$

where A_r is a multivector of grade r and n is the dimension of the space.

Natural units ($G = c = \epsilon_0 = 1$) are employed throughout this and the following paper.

3 The Field Equations

The gravitational gauge fields are a linear function $\bar{h}(a)$ mapping vectors to vectors and a linear function $\Omega(a)$ mapping vectors to bivectors. Both of these gauge fields have an arbitrary position dependence. The gauge-theoretic origin of these fields is described in [3, 4]. The gauge fields are related by the equation

$$2\Omega(a) = -\bar{h}(\nabla\wedge g(a)) + \underline{h}^{-1}(\partial_b)\wedge(a\cdot\nabla\bar{h}(b)), \tag{18}$$

where

$$g(a) \equiv \bar{h}^{-1}\underline{h}^{-1}(a). \tag{19}$$

The linear function $g(a)$ defines the line element via

$$ds^2 = e_\mu\cdot g(e_\nu)dx^\mu\, dx^\nu = g_{\mu\nu}dx^\mu\, dx^\nu, \tag{20}$$

where the $\{x^\mu\}$ are a set of arbitrary coordinates and the $\{e_\mu\}$ are the associated coordinate frame,

$$e_\mu \equiv \frac{\partial x}{\partial x^\mu}, \tag{21}$$

with x the flatspace position vector.

The field strength corresponding to the $\Omega(a)$ gauge field is defined by

$$R(a\wedge b) \equiv a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a) + \Omega(a)\times\Omega(b) \tag{22}$$

and is a linear function mapping bivectors to bivectors. From this the covariant Riemann tensor is defined by

$$\mathcal{R}(a\wedge b) \equiv R\underline{h}(a\wedge b). \tag{23}$$

The Ricci and Einstein tensors are defined from the Riemann tensor in the obvious way,

$$\text{Ricci Tensor:} \quad \mathcal{R}(b) \equiv \partial_a\cdot\mathcal{R}(a\wedge b) \tag{24}$$

$$\text{Ricci Scalar:} \quad \mathcal{R} \equiv \partial_a\cdot\mathcal{R}(a) \tag{25}$$

$$\text{Einstein Tensor:} \quad \mathcal{G}(a) \equiv \mathcal{R}(a) - \tfrac{1}{2}a\mathcal{R}. \tag{26}$$

The $\bar{h}(a)$ field ensures that one only ever has to make 'flatspace' contractions, which is an attractive feature of the gauge-theory approach.

We are interested in fields of the form

$$\bar{h}(a) = a + a\cdot l\, l \tag{27}$$

where l is a (flatspace) null vector, $l^2 = 0$. This is the gauge theory analogue of the Kerr-Schild ansatz. The function (27) extends to act on multivectors as

$$\bar{h}(A) = \underline{h}(A) = A + A\cdot l\, l, \tag{28}$$

and we see immediately that $\det(\bar{h}) = 1$. The following results are also useful:

$$\underline{h}^{-1}(A) = \bar{h}^{-1}(A) = A - A\cdot l\, l \tag{29}$$

$$g(A) = A - 2A\cdot l\, l \tag{30}$$

$$\bar{h}(l) = l. \tag{31}$$

The $\Omega(a)$ field defined by (27) has the simple form

$$\begin{aligned}\Omega(a) &= \bar{h}[\nabla\wedge(a\cdot l\, l)] \\ &= \nabla\wedge(a\cdot l\, l) - a\cdot l\, v\wedge l\end{aligned} \tag{32}$$

where

$$v \equiv l\cdot\nabla l. \tag{33}$$

It follows that $l\cdot v = 0$ and $\Omega(l) = 0$.

Following the route adopted by Chandrasekhar in Section 57 of [9], we next form the quantity

$$\begin{aligned}l\cdot\mathcal{R}(l) &= l\cdot[\partial_a\cdot\mathcal{R}(a\wedge l)] \\ &= (l\wedge\partial_a)\cdot R(a\wedge l) \\ &= (l\wedge\partial_a)\cdot[a\cdot\nabla\Omega(\check{l}) - l\cdot\nabla\Omega(a)],\end{aligned} \tag{34}$$

where the check on $\check{l}$ denotes that this vector is not differentiated. Substituting (32) into the above we find that

$$\begin{aligned} l\cdot\mathcal{R}(l) &= (l\wedge\partial_a)\cdot\big(-\dot{\nabla}[(a\cdot\nabla l)\cdot\dot{l}\,]\wedge l - l\cdot\nabla\nabla\wedge(a\cdot l\,l)\big) \\ &= \partial_a\cdot l\,(a\cdot\nabla l)\cdot v - l\cdot\nabla(\nabla\cdot l\,l + v)\cdot l \\ &= v^2 - (l\cdot\nabla v)\cdot l \\ &= 2v^2. \end{aligned} \tag{35}$$

If we were looking solely for vacuum solutions, we would conclude from this that v must be null. Since $v\cdot l = 0$, it would then follow that v must be parallel to l,

$$v = \phi l, \tag{36}$$

where ϕ is an arbitrary scalar. We restrict attention to solutions for which this relation does hold, even if matter is present. (This clearly restricts the form of matter distribution that can be described by fields of the type (27)). It follows from equation (36) that $\Omega(a)$ reduces to the simpler form

$$\Omega(a) = \nabla\wedge(a\cdot l\,l). \tag{37}$$

The Riemann tensor now splits into terms that are second-order and fourth-order in l. The fourth-order contribution is

$$\mathcal{R}_4(a\wedge b) = -\dot{\Omega}\big([(a\wedge b)\cdot l\,l]\cdot\dot{\nabla}\big) + \Omega(a)\times\Omega(b). \tag{38}$$

After some rearrangement this can be brought to the form

$$\mathcal{R}_4(B) = \tfrac{1}{4}\partial_a\cdot\partial_b\,(a\cdot\nabla l)lBl(b\cdot\nabla l) - \tfrac{1}{4}(a\cdot\nabla l)\cdot(b\cdot\nabla l)\,\partial_a lBl\partial_b. \tag{39}$$

Both the contraction, $\partial_a \cdot \mathcal{R}(a\wedge b)$, and the protraction, $\partial_a \wedge \mathcal{R}(a\wedge b)$, of this contribition to the Riemann tensor vanish, as can be seen easily from the result that

$$\partial_a B_1 a\wedge b B_2 = \partial_a B_1(ab - a\cdot b)B_2 = -bB_1B_2, \tag{40}$$

for any two bivectors B_1 and B_2. The presence of the null vector l in the analogous terms in $\mathcal{R}_4(B)$ ensures that $B_1B_2 = 0$, and hence that $\mathcal{R}_4(B)$ makes no contribution to the Ricci tensor.

The only part of $\mathcal{R}(B)$ which contributes to the Einstein tensor is therefore the second-order term

$$\mathcal{R}_2(a\wedge b) = a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a). \tag{41}$$

Contracting this and setting the result to zero we find that the vacuum Einstein equations reduce to solving the equation

$$\mathcal{R}(a) = \nabla\cdot\Omega(a) - a\cdot\nabla\,\partial_b\cdot\Omega(b) = 0. \tag{42}$$

The Ricci scalar and Einstein tensor are now straightforward to calculate:

$$\mathcal{R} = -2\nabla\cdot(\partial_a\cdot\Omega(a)) \tag{43}$$

and

$$\mathcal{G}(a) = \nabla\cdot[\Omega(a) - a\wedge(\partial_b\cdot\Omega(b))]. \tag{44}$$

The above formulae for $\Omega(a)$ (37) and $\mathcal{G}(a)$ are valid for any Kerr-Schild type solution for which $l\cdot\nabla l = \phi l$. For such fields the Einstein tensor (44) is a total divergence in Minkowski spacetime. In general, the field equations will be satisfied everywhere except for some singular region over which the fields are discontinuous. This singular region contains the source of the fields. By converting integrals over this source region to surface integrals, we learn how the source matter is distributed. (For the case of static fields, Virbhadra [10] gave a formula which agrees with (44) for $a = \gamma_0$, but the fact that the expression is a total divergence was not exploited.)

3.1 Spherically-Symmetric Solutions

For the remainder of this paper we restrict attention to spherically-symmetric solutions. For these it is useful to introduce a standard set of polar coordinates:

$$\begin{aligned} t &\equiv x\cdot\gamma_0 & \cos\theta &\equiv x\cdot\gamma^3/r \\ r &\equiv \sqrt{(x\wedge\gamma_0)^2} & \tan\phi &\equiv (x\cdot\gamma^2)/(x\cdot\gamma^1). \end{aligned} \tag{45}$$

We also define

$$e_r \equiv x\wedge\gamma_0\,\gamma_0/r, \qquad \boldsymbol{\sigma}_r \equiv e_r\gamma_0, \tag{46}$$

and

$$e_\pm \equiv \gamma_0 \pm e_r. \tag{47}$$

The solutions of interest here can be written in the form

$$l = \sqrt{\alpha}\, e_\pm, \tag{48}$$

where $\alpha = \alpha(t, r)$. For fields of this type it is a simple matter to demonstrate that the fourth-order contribution to the Riemann tensor (39) vanishes. To see this consider the case of e_+, for which we obtain

$$\begin{aligned} \mathcal{R}_4(B) &= \frac{\alpha^2}{4}\big(-\partial_a\cdot\partial_b\, a\cdot\nabla\boldsymbol{\sigma}_r(1-\boldsymbol{\sigma}_r)B(1+\boldsymbol{\sigma}_r)b\cdot\nabla\boldsymbol{\sigma}_r \\ &\quad + (a\cdot\nabla\boldsymbol{\sigma}_r)\cdot(b\cdot\nabla\boldsymbol{\sigma}_r)\,\partial_a e_+ B e_+ \partial_b\big) \\ &= \frac{\alpha^2}{4r}\big(\dot{\boldsymbol{\nabla}}(1-\boldsymbol{\sigma}_r)B(1+\boldsymbol{\sigma}_r)\dot{\boldsymbol{\sigma}}_r - \dot{\boldsymbol{\nabla}}(1-\boldsymbol{\sigma}_r)B(1+\boldsymbol{\sigma}_r)\dot{\boldsymbol{\sigma}}_r\big) \\ &= 0, \end{aligned} \tag{49}$$

with the same result holding for e_-. It follows that the Riemann tensor is given entirely by (41), which is also a total divergence and so can be analysed using Gauss' theorem. We now turn to three applications of these results.

4 The Schwarzschild Solution

The simplest solution to the field equations is the Schwarzschild solution, obtained from

$$\alpha = M/r, \qquad l = \sqrt{\alpha}\,(\gamma_0 - e_r). \tag{50}$$

The derivation of this solution is contained in [2] (see also [11]). The line element generated by this solution is that of the advanced Eddington-Finkelstein form of the Schwarzschild solution, and is therefore not time-reverse symmetric. It turns out that in the gauge theory approach the presence of a horizon implies the breakdown of time-reversal symmetry [3, 4]. The solution (50) lies in the gauge sector of solutions picked out as the endpoint of a collapse process.

The Riemann tensor for the solution (50) can be constructed using equation (41), from which we find

$$\begin{aligned}\mathcal{R}(\boldsymbol{a}) &= \boldsymbol{a}\cdot\boldsymbol{\nabla}\Omega(\gamma_0) \\ &= \boldsymbol{a}\cdot\boldsymbol{\nabla}\big(\nabla\wedge(M(\gamma_0 - e_r)/r)\big) \\ &= M\boldsymbol{a}\cdot\boldsymbol{\nabla}\frac{\boldsymbol{x}}{r^3},\end{aligned} \tag{51}$$

and

$$\begin{aligned}\mathcal{R}(i\boldsymbol{b}) &= \dot{\Omega}[(i\boldsymbol{b})\cdot\dot{\boldsymbol{\nabla}}\gamma_0] \\ &= \nabla\wedge\big(-\frac{M}{r}(i\boldsymbol{b}\wedge\boldsymbol{\sigma}_r)\cdot\boldsymbol{\nabla}\boldsymbol{\sigma}_r\gamma_0\big) \\ &= Mi\boldsymbol{\nabla}\cdot\big(\boldsymbol{b}\wedge\frac{\boldsymbol{x}}{r^3}\big).\end{aligned} \tag{52}$$

Away from the origin, these derivatives evaluate to

$$\mathcal{R}(\boldsymbol{a}) = \frac{M}{r^3}(\boldsymbol{a} - 3\boldsymbol{a}\cdot\boldsymbol{\sigma}_r\boldsymbol{\sigma}_r), \qquad \mathcal{R}(i\boldsymbol{b}) = \frac{M}{r^3}(\boldsymbol{b} - 3\boldsymbol{b}\cdot\boldsymbol{\sigma}_r\boldsymbol{\sigma}_r), \tag{53}$$

so we can write the vacuum Riemann tensor in the manifestly self-dual form

$$\mathcal{R}(B) = -\frac{M}{2r^3}(B + 3\boldsymbol{\sigma}_r B\boldsymbol{\sigma}_r). \tag{54}$$

(Self duality of the Weyl tensor has the simple expression $\mathcal{W}(iB) = i\mathcal{W}(B)$ in the STA [4]). The above form of the Riemann tensor (54) for the Schwarzschild solution was first given in [3] and [12].

The form of the Riemann tensor in equation (54) is valid everywhere away from the singularity. To study the form of the singularity, we return to the differential expressions for the Riemann tensor and integrate over a sphere of radius r_0, centered on the origin. Using Gauss' theorem to convert the volume integrals to surface integrals, we obtain

$$\int_{r\le r_o} d^3x\,\mathcal{R}(\boldsymbol{a}) = M\int_0^{2\pi} d\phi\int_0^{\pi} d\theta\,\sin\theta\,\boldsymbol{a}\cdot\boldsymbol{\sigma}_r\,\boldsymbol{\sigma}_r = \frac{4\pi M}{3}\boldsymbol{a}, \tag{55}$$

and

$$\int_{r\le r_o} d^3x\,\mathcal{R}(i\boldsymbol{b}) = \int_0^{2\pi} d\phi\int_0^{\pi} d\theta\,\sin\theta\,i\boldsymbol{\sigma}_r\cdot(\boldsymbol{b}\wedge\boldsymbol{\sigma}_r) = -\frac{8\pi M}{3}i\boldsymbol{b}. \tag{56}$$

These results combine to give

$$\int_{r\le r_o} d^3x\,\mathcal{R}(B) = \frac{4\pi M}{3}(B\cdot\gamma_0\gamma_0 - 2B\wedge\gamma_0\gamma_0) = -\frac{2\pi M}{3}(B + 3\gamma_0 B\gamma_0), \tag{57}$$

which contracts to yield

$$\int d^3x\,\mathcal{R}(a) = 4\pi M\,\gamma_0 a\gamma_0 \tag{58}$$

$$\int d^3x\,\mathcal{R} = -8\pi M \tag{59}$$

$$\int d^3x\,\mathcal{G}(a) = 8\pi M a\cdot\gamma_0\,\gamma_0. \tag{60}$$

Since $\mathcal{R}(a) = 0$ everywhere except for the origin, the integrals (58)–(60) can be taken over any region of space enclosing the origin. It is clear then that the solution represents a point source of matter, with the matter stress-energy tensor given by

$$\mathcal{T}(a) = M\delta(\boldsymbol{x})a\cdot\gamma_0\gamma_0. \tag{61}$$

The same conclusion was reached in [4], where the calculations were performed in a different gauge. This result confirms Feynman's speculation in Lecture 15 of [13]. The integrals performed above are not gauge invariant, but gauge-invariant information is extracted from them in the form of the matter stress-energy tensor (61). Furthermore, the integral of the Ricci scalar provides a direct measure of the mass of the source, without the need to resort to constructing integrals in an asymptotically flat region of spacetime.

5 The Reissner-Nordstrom Solution

The Reissner-Nordstrom solution can be written in the form

$$\bar{h}(a) = a + \eta a\cdot e_-\,e_- \tag{62}$$

where (in natural units)

$$\eta \equiv \frac{M}{r} - \frac{q^2}{8\pi r^2}, \tag{63}$$

q the charge of the source. The Einstein tensor for this solution is

$$\mathcal{G}(a) = \nabla\cdot\big(\nabla(\eta a\cdot e_-)\wedge e_- - \nabla\cdot(\eta e_-)a\wedge e_-\big). \tag{64}$$

Away from the origin we know that the mass term can be ignored, which leaves

$$\mathcal{G}(\gamma_0) = -\frac{q^2}{4\pi}\nabla\cdot\big(\frac{\boldsymbol{\sigma}_r}{r^3}\big) = \frac{q^2}{4\pi r^4}\,\gamma_0, \tag{65}$$

$$\mathcal{G}(\boldsymbol{a}\gamma_0) = \frac{q^2}{8\pi}\nabla\cdot\Big(\nabla\big(\frac{\boldsymbol{a}\cdot\boldsymbol{\sigma}_r}{r^2}\big)\wedge e_-\Big) = -\frac{q^2}{4\pi r^4}\,\boldsymbol{\sigma}_r\boldsymbol{a}\gamma_0\boldsymbol{\sigma}_r. \tag{66}$$

These combine to give a corresponding matter stress-energy tensor of

$$\mathcal{T}(a) = \frac{1}{8\pi}\mathcal{G}(a) = -\tfrac{1}{2}\mathcal{F}a\mathcal{F} \tag{67}$$

where $\mathcal{F} \equiv q\boldsymbol{\sigma}_r/(4\pi r^2)$. This is the expected form for the electromagnetic stress-energy tensor due to a point source of charge q. (See [4] for a detailed explanation of how to handle electromagnetism in gauge-theory gravity.)

To study the behaviour of the fields near the origin we return to the differential form for $\mathcal{G}(a)$ and again construct integrals over a sphere of radius r_0. For this case we find that

$$\int_{r\leq r_o} d^3x\, \mathcal{G}(\gamma_0) = \int_{r\leq r_o} d^3x\, \boldsymbol{\nabla}\cdot\big(\frac{2M}{r^2}\,\boldsymbol{\sigma}_r - \frac{q^2}{4\pi r^3}\,\boldsymbol{\sigma}_r\big)\,\gamma_0 = (8\pi M - \frac{q^2}{r^0})\,\gamma_0, \quad (68)$$

$$\int_{r\leq r_o} d^3x\, \mathcal{G}(\boldsymbol{a}\gamma_0) = \int_{r\leq r_o} d^3x\, \frac{q^2}{8\pi}\gamma_0\boldsymbol{\nabla}\cdot\big(\frac{1}{r^3}\boldsymbol{a}\wedge\boldsymbol{\sigma}_r\big) = \frac{q^2}{3r_0}\,\boldsymbol{a}\gamma_0, \quad (69)$$

which combine to give

$$\int_{r\leq r_o} d^3x\, \mathcal{T}(a) = M a\cdot\gamma_0\,\gamma_0 + \frac{q^2}{24\pi r_0}(a - 4a\cdot\gamma_0\,\gamma_0). \quad (70)$$

The mass term here is precisely as expected and shows again that a point source is located at the origin. The electromagnetic contribution is traceless, as one expects for the electromagnetic stress-energy tensor. Focusing attention on the γ_0-frame energy component of the stress-energy tensor, we see that

$$\int_{r\leq r_o} d^3x\, \gamma_0\cdot\mathcal{T}(\gamma_0) = M - \frac{q^2}{8\pi r_0}. \quad (71)$$

This result was also obtained by Tod [14], who calculated the quasi-local mass for the Reissner-Nordstrom solution as defined by Penrose [15]. Tod argued that this result implies that a source for the Reissner-Nordstrom solution should have $r > q^2/(8\pi M)$ at the surface in order to meet the dominant energy condition. However, this misses the point that the negative contribution to the integral comes entirely from the origin. Everywhere off the origin the stress-energy tensor satisfies the dominant energy condition. Taking the integrals over the volume defined by $r_0 < r < \infty$ we find that the electromagnetic field energy is $q^2/(8\pi r_0)$, which agrees with the formula given by Virbhadra [10] and is simply the classical result.

The electromagnetic contribution to (71) is negative and finite for all finite r_0, and tends to zero as the integral extends over all space. This is in stark contrast to the standard picture from classical electromagnetism, where the integral of the γ_0-frame energy $\boldsymbol{E}^2/2$ diverges for the interior of any surface enclosing the origin — the classical self-energy problem discussed by many authors (see [8, 16], for example). Inclusion of the gravitational field has removed this divergence, ensuring that the total electromagnetic self-energy is zero. The manner in which this regularisation is achieved is both simple and instructive. The electromagnetic energy density is rewritten as

$$\tfrac{1}{2}\boldsymbol{E}^2 = \frac{q^2}{32\pi r^4} = -\frac{q^2}{32\pi}\boldsymbol{\nabla}\cdot\big(\frac{\boldsymbol{x}}{r^4}\big), \quad (72)$$

so that the integral over space of the electromagnetic energy density can be converted to a surface integral, recovering the contribution to (71). Since the electromagnetic energy density near a point source is very large, it is unsurprising that the inclusion of gravity has significant consequences, and these clearly have implications for the status of self-energies in classical field theory. However, since only classical fields are employed above, it is not clear whether this result has similar implications for the divergent self-energies encountered in QED.

6 The Vaidya Solution

As a final example of the use of integral equations for spherically-symmetric Kerr-Schild fields, we consider Vaidya's 'shining star' solution [1]. This is generated by the field

$$\bar{h}(a) = a + \frac{\mu(t-r)}{r} a \cdot e_+ \, e_+, \tag{73}$$

which is clearly similar to the Schwarzschild solution, except that now the mass $\mu = \mu(t-r)$ is variable and the null geodesics e_+ are outgoing rather than incoming. The solution (73) is clearly of Kerr-Schild type, and defining $l = \sqrt{\mu/r}\, e_+$ we find that

$$l \cdot \nabla l = \big(\frac{\mu}{r}\big)^{1/2} e_+ \cdot \nabla \, [\big(\frac{\mu}{r}\big)^{1/2} e_+] = -\tfrac{1}{2} \big(\frac{\mu}{r^3}\big)^{1/2} l, \tag{74}$$

so that equation (36) is satisfied. The Einstein tensor for (73) is

$$\mathcal{G}(a) = \nabla \cdot \big(\frac{2\mu}{r^2} a \cdot e_+ \, \boldsymbol{\sigma}_r \big) \tag{75}$$

and away from the origin (where we can set $\nabla \cdot (\boldsymbol{\sigma}_r / r^2) = 0$) this becomes

$$\mathcal{G}(a) = -\frac{2\dot{\mu}}{r^2} a \cdot e_+ \, e_+, \tag{76}$$

where $\dot{\mu} = \partial_t \mu$. This tensor represents a radially-symmetric flux of outgoing massless particles, though it does not have the form expected for purely electromagnetic radiation. Again, the presence of a δ-function point source at the origin can be inferred from the differential form of the Einstein tensor. By evaluating the integral of $\mathcal{G}(a)$ over a sphere centred on the origin, and shrinking the radius to zero, we find that

$$\mathcal{G}(a) = -\frac{2\dot{\mu}}{r^2} a \cdot e_+ \, e_+ + 8\pi\mu\delta(\boldsymbol{x}) a \cdot \gamma_0 \, \gamma_0. \tag{77}$$

The solution (73) therefore describes a point mass at rest at the origin which is losing mass at some arbitrary rate. This is borne out by the Riemann tensor,

$$\mathcal{R}(B) = -\frac{\dot{\mu}}{r^2} B \cdot e_+ \, e_+ - \frac{\mu}{2r^3} (B + 3\boldsymbol{\sigma}_r B \boldsymbol{\sigma}_r), \tag{78}$$

which exhibits a neat split into a source term describing the energy outflow and a Weyl term due to the point mass at the origin.

The fact that the Einstein tensor is given by the divergence of a bivector implies that

$$\nabla \cdot \mathcal{G}(a) = 0. \tag{79}$$

We can therefore define a conserved total energy by

$$\begin{aligned} E &\equiv \int d^3x \, \gamma_0 \cdot \mathcal{G}(\gamma_0) \\ &= \int d^3x \, \boldsymbol{\nabla} \cdot \big(2\mu \frac{\boldsymbol{\sigma}_r}{r^2} \big) \\ &= 8\pi\mu(-\infty). \end{aligned} \tag{80}$$

The total conserved energy is therefore determined by the mass of the source at $t = -\infty$, before it began radiating, which is clearly a sensible result. A conserved energy of this form will exist for any Kerr-Schild field of the type (27), provided that the null vector l satisfies $l \cdot \nabla l = \phi l$.

7 Conclusions

Many of the significant solutions to the Einstein equations can be represented in Kerr-Schild form and the STA-based gauge-theoretic approach of [3, 4] is well suited to their analysis. For all solutions of Kerr-Schild type where the null vector l satisfies $l\cdot\nabla l = \phi l$ the Einstein tensor is a total divergence. The structure of the sources generating the fields can therefore be elucidated by employing Gauss theorem to transform volume integrals to surface integrals. This approach is fully justified within the gauge-theory formulation, since one only ever deals with fields defined over a flat spacetime.

For the case of the Schwarzschild, Reissner-Nordstrom and Vaidya solutions the gravitational fields are seen to result from a δ-function point source of mass at the origin. This result is not surprising, though it does differ from the accepted picture in general relativity. For the Schwarzschild solution this picture involves two distinct universes connected by a 'throat', with separate future and past singularities [17, 18]. No such picture is available in the gauge theory approach, where one is forced to work with time-reverse asymmetric fields generated by a single δ-function [3, 4]. For the Reissner-Nordstrom solution the result of maximum analytic extension is an infinite ladder of possible 'universes' connected by wormholes [18]. Again, the gauge theory approach does not admit such a possibility, forcing us instead to a less elaborate picture of a single δ-function point source surrounded by a Coulomb field. An unexpected bonus of this approach is that the infinite self-energy of the Coulomb field is removed by the gravitational field.

From these comments it is clear that the results of this paper enjoy a somewhat ambiguous status. If one accepts that gravitational forces should be described by gauge fields, and follows the derivation of these fields given in [3, 4], then one is forced to accept the validity of results obtained from applying Gauss' theorem to fields defined over the background Minkowski spacetime. If, on the other hand, one believes that a theory of gravity should be free from the topological constraints implied by the gauge theory formalism, then one could argue that Gauss' theorem is not applicable in the way it is used here. Given that both approaches agree for all experimentally-testable effects, there are as yet no physical grounds for preferring one approach over the other. Ultimately, however, the interface with quantum theory is likely to favour one approach, and in this respect there are clear reasons for preferring the gauge theory route to describing gravitational interactions.

In the following paper [2] the techniques developed here are applied to the Kerr solution, revealing the presence of a 'tension disk' surrounded by the ring singularity. Elsewhere these techniques will be applied to Kinnersley's and Bonnor's work on accelerating and radiating masses [19, 20].

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