EFFECTS OF SPIN-TORSION IN GAUGE THEORY GRAVITY

Chris Doran*, Anthony Lasenby, Anthony Challinor and Stephen Gull
MRAO, Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, UK.

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Abstract

The spin-torsion sector of a new gauge-theoretic formulation of gravity is analysed and the relationship to the Einstein-Cartan-Kibble-Sciama theory of gravity is discussed. The symmetries of the Riemann tensor and the conservation laws of the theory are derived. This formalism is applied to the problem of a Dirac field coupled self-consistently to gravity. The equations derived from a minimally-coupled gauge-invariant Lagrangian naturally give the gauge-theoretic analogues of the Einstein-Cartan-Dirac equations. Finally, a semi-classical model for a spinning point-particle moving in a gravitational background with torsion is considered.

1 Introduction

The problem of formulating gravity as a gauge theory has been considered by many authors (see [1]–[8] for a representative sample). The gauge-theoretic approach leads naturally to an extension of General Relativity known as spin-torsion theory [9]. Such theories build upon Cartan's suggestion [10] that torsion (the antisymmetric part of the connection) should be identified as a possible physical field. The connection between torsion and quantum spin emerged later [2, 11, 12] when it was realised that the stress-energy tensor for a massive fermion field was asymmetric [13, 14].

A new approach to gauge theory gravity (GTG) was developed in [15, 16]. In this approach, gravitational effects are described by a pair of gauge fields defined over a flat Minkowski background spacetime. The gauge fields ensure the invariance of the theory under arbitrary local displacements and rotations in the background spacetime. All physical predictions are extracted in a gauge-invariant manner, which ensures that the background spacetime plays no dynamic role in the physics. Most of the discussion in [15] was simplified by setting the torsion to zero. Even if the torsion does vanish, differences between GTG and General Relativity still arise over global issues such as the role of topology and horizons. These differences are highlighted by

*E-mail: C.Doran@mrao.cam.ac.uk

the discussion of the Kerr singularity in Doran *et al.* [17]–[19]. Further applications of GTG to situations with vanishing torsion can be found in [20].

In this paper we consider the spin-torsion sector of GTG in more detail. This is important both for theoretical considerations, such as spinor-driven inflationary models, and for comparisons with possible experiments or observations. Within GTG, torsion is viewed as a physical field derived from the gravitational gauge fields. This viewpoint has some conceptual advantages over that used in differential geometry, where torsion is regarded as a property of a non-Riemannian manifold. A further feature of the GTG approach is that the type of torsion which can arise physically is constrained [15]. This constraint follows from the requirement that minimally-coupled equations for the matter fields should be derivable from a minimally-coupled, gauge-invariant action.

The paper is arranged as follows. We begin with an introduction to the 'spacetime algebra' [21, 22]. This algebraic structure is well suited to many aspects of relativistic physics, enabling manipulations to be carried out in a coordinate-free manner. The theory developed in [15] takes on its most compelling form when written in the spacetime algebra, so we employ it here. One aim of this paper is to advertise the advantages of the algebraic techniques provided by spacetime algebra, which extends the language of differential forms to incorporate spinors and quaternions, and provides very clean and clear methods for encoding rotations. These advantages are perhaps not as widely known as they deserve to be. The relations derived in this paper may be translated into more conventional languages using the schemes described in Appendix D and in [15].

We next introduce the equations of GTG and discuss the relationship between GTG and the Einstein-Cartan-Kibble-Sciama (ECKS) theory of gravity. We then proceed to a covariant expression for the Riemann tensor in the presence of torsion. From this we are able to relate the Riemann tensor, and its contractions, to the equivalent tensors in the absence of torsion. We then consider the symmetries of the Riemann tensor and the gauge theory version of the Bianchi identity. As is well known, much of the symmetry of the Riemann tensor is lost when torsion is included.

The Dirac field coupled self-consistently to gravity provides an example of a system with torsion, which may be relevant to discussions of the early universe. The equations for this setup are derived from a minimally-coupled, gauge-invariant action. The resulting equations are the GTG analogues of the Einstein-Cartan-Dirac equations. Appendix A contains a number of results for the gravitational fields particular to the case of spin generated by a single Dirac fermion. The analogues of the Einstein-Dirac equations are also given. These equations describe the Dirac field coupled to gravity via the (symmetric) stress-energy tensor only. In this arrangement the torsion vanishes. The Einstein-Dirac equations can also be derived from an action, but only if one is prepared to sacrifice the gauge structure of the theory. In Appendix B we give the GTG proof of the useful result, first given by Seitz [23], relating solutions of the Einstein-Cartan-Dirac equations and solutions of the Einstein-Dirac equations.

Finally, we consider a simple, semi-classical model for a spin-1/2 point-particle moving in a gravitational background with torsion, which builds upon the model discussed in [15]. This displays a coupling of the particle's motion to the torsion, unlike some previous models [24]. This model is used in a separate paper [25] to discuss the anisotropy of a new cosmological solution of the Einstein-Cartan-Dirac equations. Throughout the main part of this paper we assume that the spin tensor has a vanishing contraction, which is necessary for consistency with the minimal coupling procedure. Results for the case where the contraction of the spin tensor does not vanish are contained in Appendix C.

Natural units ($G = c = \hbar = 1$) are used throughout this paper.

2 Gauge Theory Gravity

We begin with a brief introduction to the 'spacetime algebra' — the geometric (or Clifford) algebra of spacetime. This is familiar to physicists in the guise of the algebra generated from the Dirac γ-matrices. Further details may be found in [21, 22]. The spacetime algebra (STA) is generated by four vectors $\{\gamma_\mu\}, \mu = 0 \ldots 3$, equipped with an associative (Clifford) product, denoted by juxtaposition. The symmetrised and antisymmetrised products define the inner and outer products between vectors, denoted by a dot and a wedge respectively:

$$\begin{aligned} \gamma_\mu \cdot \gamma_\nu &\equiv \tfrac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = \eta_{\mu\nu} = \operatorname{diag}(+ - - -) \\ \gamma_\mu \wedge \gamma_\nu &\equiv \tfrac{1}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu). \end{aligned} \tag{2.1}$$

The outer product of two vectors defines a bivector — a directed plane segment, representing the plane including the two vectors.

A full basis for the STA is provided by the set

$$\begin{array}{ccccc} 1 & \{\gamma_\mu\} & \{\sigma_k, i\sigma_k\} & \{i\gamma_\mu\} & i \\ \text{1 scalar} & \text{4 vectors} & \text{6 bivectors} & \text{4 trivectors} & \text{1 pseudoscalar} \\ \text{grade 0} & \text{grade 1} & \text{grade 2} & \text{grade 3} & \text{grade 4} \end{array} \tag{2.2}$$

where $\sigma_k \equiv \gamma_k \gamma_0, k = 1 \ldots 3$, and $i \equiv \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \sigma_1 \sigma_2 \sigma_3$. The pseudoscalar i squares to -1 and anticommutes with all odd-grade elements. The $\{\sigma_k\}$ generate the geometric algebra of Euclidean 3-space, which will be familiar as the algebra of the Pauli spin matrices. An arbitrary real superposition of the basis elements (2.2) is called a 'multivector', and these inherit the associative Clifford product of the $\{\gamma_\mu\}$ generators. For a grade-r multivector A_r and a grade-s multivector B_s we define the inner and outer products via

$$A_r \cdot B_s \equiv \langle A_r B_s \rangle_{|r-s|}, \qquad A_r \wedge B_s \equiv \langle A_r B_s \rangle_{r+s}, \tag{2.3}$$

where $\langle M\rangle_r$ denotes the grade-r part of M. The subscript 0 is dropped when denoting the scalar part of a multivector, $\langle M\rangle = \langle M\rangle_0$. We shall also make use of the commutator product,

$$A\times B \equiv \tfrac{1}{2}(AB - BA). \tag{2.4}$$

The operation of reversion, denoted by a tilde, is defined by

$$(AB)\tilde{} \equiv \tilde{B}\tilde{A}, \tag{2.5}$$

and the rule that vectors are unchanged under reversion. We adopt the convention that in the absence of brackets, inner, outer and commutator products take precedence over Clifford products.

Vectors are usually denoted in lower case Latin, or Greek for a frame of orthonormal vectors. Let $\{e_\mu\}$ be an arbitrary basis for the grade-1 vector space. The reciprocal basis, denoted by $\{e^\mu\}$, satisfies $e_\mu\cdot e^\nu = \delta_\mu^\nu$. The vector derivative $\partial_a F$ of a multivector-valued function $F(a)$, with respect to the vector argument a, is defined by

$$\partial_a F(a) \equiv e^\mu e_\mu\cdot\partial_a F(a), \tag{2.6}$$

with the directional derivative operator $e_\mu\cdot\partial_a$ defined by the limit

$$e_\mu\cdot\partial_a F(a) \equiv \lim_{\epsilon\rightarrow 0}\left(\frac{F(a+\epsilon e_\mu) - F(a)}{\epsilon}\right). \tag{2.7}$$

We shall use the symbol ∇ for the vector derivative with respect to spacetime position, so that $\nabla \equiv \partial_x$.

Linear functions mapping vectors to vectors are usually denoted with an underbar, $\underline{f}(a)$ (where a is the vector argument). The adjoint function $\bar{f}(a)$ satisfies

$$\underline{f}(a)\cdot b = a\cdot\bar{f}(b). \tag{2.8}$$

Linear functions extend to act on multivectors via the rule

$$\underline{f}(a\wedge b\wedge\cdots\wedge c) \equiv \underline{f}(a)\wedge\underline{f}(b)\wedge\cdots\wedge\underline{f}(c), \tag{2.9}$$

which defines a grade-preserving linear operation. The determinant of a linear function is defined by

$$\underline{f}(i) \equiv \det(\underline{f})i. \tag{2.10}$$

Contractions of a linear function may be written as $\partial_a\cdot\underline{f}(a)$, and the protraction of a linear function is defined to be $\partial_a\wedge\underline{f}(a)$. Throughout, the argument of a linear function is assumed to independent of position, unless stated otherwise.

Gravitational effects are introduced via two gauge fields, $\bar{h}(a) = \bar{h}(a,x)$ and $\Omega(a) = \Omega(a,x)$, where x is the STA position vector $x = x^\mu \gamma_\mu$ and the dependence on x is often left implicit. The first of these, $\bar{h}(a)$, is a position-dependent linear function mapping the vector argument a to vectors. The gauge-theoretic purpose of $\bar{h}(a)$ is to ensure covariance of the equations under arbitrary local displacements of the matter fields in the background spacetime [15, 16]. The second gauge field, $\Omega(a)$, is a position-dependent linear function which maps the vector a onto the space of bivectors. Its introduction ensures covariance of the equations under local (Lorentz) rotations of fields, at a point, in the background spacetime. It is useful to define another gauge field $\omega(a)$ by

$$\omega(a,x) \equiv \Omega(\underline{h}(a),x) \tag{2.11}$$

which also maps vectors to bivectors. This field has the advantage of transforming covariantly under position-gauge transformations $x \mapsto x'$:

$$\omega(a,x) \mapsto \omega'(a,x) \equiv \omega(a,x'). \tag{2.12}$$

From these gauge fields we assemble the covariant derivative $\mathcal{D}$, whose action on a multivector M is given by

$$\begin{aligned} \mathcal{D}M &\equiv \partial_a a\cdot\mathcal{D}M \\ &= \partial_a(a\cdot\bar{h}(\nabla)M + \omega(a)\times M), \end{aligned} \tag{2.13}$$

which also defines the operator $a\cdot\mathcal{D}$. We shall also make use of the operator $\mathcal{D}_a$ defined by

$$\mathcal{D}_a M \equiv a\cdot\nabla M + \Omega(a)\times M. \tag{2.14}$$

The covariant derivative contains a grade-raising and lowering component, so that we may write

$$\mathcal{D}M = \mathcal{D}\cdot M + \mathcal{D}\wedge M, \tag{2.15}$$

where

$$\mathcal{D}\cdot M \equiv \partial_a\cdot(a\cdot\mathcal{D}M), \qquad \mathcal{D}\wedge M \equiv \partial_a\wedge(a\cdot\mathcal{D}M). \tag{2.16}$$

The field strength corresponding to the $\Omega(a)$ gauge field is defined by

$$R(a\wedge b) \equiv a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a) + \Omega(a)\times\Omega(b). \tag{2.17}$$

From this we define the covariant Riemann tensor

$$\mathcal{R}(a\wedge b) \equiv R[\underline{h}(a\wedge b)]. \tag{2.18}$$

The Ricci tensor, Ricci scalar and Einstein tensor are formed from contractions of the Riemann tensor:

$$\begin{aligned} \text{Ricci Tensor:} \quad & \mathcal{R}(a) = \partial_b \cdot \mathcal{R}(b \wedge a) && (2.19) \\ \text{Ricci Scalar:} \quad & \mathcal{R} = \partial_a \cdot \mathcal{R}(a) && (2.20) \\ \text{Einstein Tensor:} \quad & \mathcal{G}(a) = \mathcal{R}(a) - \tfrac{1}{2} a \mathcal{R}. && (2.21) \end{aligned}$$

In any expression containing the symbol $\mathcal{R}$ the argument determines whether the Riemann or Ricci tensors, or Ricci scalar is implied.

Differential operators act on the multivector to their immediate right unless brackets are present. When acting on multivectors to which they are not immediately adjacent, the 'overdot' notation is employed to denote the scope of a differential operator, so that

$$\dot{\nabla} A \dot{B} = \partial_a A a \cdot \nabla B. \tag{2.22}$$

As an example, Leibniz' rule can be written

$$\nabla(AB) = \nabla AB + \dot{\nabla} A \dot{B}. \tag{2.23}$$

The overdot notation extends to covariant derivatives. For example, for a general multivector-valued tensor $T(b, \ldots c)$ we define

$$a \cdot \dot{\mathcal{D}} \dot{T}(b, \ldots c) \equiv a \cdot \mathcal{D} T(b, \ldots c) - T(a \cdot \mathcal{D} b, \ldots c) - \cdots - T(b, \ldots a \cdot \mathcal{D} c). \tag{2.24}$$

The resultant object $a \cdot \dot{\mathcal{D}} \dot{T}(b, \ldots c)$ is also a covariant tensor. The utility of the $a \cdot \dot{\mathcal{D}}$ operator for tensors is that it commutes with contractions and protractions. The proof is straightforward, involving only the results that

$$\partial_b a \cdot \dot{\mathcal{D}} \dot{T}(b, c, \ldots d) = a \cdot \dot{\mathcal{D}} \dot{T}'(c, \ldots d) - \omega(a) \cdot \partial_b T(b, c, \ldots d) - \partial_b T(\omega(a) \cdot b, c, \ldots d) \tag{2.25}$$

where $T'(c, \ldots d) \equiv \partial_b T(b, c, \ldots d)$, and

$$\begin{aligned} \partial_b \cdot \omega(a) T(b, \ldots) - \partial_b T(\omega(a) \cdot b, \ldots) &= \partial_b T((\omega(a) \cdot b) \cdot \partial_e e, \ldots) - \partial_b T(\omega(a) \cdot b, \ldots) \\ &= 0. \end{aligned} \tag{2.26}$$

The field equations are obtained from variation of a suitable gauge-invariant action with respect to the gravitational gauge fields and the matter fields [15]. Varying with respect to the $\bar{h}$-field leads to the GTG analogue of the Einstein equation:

$$\mathcal{G}(a) = \kappa \mathcal{T}(a), \tag{2.27}$$

where $\mathcal{T}(a)$ is the covariant matter stress-energy tensor (which need not be symmetric if spin is present) and $\kappa \equiv 8\pi$. On varying the $\Omega(a)$-field we find that

$$\mathcal{D}\wedge\bar{h}(a) = \kappa\mathcal{S}[\bar{h}(a)] + \tfrac{1}{2}\kappa[\partial_b\cdot\mathcal{S}(b)]\wedge\bar{h}(a), \tag{2.28}$$

where $\mathcal{S}(a)$ is the covariant spin tensor and a is independent of position. The definition of $\mathcal{S}(a)$ in terms of the matter Lagrangian is given in [15] and for the Dirac field it is discussed in Section 5 below.

The variation principle leading to (2.28) is different to that usually considered in the context of Poincaré gauge theory [5, 26]. There one varies the action with respect to the metric and the torsion tensors, which are considered as separate dynamical variables. In GTG we vary with respect to the $\bar{h}(a)$ and $\Omega(a)$ gauge fields, the latter of which contains information about both the Christoffel connection and the torsion. In this respect the GTG approach is closer to the Palatini principle. The fact that $\Omega(a)$ is bivector-valued and coordinate-free removes the problems with the Palatini approach in ECKS theory first discussed by Sandberg [27]. This is because the 'projective transformation' discussed in [27] is not a permissible transformation in GTG. Treating $\Omega(a)$ as a separate dynamical variable also recovers the link with a genuine gauge theory since it is the full $\Omega(a)$ that appears in the expression for minimal coupling to a Dirac wavefunction.

An important consequence of equation (2.28) is that minimally-coupled equations for the matter fields are obtained from a minimally-coupled Lagrangian only if the contraction of the spin tensor vanishes, that is, if

$$\partial_a\cdot\mathcal{S}(a) = 0. \tag{2.29}$$

This is because equation (2.28) implies

$$\mathcal{D}_a[\bar{h}(\partial_a)\det(\underline{h})^{-1}] = -\tfrac{1}{2}\kappa\partial_a\cdot\mathcal{S}(a), \tag{2.30}$$

and the left-hand side must vanish for the minimally-coupled equations to arise from the minimally-coupled action. Minimal coupling at the level of the field equations is crucial in ensuring that GTG incorporates the equivalence principle (both classical and quantum). Minimal coupling at the level of the action, on the other hand, ensures that one has a genuine gauge theory and is important for constructing a quantum field theory. We therefore believe that equation (2.29) is a necessary requirement of all physical fields and this condition is assumed throughout the main part of this paper. Results for the case when $\partial_a\cdot\mathcal{S}(a)$ is not zero are given in Appendix C. For scalar fields and Yang-Mills gauge fields the spin tensor vanishes. For spin-1/2 fields the spin tensor is of the form

$$\mathcal{S}(a) = T\cdot a, \tag{2.31}$$

where T is the spin trivector. This form necessarily has vanishing contraction. The picture is slightly more complicated for spin-1 and spin-3/2 matter fields, where there is some freedom in the equations that one can work with. In both these cases, however, it is possible to construct theories in which equation (2.29) does hold. This work will be presented elsewhere.

Given that the spin tensor satisfies equation (2.29), equation (2.28) reduces to the 'wedge' equation

$$\mathcal{D}\wedge\bar{h}(a) = \partial_b\wedge[b\cdot\mathcal{D}\bar{h}(a)] = \kappa\mathcal{S}[\bar{h}(a)], \tag{2.32}$$

or, in the case where a is position-dependent,

$$\mathcal{D}\wedge\bar{h}(a) = \bar{h}(\nabla\wedge a) + \kappa\mathcal{S}[\bar{h}(a)]. \tag{2.33}$$

We refer to the left-hand side of equation (2.32) as the torsion. In the spin-torsion extension of General Relativity, the torsion is measured by the antisymmetric part of the connection (the symmetric part is fixed by the requirement of metric compatibility) on the spacetime manifold. The relationship between these two expressions is described in Appendix D.

The criterion that led us to restrict the torsion type also constrains the Lagrangian for the free gravitational field. In particular, it rules out the quadratic curvature terms in the Lagrangian often considered in the context of Poincaré gauge theory [5], [28]–[30]. This is because the higher-order terms in the field equations prevent one from deriving $\mathcal{D}_a\bar{h}(\partial_a \det(\underline{h})^{-1}) = 0$. The only additional terms that can be added to the Lagrangian are quadratic in the torsion [5, 31]. Such terms contain derivatives of the $\bar{h}$-field, however, so considerably complicate the theory. Indeed, the theory looks so simple and compelling with no derivatives of the $\bar{h}$-function present that it is tempting to view their absence as suggestive of a deeper principle. We discuss some ways that derivatives of $\bar{h}$-field can be incorporated into the action in Section 5, however, and it is a simple matter to adapt the present work for the presence of quadratic torsion terms.

Equations (2.27) and (2.28) are locally the same as those of ECKS theory. This can be made clear by introducing a coordinate frame and following the scheme described in Appendix D. Globally, however, the theories are not equivalent. This is because GTG is explicitly constructed as a gauge theory in a flat spacetime and there is no possibility of the fields altering the topology of this background spacetime. This leeds to a number of differences between the two theories [15, 17, 18], though these differences are not the subject of the present paper. Our aim here is to show that GTG, together with the techniques of spacetime algebra, simplify many otherwise difficult derivations required in all of GTG, ECKS theory and General Relativity.

This completes the definitions of the quantities required in this paper. Further details may be found in [15].

3 The Riemann Tensor

The definition (2.18) for the Riemann tensor can be manipulated into a manifestly covariant form. We begin by introducing the derivatives

$$L_a \equiv a\cdot\bar{h}(\nabla) \tag{3.1}$$

and writing

$$\mathcal{R}(a\wedge b) = L_a\omega(b) - L_b\omega(a) - \omega[\underline{h}^{-1}(L_a\underline{h}(b) - L_b\underline{h}(a))] + \omega(a)\times\omega(b), \tag{3.2}$$

where in this section the vectors a, b *etc.* are assumed to be arbitrary functions of position. To eliminate the derivatives of the $\underline{h}$-function, we first write the 'wedge' equation (2.32) in the form

$$\begin{aligned} \bar{h}(\dot{\nabla})\wedge\dot{\bar{h}}(c) &= -\partial_d\wedge[\omega(d)\cdot\bar{h}(c)] + \kappa\mathcal{S}[\bar{h}(c)] \\ \implies \quad \langle b\wedge a\,\bar{h}(\dot{\nabla})\wedge\dot{\bar{h}}(c)\rangle &= -\langle b\wedge a\,\partial_d\wedge[\omega(d)\cdot\bar{h}(c)]\rangle + \kappa\langle b\wedge a\,\mathcal{S}[\bar{h}(c)]\rangle \\ \implies \quad [\dot{L}_a\dot{\underline{h}}(b) - \dot{L}_b\dot{\underline{h}}(a)]\cdot c &= [a\cdot\omega(b) - b\cdot\omega(a)]\cdot\bar{h}(c) - \kappa\bar{\mathcal{S}}(a\wedge b)\cdot\bar{h}(c), \end{aligned} \tag{3.3}$$

where overdots denote the scope of the differential operator and $\bar{\mathcal{S}}(a)$ is the adjoint to the spin tensor, defined by

$$\bar{\mathcal{S}}(B)\cdot a = B\cdot\mathcal{S}(a), \tag{3.4}$$

for an arbitrary vector a and bivector B. The adjoint function $\bar{\mathcal{S}}(B)$ is a vector-valued linear function of its bivector argument B. It follows that

$$\begin{aligned} \underline{h}^{-1}(L_a\underline{h}(b) - L_b\underline{h}(a)) &= L_a b - L_b a + a\cdot\omega(b) - b\cdot\omega(a) - \kappa\bar{\mathcal{S}}(a\wedge b) \\ &= a\cdot\mathcal{D}b - b\cdot\mathcal{D}a - \kappa\bar{\mathcal{S}}(a\wedge b). \end{aligned} \tag{3.5}$$

If we denote the right-hand side of (3.5) by c then we can write the Riemann tensor as

$$\mathcal{R}(a\wedge b) = L_a\omega(b) - L_b\omega(a) + \omega(a)\times\omega(b) - \omega(c), \tag{3.6}$$

which only differs from the torsion-free expression given in [15] by the definition of the vector c.

Evaluating the commutator of L_a and L_b gives

$$\begin{aligned} [L_a, L_b] &= [L_a\underline{h}(b) - L_b\underline{h}(a)]\cdot\nabla \\ &= c\cdot\bar{h}(\nabla) \\ &= L_c, \end{aligned} \tag{3.7}$$

where c is given by the right-hand side of (3.5). This bracket structure summarises the content of the 'wedge' equation (2.32) in a form that is independent of the torsion. The structure is essentially that of the Lie bracket, though its use in GTG is somewhat different. A new method of solving the field equations, which exploits this bracket structure, was given in [15]. This 'intrinsic' method has the advantage of yielding a set of first-order equations which, although still non-linear, are generally more tractable than their second-order counterparts in General Relativity.

When solving the field equations in the presence of torsion it proves useful to introduce the new field

$$\omega'(a) = \omega(a) + \kappa\mathcal{S}(a) - \tfrac{3}{2}\kappa a{\cdot}T \tag{3.8}$$

where

$$T \equiv \tfrac{1}{3}\partial_a\wedge\mathcal{S}(a) \tag{3.9}$$

is the spin trivector. The $\omega'(a)$-field satisfies the equation

$$\mathcal{D}'\wedge\bar{h}(a) \equiv \partial_b\wedge(\dot{L}_b\dot{\bar{h}}(a) + \omega'(b){\cdot}\bar{h}(a)) = 0 \tag{3.10}$$

and is the ω-function in the absence of torsion. We can derive an expression for $\omega'(a)$ in terms of the $\bar{h}$-function alone by inverting (3.10) to yield [15]

$$\omega'(a) = -H(a) + \tfrac{1}{2}a{\cdot}(\partial_b\wedge H(b)), \tag{3.11}$$

where

$$H(a) \equiv \bar{h}(\nabla\wedge\bar{h}^{-1}(a)) = -\bar{h}(\dot{\nabla})\wedge\dot{\bar{h}}(\bar{h}^{-1}(a)). \tag{3.12}$$

In GTG, $\omega'(a)$ plays the same role as the Christoffel symbol (the symmetric part of the connection) in the spin-torsion extension of General Relativity.

On defining

$$\varpi(a) \equiv \omega(a) - \omega'(a) = -\kappa\mathcal{S}(a) + \tfrac{3}{2}\kappa a{\cdot}T \tag{3.13}$$

we see that the Riemann tensor can be written as

$$\mathcal{R}(a\wedge b) = \mathcal{R}'(a\wedge b) + a{\cdot}\dot{\mathcal{D}}'\dot{\varpi}(b) - b{\cdot}\dot{\mathcal{D}}'\dot{\varpi}(a) + \varpi(a)\times\varpi(b), \tag{3.14}$$

where the primes denote the equivalent quantities in the absence of torsion. (In the literature these are often denoted with a pair of curly braces to denote the use of the Christoffel connection.)

On contracting equation (3.14) with ∂_a we find the following expression for the Ricci tensor:

$$\begin{aligned}\mathcal{R}(b) =&\mathcal{R}'(b) + \dot{\mathcal{D}}'{\cdot}\dot{\varpi}(b) - b{\cdot}\mathcal{D}'[\partial_a{\cdot}\varpi(a)] + \partial_a{\cdot}[\varpi(a)\times\varpi(b)]\\ =&\mathcal{R}'(b) - \kappa\dot{\mathcal{D}}'{\cdot}\dot{\mathcal{S}}(b) - \tfrac{3}{2}\kappa b{\cdot}(\mathcal{D}'{\cdot}T) + \langle\partial_a\varpi(a)\varpi(b)\rangle_1 - \partial_a\,\varpi(a){\cdot}\varpi(b)\\ =&\mathcal{R}'(b) - \kappa\dot{\mathcal{D}}'{\cdot}\dot{\mathcal{S}}(b) - \tfrac{3}{2}\kappa b{\cdot}(\mathcal{D}'{\cdot}T) - \kappa^2\bar{\mathcal{S}}[\mathcal{S}(b)] + \tfrac{3}{2}\kappa^2\bar{\mathcal{S}}(b{\cdot}T),\end{aligned} \tag{3.15}$$

where again $\mathcal{R}'(a)$ is the Ricci tensor in the absence of torsion and we have assumed a vanishing contraction of the torsion tensor. Finally, we make a further contraction to obtain the following expression for the Ricci scalar:

$$\mathcal{R} = \mathcal{R}' + \tfrac{9}{2}\kappa^2 T^2 - \kappa^2 \mathcal{S}(\partial_a)\cdot\mathcal{S}(a). \tag{3.16}$$

No derivatives of $\mathcal{S}(a)$ appear in this expression, which is a feature of assuming that the spin tensor has a zero contraction [5, 31].

4 Symmetries and Conservation Laws

In this section we discuss some of the symmetries of the Riemann tensor and conservation laws of GTG when torsion is present. These have already been given in [15], for the case of vanishing torsion. The zero torsion results may be combined with the expressions of the previous section to extend them to the torsion sector. However, for reasons of completeness, we shall not follow this route but shall give derivations from 'first principles'. Many of these results have immediate counterparts in differential geometry, though the conciseness of their derivations here compares favourably with tensor-calculus techniques.

Our starting point is the equation

$$\mathcal{D}\wedge\bar{h}(c) = \kappa\mathcal{S}[\bar{h}(c)], \tag{4.1}$$

where c is independent of position. Taking the covariant exterior derivative of this equation gives

$$\mathcal{D}\wedge(\mathcal{D}\wedge\bar{h}(c)) = \kappa\mathcal{D}\wedge\mathcal{S}[\bar{h}(c)]. \tag{4.2}$$

But,

$$\begin{aligned}\mathcal{D}\wedge(\mathcal{D}\wedge\bar{h}(c)) &= \bar{h}(\partial_a)\wedge\mathcal{D}_a[\bar{h}(\partial_b)\wedge(\mathcal{D}_b\bar{h}(c))]\\ &= [\mathcal{D}\wedge\bar{h}(\partial_b)]\wedge[\mathcal{D}_b\bar{h}(c)] + \bar{h}(\partial_a)\wedge\bar{h}(\partial_b)\wedge[\mathcal{D}_a\mathcal{D}_b\bar{h}(c)]\\ &= \kappa\mathcal{S}[\bar{h}(\partial_b)]\wedge[\mathcal{D}_b\bar{h}(c)] + \tfrac{1}{2}\bar{h}(\partial_a)\wedge\bar{h}(\partial_b)\wedge[R(a\wedge b)\cdot\bar{h}(c)],\end{aligned} \tag{4.3}$$

so that

$$\begin{aligned}(2\kappa)^{-1}\partial_a\wedge\partial_b\wedge[\mathcal{R}(a\wedge b)\cdot\bar{h}(c)] &= \mathcal{D}\wedge\mathcal{S}[\bar{h}(c)] - \mathcal{S}[\bar{h}(\partial_b)]\wedge[\mathcal{D}_b\bar{h}(c)]\\ &= \dot{\mathcal{D}}\wedge\dot{\mathcal{S}}[\bar{h}(c)] + \partial_a\wedge\mathcal{S}[a\cdot\mathcal{D}\bar{h}(c)] - \mathcal{S}(\partial_b)\wedge[b\cdot\mathcal{D}\bar{h}(c)].\end{aligned} \tag{4.4}$$

The final two terms on the right-hand side of equation (4.4) are

$$\begin{aligned}\partial_a\wedge\mathcal{S}[a\cdot\mathcal{D}\bar{h}(c)] - \mathcal{S}(\partial_b)\wedge[b\cdot\mathcal{D}\bar{h}(c)] &= \mathcal{S}(\partial_a)\wedge[\partial_b\, a\cdot(b\cdot\mathcal{D}\bar{h}(c)) - a\cdot\mathcal{D}\bar{h}(c)]\\ &= -\mathcal{S}(\partial_a)\wedge[a\cdot(\mathcal{D}\wedge\bar{h}(c))]\\ &= -\kappa\mathcal{S}(\partial_a)\wedge[a\cdot\mathcal{S}(\bar{h}(c))],\end{aligned} \tag{4.5}$$

so equation (4.4) reduces to the covariant equation

$$\partial_a\wedge\partial_b\wedge[\mathcal{R}(a\wedge b)\cdot c] = 2\kappa\dot{\mathcal{D}}\wedge\dot{\mathcal{S}}(c) - 2\kappa^2\mathcal{S}(\partial_a)\wedge[a\cdot\mathcal{S}(c)]. \tag{4.6}$$

Protracting equation (4.6) with ∂_c yields

$$\partial_a\wedge\partial_b\wedge\mathcal{R}(a\wedge b) = 3\kappa\mathcal{D}\wedge T + \kappa^2\mathcal{S}(\partial_a)\wedge\mathcal{S}(a), \tag{4.7}$$

and dotting back with c gives

$$2\partial_b\wedge\mathcal{R}(c\wedge b) - \partial_a\wedge\partial_b\wedge[\mathcal{R}(a\wedge b)\cdot c] = 3\kappa c\,\mathcal{D}\wedge T + \kappa^2 c\,\mathcal{S}(\partial_a)\wedge\mathcal{S}(a). \tag{4.8}$$

Hence, applying equation (4.6) again, we find that

$$\partial_a\wedge\mathcal{R}(a\wedge b) = -\kappa[\dot{\mathcal{D}}\wedge\dot{\mathcal{S}}(b) + \tfrac{3}{2}b\,\mathcal{D}\wedge T] + \kappa^2\mathcal{S}(\partial_a)\wedge[a\cdot\mathcal{S}(b)] - \tfrac{1}{2}\kappa^2 b\,\mathcal{S}(\partial_a)\wedge\mathcal{S}(a). \tag{4.9}$$

In the absence of spin the right-hand side of (4.9) vanishes, leaving the simple equation $\partial_a\wedge\mathcal{R}(a\wedge b) = 0$ which, as noted in [15], encodes all the symmetries of the Riemann tensor in the absence of torsion.

The adjoint function to $\mathcal{R}(B)$ is written $\bar{\mathcal{R}}(B)$ and is defined by

$$A\cdot\mathcal{R}(B) = B\cdot\bar{\mathcal{R}}(A) \tag{4.10}$$

for any two bivectors A and B. The Riemann tensor and its adjoint are related by

$$\mathcal{R}(B) - \bar{\mathcal{R}}(B) = -\partial_b\wedge[B\cdot(\partial_a\wedge\mathcal{R}(a\wedge b))] + \tfrac{1}{2}B\,\partial_b\wedge\partial_a\wedge\mathcal{R}(a\wedge b) \tag{4.11}$$

and equation (4.9) can be used to express the right-hand side in terms of the spin. Clearly a vanishing spin tensor implies that $\mathcal{R}(B) = \bar{\mathcal{R}}(B)$, though the relation $\partial_a\wedge\mathcal{R}(a\wedge b) = 0$ contains more information and is more useful in practice.

Returning to equation (4.9) and contracting with ∂_b we obtain

$$\partial_a\wedge\mathcal{R}(a) = \partial_a\wedge\mathcal{G}(a) = -\kappa\dot{\mathcal{S}}(\dot{\mathcal{D}}) + \kappa^2[\partial_b\cdot\mathcal{S}(\partial_a)]\wedge[a\cdot\mathcal{S}(b)], \tag{4.12}$$

and the same expression must hold for the matter stress-energy tensor. The final term can be simplified by noting that

$$i\bar{\mathcal{S}}(iB) = i\partial_a\langle iB\mathcal{S}(a)\rangle = \mathcal{S}(a)\wedge(\partial_a\cdot B) \tag{4.13}$$

(since $\partial_a\cdot\mathcal{S}(a)$=0). We can therefore write

$$[\partial_b\cdot\mathcal{S}(\partial_a)]\wedge[a\cdot\mathcal{S}(b)] = \partial_b\cdot[i\bar{\mathcal{S}}(i\mathcal{S}(b))] = -i\partial_b\wedge\bar{\mathcal{S}}(i\mathcal{S}(b)). \tag{4.14}$$

The gauge theory version of the Bianchi identity follows from the Jacobi identity applied to covariant derivatives of an arbitrary multivector M:

$$[\mathcal{D}_a, [\mathcal{D}_b, \mathcal{D}_c]]M + \text{cyclic permutations} = 0. \tag{4.15}$$

Evaluating the commutators we find that

$$\mathcal{D}_a R(b\wedge c) + \text{cyclic permutations} = 0, \tag{4.16}$$

which we want to replace by a covariant expression. We first extend to the case where a, b and c are position-dependent:

$$\mathcal{D}_a R(b\wedge c) - R[(a\cdot\nabla b - b\cdot\nabla a)\wedge c] + \text{cyclic permutations} = 0. \tag{4.17}$$

Next, we replace a with $\underline{h}(a)$ *etc.* to find

$$a\cdot\mathcal{D}\mathcal{R}(b\wedge c) - \mathcal{R}[(\underline{h}^{-1}L_a\underline{h}(b) - \underline{h}^{-1}L_b\underline{h}(a))\wedge c] + \text{cyclic permutations} = 0. \tag{4.18}$$

Recalling equation (3.5), equation (4.18) becomes

$$\begin{aligned} a\cdot\mathcal{D}\mathcal{R}(b\wedge c) &- \mathcal{R}[(a\cdot\mathcal{D}b - b\cdot\mathcal{D}a)\wedge c] \\ &+ \kappa\mathcal{R}[\bar{\mathcal{S}}(a\wedge b)\wedge c] + \text{cyclic permutations} = 0, \end{aligned} \tag{4.19}$$

which simplifies to give

$$a\cdot\dot{\mathcal{D}}\dot{\mathcal{R}}(b\wedge c) + \kappa\mathcal{R}[\bar{\mathcal{S}}(a\wedge b)\wedge c] + \text{cyclic permutations} = 0. \tag{4.20}$$

The adjoint form of this equation is

$$\partial_c\wedge\partial_b\wedge\partial_a\langle a\cdot\dot{\mathcal{D}}\dot{\mathcal{R}}(b\wedge c)B\rangle = -\kappa\partial_c\wedge\partial_b\wedge\partial_a\langle\mathcal{R}[\bar{\mathcal{S}}(a\wedge b)\wedge c]B\rangle, \tag{4.21}$$

where B is an arbitrary bivector. Equation (4.21) evaluates to

$$\dot{\mathcal{D}}\wedge\dot{\bar{\mathcal{R}}}(B) = \kappa\partial_a\,\mathcal{S}(a)\wedge\bar{\mathcal{R}}(B), \tag{4.22}$$

or, alternatively,

$$\partial_a\wedge[a\cdot\mathcal{D}\bar{\mathcal{R}}(B) - \bar{\mathcal{R}}(a\cdot\mathcal{D}B)] = \kappa\partial_a\,\mathcal{S}(a)\wedge\bar{\mathcal{R}}(B), \tag{4.23}$$

which is our gauge theory version of the Bianchi identity.

On contracting equation (4.22) we obtain

$$\begin{aligned} (\partial_a\wedge\partial_b)\cdot[\dot{\mathcal{D}}\wedge\dot{\bar{\mathcal{R}}}(a\wedge b)] &= \kappa(\partial_a\wedge\partial_b\wedge\partial_c)\,\mathcal{S}(c)\wedge\bar{\mathcal{R}}(a\wedge b), \\ \implies \quad 2b\cdot\dot{\mathcal{D}}\,\dot{\bar{\mathcal{R}}}(\partial_b) - \mathcal{D}\mathcal{R} &= \kappa(\partial_a\wedge\partial_b\wedge\partial_c)\,\mathcal{S}(c)\wedge\bar{\mathcal{R}}(a\wedge b), \end{aligned} \tag{4.24}$$

which may be written as

$$\dot{\bar{\mathcal{G}}}(\dot{\mathcal{D}}) \equiv a\cdot\dot{\mathcal{D}}\dot{\bar{\mathcal{G}}}(\partial_a) = -\kappa\partial_a\cdot\bar{\mathcal{R}}[\mathcal{S}(a)] + \kappa\partial_a\cdot\mathcal{S}[\bar{\mathcal{R}}(a)]. \tag{4.25}$$

Taking the inner product with an arbitrary vector b, we obtain the adjoint form

$$\dot{\mathcal{D}}\cdot\dot{\mathcal{G}}(a) = \kappa\mathcal{S}(\partial_b)\cdot\mathcal{R}(b\wedge a) - \kappa[a\cdot\mathcal{S}(\partial_b)]\cdot\mathcal{R}(b). \tag{4.26}$$

In the absence of torsion, we obtain the simple result

$$\dot{\mathcal{G}}(\dot{\mathcal{D}}) = 0, \tag{4.27}$$

which is the GTG version of covariant conservation of the Einstein tensor.

5 The Self-Consistent Dirac Field

A simple way to produce gauge fields with torsion is to consider matter described by a Dirac spinor field $|\psi(x)\rangle$. In the STA, it is convenient to represent such a spinor by an even multivector ψ (see [32, 33] for an explicit map for the Dirac-Pauli representation). The $\{\hat{\gamma}_\mu\}$ operators, and the conventional unit scalar imaginary j have actions which are represented by:

$$\begin{aligned} \hat{\gamma}_\mu|\psi\rangle &\leftrightarrow \gamma_\mu\psi\gamma_0 \qquad (\mu = 0\ldots 3) \\ j|\psi\rangle &\leftrightarrow \psi i\sigma_3. \end{aligned} \tag{5.1}$$

In this manner, all matrix manipulations are eliminated, and the geometric content of the Dirac theory is made manifest [34].

Spinors transform single sidedly under Lorentz rotations, which leads to the introduction of the spinor covariant derivative $D\psi \equiv \partial_a a\cdot D\psi$ where

$$a\cdot D\psi \equiv L_a\psi + \tfrac{1}{2}\omega(a)\psi. \tag{5.2}$$

The equations of motion for the Dirac field and the gauge fields are derived from the minimally coupled action [15]

$$S_{\mathrm{ECD}} = \int |d^4x| \det(\underline{h})^{-1}\,[\tfrac{1}{2}\mathcal{R} - \kappa\langle D\psi i\gamma_3\tilde{\psi} - m\psi\tilde{\psi}\rangle], \tag{5.3}$$

where m is the mass of the particle. Varying with respect to ψ and the gravitational gauge fields leads to the system of equations

$$\begin{aligned} \text{`wedge':} \qquad & \mathcal{D}\wedge\bar{h}(a) = \kappa T\cdot\bar{h}(a) \\ \text{Einstein:} \qquad & \mathcal{G}(a) = \kappa\mathcal{T}(a) \\ \text{Dirac:} \qquad & D\psi i\sigma_3 = m\psi\gamma_0, \end{aligned} \tag{5.4}$$

where

$$T \equiv \tfrac{1}{2}\psi i\gamma_3\tilde{\psi} \tag{5.5}$$

is the spin trivector, and the matter stress-energy tensor $\mathcal{T}(a)$ is given by [15]

$$\mathcal{T}(a) = \langle a\cdot D\psi i\gamma_3\tilde{\psi}\rangle_1. \tag{5.6}$$

These equations are the gauge theory equivalents of the Einstein-Cartan-Dirac equations. The 'wedge' equation summarises Cartan's suggestion [10] that the torsion be identified with the spin of the matter present. Note that the minimally-coupled Lagrangian has given rise to the minimally-coupled Dirac equation, on variation of ψ. As noted in the introduction, this is because 'trivector' spin of the type $\mathcal{S}(a) = T\cdot a$ automatically has a vanishing contraction. A number of results for the gravitational field particular to this case of trivector spin are contained in Appendix A.

Protracting the stress-energy tensor (5.6) with ∂_a, we find that

$$\begin{aligned}\partial_a\wedge\mathcal{T}(a) &= \partial_a\wedge\langle a\cdot D\psi i\gamma_3\tilde{\psi}\rangle_1 \\ &= \tfrac{1}{2}\langle D\psi i\gamma_3\tilde{\psi} - \partial_a\psi i\gamma_3(a\cdot D\psi)\tilde{}\,\rangle_2.\end{aligned} \tag{5.7}$$

However, the Dirac equation implies that $\langle D\psi i\gamma_3\tilde{\psi}\rangle_2 = 0$, so we can write

$$\begin{aligned}\partial_a\wedge\mathcal{T}(a) &= -\tfrac{1}{2}\langle\partial_a[a\cdot D\psi i\gamma_3\tilde{\psi} + \psi i\gamma_3(a\cdot D\psi)\tilde{}\,]\rangle_2 \\ &= -\langle\partial_a(a\cdot\mathcal{D}T)\rangle_2 \\ &= -\mathcal{D}\cdot T.\end{aligned} \tag{5.8}$$

This demonstrates that the Dirac equation and the definition (5.6) for the stress-energy tensor are consistent with equation (4.12), since the final term on the right-hand side of equation (4.12) vanishes for the case of trivector spin. It can also be shown that the Dirac equation is consistent with the contracted Bianchi identity (4.25). The proof of this is simplified by employing the results of Appendix A. A new, exact solution to the Einstein-Cartan-Dirac equations in GTG, describing an almost homogeneous universe with a spin-induced anisotropy, is given in [25].

The gauge-invariant action (5.3) naturally gives rise to a spin-torsion theory, with the torsion (measured by the left-hand side of the 'wedge' equation) given by the spin of the matter. Such an interaction is inevitable if the gravitational fields are introduced via a gauge theoretic route. This is good reason to prefer the Einstein-Cartan-Dirac equations over the Einstein-Dirac equations, which result from coupling the Dirac equation with the torsion-free spacetime of General Relativity. However, as most of the literature focusus on solutions of the Einstein-Dirac equations (see, for example, [35]–[38]), it is interesting to see how the present techniques can be adapted to such a setup. We start by replacing $\omega(a)$ in the action by the equivalent expression in terms of $\bar{h}(a)$ and its derivatives. The action then becomes

$$S_{\mathrm{ED}} = \int |d^4x| \det(\underline{h})^{-1} \left[\tfrac{1}{2}\mathcal{R}' - \kappa\langle\bar{h}(\nabla)\psi i\gamma_3\tilde{\psi} + \tfrac{1}{4}\partial_a\wedge H(a)\psi i\gamma_3\tilde{\psi} - m\psi\tilde{\psi}\rangle\right], \tag{5.9}$$

where $H(a)$ is as defined in equation (3.12) and $\mathcal{R}'$ is now a function of $\bar{h}(a)$ and its first and second derivatives. The dynamical variables are now $\bar{h}(a)$ and ψ, with the 'wedge' equation replaced by the identity

$$\mathcal{D}'\wedge\bar{h}(a) = 0. \tag{5.10}$$

Variation with respect to ψ produces the equation

$$D'\psi i\sigma_3 = m\psi\gamma_0, \tag{5.11}$$

where the prime again denotes that the covariant derivative does not contain a contribution from the spin. Variation with respect to $\bar{h}(a)$ yields

$$\mathcal{G}'(a) = \kappa\mathcal{T}'(a) \tag{5.12}$$

where $\mathcal{T}'(a)$ is obtained by variation of the matter action with respect to $\bar{h}(a)$. The derivation of $\mathcal{T}'(a)$ is more complicated now due to the presence of derivatives of the $\bar{h}(a)$-function in the matter Lagrangian. The calculations are not too difficult, however, and yield

$$\mathcal{T}'(a) = \langle a\cdot D'\psi i\gamma_3\tilde{\psi}\rangle_1 + \tfrac{1}{2}a\cdot(\mathcal{D}'\cdot T). \tag{5.13}$$

Satisfyingly $\mathcal{T}'(a)$ is a symmetric tensor, which it has to be since $\mathcal{G}'(a)$ is automatically symmetric. The pair of equations (5.11) and (5.12) constitute the GTG version of the Einstein-Dirac equations. These are usually obtained from the system (5.4) by the Belifante-Rosenfield procedure [39, 40]. It is perhaps surprising that to derive the equations rigorously from an action involves replacing the symmetric metric tensor $g_{\mu\nu}$ with $\bar{h}(a)$ as the dynamical variable, since it is the latter that unambiguously yields a symmetric stress-energy tensor.

Despite the neat features of the GTG derivation of the Einstein-Dirac set of equations, there are still good reasons to prefer the Einstein-Cartan-Dirac set, though of course no experiment has yet been devised which could distinguish between them. The key reason, as stated earlier, is that the Einstein-Cartan-Dirac set maintain the link with gauge theories, which is lost if one pushes the Riemannian geometry viewpoint of the Einstein-Dirac set. Such considerations are clearly important in attempting to construct a fully quantum theory. Furthermore, the clean separation into the $\bar{h}(a)$ and $\Omega(a)$ gauge fields of the GTG approach is particularly appealing in looking for extensions to a multiparticle quantum theory of gravity.

It is worth pointing out that if $\bar{h}(a), \omega(a)$ and ψ satisfy equation (5.4), then the fields $\bar{h}(a), \omega'(a)$ and ψ satisfy equations (5.11) and (5.12) if the condition

$$T\psi = \tfrac{1}{2}\psi i\gamma_3\tilde{\psi}\psi = 0 \tag{5.14}$$

is satisfied. It is a simple matter to show that this is equivalent to the condition [21]

$$\tilde{\psi}\psi = \psi\tilde{\psi} = 0. \tag{5.15}$$

The proof of the analogous result between Einstein-Dirac and Einstein-Cartan-Dirac theory was given by Seitz in [23]. The gauge theory proof is given in Appendix B. Condition (5.15) is always satisfied for neutrino solutions.

6 Spinning Point-Particle Models

In recent years there has been considerable interest in obtaining semi-classical equations describing the dynamics of a fermion, while maintaining the concept of a definite particle trajectory $x(\lambda)$ (see, for example, [41, 42]). One of the aims of this work is to obtain a quantum theory via a path-integral quantisation of a suitable classical model [43, 44]. In [15], an action was proposed to describe the dynamics of a spin-1/2 particle in a gravitational background, but the resulting equations were only given in a full classical approximation, which neglected the spin. In this section we derive the semi-classical equations of motion resulting from this action, including the effects of torsion in the gravitational background.

The action is given by

$$S = \int d\lambda \, \langle \dot{\Psi} i\sigma_3 \tilde{\Psi} + \tfrac{1}{2}\Omega(\dot{x})\Psi i\sigma_3 \tilde{\Psi} + p(v - m_p e \Psi\gamma_0\tilde{\Psi}) + m_p^2 e \rangle, \tag{6.1}$$

where $v \equiv \underline{h}^{-1}(\dot{x})$ is the covariant tangent vector to the trajectory and, for this section, overdots denote differentiation with respect to λ. The Lagrange multiplier p is identified with the momentum of the particle (which has mass m_p) and an einbein e is included to ensure invariance of the equations under reparameterisation of the path. The spinor $\Psi(\lambda)$ contains information about the spin and velocity of the particle. In general, p and v will not be collinear. The dynamical variables are $x(\lambda), \Psi(\lambda), p(\lambda)$ and $e(\lambda)$. Varying Ψ, p and e gives the equations

$$\dot{\Psi} i\sigma_3 + \tfrac{1}{2}\omega(v)\Psi i\sigma_3 = m_p e p \Psi\gamma_0 \tag{6.2}$$

$$v = m_p e \Psi\gamma_0\tilde{\Psi} \tag{6.3}$$

$$p\cdot v = e m_p^2. \tag{6.4}$$

Equation (6.2) may be written in manifestly covariant form

$$v\cdot D\Psi i\sigma_3 = m_p e p \Psi\gamma_0. \tag{6.5}$$

Equations of motion for v and the spin bivector $S \equiv \Psi i\sigma_3\tilde{\Psi}$ follow from (6.2). We find that

$$e v\cdot\mathcal{D}(v/e) = -2m_p^2 e^2 p\cdot S \tag{6.6}$$

$$v\cdot\mathcal{D}S = 2p\wedge v. \tag{6.7}$$

In the presence of trivector type torsion the geodesic equation reduces to $v\cdot\mathcal{D}v = 0$, where v is the unit tangent vector to the path (this is not true for more general types of torsion, in which case it is necessary to distinguish between geodesics and auto-parallels). It follows from (6.6) that, in general, the motion is not geodesic. Furthermore, if we construct the unit spin vector along $\Psi\gamma_3\tilde{\Psi}$, we find that this is not Fermi transported in general. These effects, which are present even in the absence of gravitation [33], were present in the classical dipole model of Papapetrou [45].

The equation of motion for p is obtained by variation of x. We find that

$$\partial_\lambda \bar{h}^{-1}(p) + \tfrac{1}{2}\partial_\lambda\partial_a(\Omega(a)\cdot S) = \overset{*}{\nabla}(\dot{x}\cdot\overset{*}{\bar{h}}{}^{-1}(p)) + \tfrac{1}{2}\partial_a\langle a\cdot\overset{*}{\nabla}\overset{*}{\Omega}(\dot{x})S\rangle, \tag{6.8}$$

where, for this section, we employ overstars in place of overdots for the scope of a differential operator. This equation yields

$$\begin{aligned}\bar{h}^{-1}(\dot{p}) + \underline{h}(v)\cdot(\overset{*}{\nabla}\wedge\overset{*}{\bar{h}}{}^{-1}(p)) &= \tfrac{1}{2}\partial_a\langle(a\cdot\overset{*}{\nabla}\overset{*}{\Omega}(\dot{x}) - \dot{x}\cdot\overset{*}{\nabla}\overset{*}{\Omega}(a))S - \Omega(a)\dot{S}\rangle \\ &= \tfrac{1}{2}\bar{h}^{-1}[v\cdot\bar{\mathcal{R}}(S)] - \tfrac{1}{2}\partial_a\,\Omega(a)\cdot[\dot{S} + \omega(v)\times S].\end{aligned} \tag{6.9}$$

We now employ equation (6.7) in the right-hand side of (6.9) to find (after application of $\bar{h}$ to both sides)

$$\dot{p} - v\cdot[\bar{h}(\overset{*}{\nabla})\wedge\overset{*}{\bar{h}}(\bar{h}^{-1}(p))] = \tfrac{1}{2}v\cdot\bar{\mathcal{R}}(S) - \partial_a\,\omega(a)\cdot(p\wedge v). \tag{6.10}$$

Hence, recalling equation (2.32), we find the manifestly covariant equation of motion for p

$$v\cdot\mathcal{D}p = \tfrac{1}{2}v\cdot\bar{\mathcal{R}}(S) + \kappa v\cdot\mathcal{S}(p). \tag{6.11}$$

The presence of $\bar{\mathcal{R}}(S)$ and $\mathcal{S}(p)$ in this equation imply a coupling of the particle to the torsion, unlike some previous models [24]. We exploit this coupling in another paper [25] to discuss the torsion-induced anisotropy of a cosmological solution for a self-consistent Dirac field. We can use equation (6.7) to substitute for p in (6.11). This gives the equation

$$v\cdot\mathcal{D}\left(\frac{em_p^2 v}{v^2} - \frac{v\cdot(v\cdot\mathcal{D}S)}{2v^2}\right) = \tfrac{1}{2}v\cdot\bar{\mathcal{R}}(S) + \kappa v\cdot\mathcal{S}(p), \tag{6.12}$$

which is the analogue of the result found by Papapetrou [45] in the absence of torsion (his equation (5.7)).

The constraint equation (6.4) is consistent with the equations of motion for v and p since

$$\begin{aligned}\partial_\lambda(p\cdot v) &= (v\cdot\mathcal{D}p)\cdot v + p\cdot(v\cdot\mathcal{D}v) = (\dot{e}/e)p\cdot v \\ \implies &\quad (p\cdot v)/e = \text{constant}.\end{aligned} \tag{6.13}$$

We thus have a complete, consistent set of semi-classical equations describing the dynamics of a massive spin-1/2 point-particle in a gravitational background. The equations are under-determined until a choice is made for the einbein. A convenient choice for comparison to classical models is to take $v^2 = 1$, so that λ is the proper time for the particle.

7 Conclusions

In this paper we have discussed the symmetry of the Riemann tensor and the conservation laws of GTG in the presence of torsion. This extends many of the results discussed in [15], where the effects of torsion were not considered. Many of the results presented here have counterparts in the standard tensor-calculus formulation of ECKS theory. A full translation between the two approaches is contained in Appendix D.

As an example of a model with torsion, we considered the Dirac field coupled self-consistently to gravity. A minimally-coupled gauge-invariant Lagrangian leeds to the GTG version of the Einstein-Cartan-Dirac equations. The Einstein-Dirac equations may also be formulated consistently but they cannot be derived from minimal coupling and gauge-theoretic arguments alone. This is a consequence of the natural association of torsion with the spin of the matter field which arises in GTG. Finally, we considered a semi-classical model for a massive spinning point-particle, moving in a gravitational background with torsion. This showed that the motion is not generally geodesic, the spin vector is not Fermi-transported and the particle couples to the torsion.

In a separate paper [25] we apply much of this formalism to obtain a new self-consistent cosmological solution describing a universe arising from a massive Dirac field. This solution provides a counterexample to the claim of [26] that Einstein-Cartan theories of gravity and General Relativity are observationally equivalent. The model presented in [25] contains an isotropic line element, but spinning point particles see a preferred direction in space due to the spin of the matter field. It is certainly true that for every solution of the Einstein-Cartan (or GTG) equations there is an equivalent solution in General Relativity. There must be because every metric gives a solution to the Einstein field equations; the matter distribution is whatever the Einstein tensor dictates. This misses the point, however, that in theories with spin there are extra physical fields present which have observable consequences. These cannot just be hidden in an effective stress-energy tensor.

A Results for Trivector Spin

The specialisation of the results in the main text to the case of trivector spin is useful for studying the Einstein-Cartan-Dirac set of equations. In this setup the spin tensor is given entirely by

$$\mathcal{S}(a) = T \cdot a \tag{A.1}$$

where T is the spin trivector. First we have the following expressions for the Riemann tensor:

$$\mathcal{R}(a\wedge b) = \mathcal{R}'(a\wedge b) + \tfrac{1}{4}\kappa^2 (a\wedge b)\cdot T\, T - \tfrac{1}{2}\kappa[(a\wedge b)\cdot\partial_c]\cdot(c\cdot\mathcal{D}T), \tag{A.2}$$

with the prime denoting the equivalent tensor in the absence of spin. Contracting equation (A.2) we obtain expressions for the Ricci tensor and scalar:

$$\mathcal{R}(a) = \mathcal{R}'(a) + \tfrac{1}{2}\kappa^2 (a\cdot T)T - \tfrac{1}{2}\kappa a\cdot(\mathcal{D}\cdot T) \tag{A.3}$$

$$\mathcal{R} = \mathcal{R}' + \tfrac{3}{2}\kappa^2 T^2. \tag{A.4}$$

These expressions are equivalent to those used in [23], but express the intrinsic content in a coordinate-free fashion.

The protraction of the Riemann tensor is

$$\partial_a \wedge \mathcal{R}(a \wedge b) = -\kappa(b \cdot \mathcal{D}T + \tfrac{1}{2} b\, \mathcal{D} \wedge T), \tag{A.5}$$

which produces an antisymmetric term of

$$\bar{\mathcal{R}}(B) - \mathcal{R}(B) = \kappa(B \cdot \mathcal{D}) \cdot T - \tfrac{1}{2}\kappa B\, \mathcal{D} \wedge T. \tag{A.6}$$

Equation (A.5) contracts to give

$$\partial_a \wedge \mathcal{R}(a) = \partial_a \wedge \mathcal{G}(a) = -\kappa \mathcal{D} \cdot T, \tag{A.7}$$

so the antisymmetric part of the Einstein tensor is determined by the covariant divergence of the spin trivector.

The Bianchi identity simplifies to

$$\dot{\mathcal{D}} \wedge \dot{\bar{\mathcal{R}}}(B) = -\kappa \bar{\mathcal{R}}(B) \times T. \tag{A.8}$$

The contracted result is

$$\dot{\bar{\mathcal{G}}}(\dot{\mathcal{D}}) = \tfrac{1}{2}\kappa(\partial_b \wedge \partial_a) \cdot [\bar{\mathcal{R}}(a \wedge b) \times T]. \tag{A.9}$$

Taking the inner product with an arbitrary vector b, we obtain the adjoint form

$$\dot{\mathcal{D}} \cdot \dot{\mathcal{G}}(b) = \tfrac{1}{2}\kappa(\partial_c \wedge \partial_a) \cdot \mathcal{R}[(a \wedge b \wedge c) \times T]. \tag{A.10}$$

Equation (A.9) may be written as

$$\dot{\bar{\mathcal{G}}}(\dot{\mathcal{D}}) = \kappa i \partial_a \wedge \mathcal{R}(a \wedge (iT)), \tag{A.11}$$

which, when combined with (A.5) gives

$$\dot{\bar{\mathcal{G}}}(\dot{\mathcal{D}}) = -\tfrac{1}{2}\kappa^2(s\mathcal{D} \cdot s + 2s \cdot \mathcal{D}s), \tag{A.12}$$

where the 'Pauli-Lubansky' vector $s \equiv -iT$ is the dual of the spin trivector. Equation (A.12) shows that $\dot{\bar{\mathcal{G}}}(\dot{\mathcal{D}})$ is second-order in the spin.

B Solutions of the Einstein-Cartan-Dirac equations and the Einstein-Dirac equations

In this appendix we give the GTG proof that solutions of the equations (5.4) satisfying condition (5.15) are also solutions of the pair of equations (5.11) and (5.12). The proof is similar to that of Seitz [23].

Assume that $\bar{h}(a), \omega(a)$ and ψ satisfy (5.4) and define $\omega'(a)$ via equation (3.11). The Dirac stress-energy tensor $\mathcal{T}(a)$, defined in equation (5.6), decomposes as follows:

$$
\begin{aligned}
\mathcal{T}(a) &= \langle a\cdot D\psi i\gamma_3\tilde{\psi}\rangle_1 \\
&= \langle a\cdot D'\psi i\gamma_3\tilde{\psi}\rangle_1 + \tfrac{1}{2}\langle\varpi(a)\psi i\gamma_3\tilde{\psi}\rangle_1 \\
&= \mathcal{T}'(a) + \tfrac{1}{2}\kappa^2(a\cdot T)\cdot T - \tfrac{1}{2}\kappa a\cdot(\mathcal{D}\cdot T).
\end{aligned} \tag{B.1}
$$

We next use equations (A.3) and (A.4) to write

$$
\mathcal{G}(a) = \mathcal{G}'(a) - \tfrac{1}{2}\kappa a\cdot(\mathcal{D}\cdot T) + \tfrac{1}{2}\kappa^2(a\cdot T)\cdot T - \tfrac{3}{4}\kappa^2 aT^2 \tag{B.2}
$$

So, given that $\mathcal{G}(a) = \kappa\mathcal{T}(a)$, the equivalent equation for the primed tensors will also be satisfied provided that $T^2 = 0$. This holds if and only if $T\psi = 0$.

It remains to consider the Dirac equation. Noting that

$$
\begin{aligned}
D'\psi &= D\psi - \tfrac{1}{4}\kappa\partial_a a\cdot T\psi \\
&= D\psi - \tfrac{3}{4}\kappa T\psi,
\end{aligned} \tag{B.3}
$$

we see that equation (5.11) will also be satisfied provided that the condition $T\psi = 0$ is satisfied. This completes the proof. Clearly, the converse is also true.

C Results for Arbitrary Spin Tensors

In this appendix we generalise the main results of this paper to a general spin tensor $\mathcal{S}(a)$ which is not required to have a vanishing contraction. The results are quoted without proof since the derivations follow those given in the main text.

For consistency with the previous results, we continue to denote the protraction of $\mathcal{S}(a)$ by $3T$. The contraction of the spin tensor is denoted t, so we have

$$
\partial_a\mathcal{S}(a) = \partial_a\cdot\mathcal{S}(a) + \partial_a\wedge\mathcal{S}(a) = t + 3T. \tag{C.1}
$$

Equation (3.5) generalises to

$$
\underline{h}^{-1}(L_a\underline{h}(b) - L_b\underline{h}(a)) = a\cdot\mathcal{D}b - b\cdot\mathcal{D}a - \kappa\bar{\mathcal{S}}(a\wedge b) - \tfrac{1}{2}\kappa(a\wedge b)\cdot t. \tag{C.2}
$$

where $\bar{\mathcal{S}}(a)$ is the adjoint to the spin tensor, as defined at equation (3.4). The right-hand side of (C.2) now defines the vector c which appears in the covariant expression (3.6) for the Riemann tensor. The $\omega'(a)$ field (the equivalent field in the absence of torsion) is given by

$$
\omega'(a) = \omega(a) + \kappa\mathcal{S}(a) - \tfrac{1}{2}\kappa(3a\cdot T + a\wedge t). \tag{C.3}
$$

The Riemann tensor is still given by equation (3.14). On contracting we arrive at the following expression for the Ricci tensor:

$$\begin{aligned}\mathcal{R}(a) =&\mathcal{R}'(a) - \kappa\dot{\mathcal{D}}'\cdot\dot{\mathcal{S}}(a) - \tfrac{3}{2}\kappa a\cdot(\mathcal{D}'\cdot T) - \tfrac{1}{2}\kappa a\mathcal{D}'\cdot t \\ &- \kappa^2\bar{\mathcal{S}}(\mathcal{S}(a)) + \tfrac{3}{2}\kappa^2\bar{\mathcal{S}}(a\cdot T) + \tfrac{1}{2}\kappa^2\bar{\mathcal{S}}(a\wedge t),\end{aligned} \tag{C.4}$$

which is the generalisation of (3.15). Contracting this expression with ∂_a, we find the generalisation of (3.16):

$$\mathcal{R} = \mathcal{R}' - \kappa\mathcal{D}'\cdot t + \tfrac{9}{2}\kappa^2 T^2 - \tfrac{1}{2}\kappa^2 t^2 - \kappa^2\mathcal{S}(\partial_a)\cdot\mathcal{S}(a). \tag{C.5}$$

Of the full spin tensor $\mathcal{S}(a)$ only t is differentiated in this expression, though the corresponding term in the action is a total divergence, so does not contribute to the field equations.

The protraction of the Riemann tensor (equation (4.9)) is given by

$$\begin{aligned}\partial_a\wedge\mathcal{R}(a\wedge b) = -\tfrac{1}{2}\kappa(b\wedge\mathcal{D}\wedge t + 2\dot{\mathcal{D}}\wedge\dot{\mathcal{S}}(b) + 3b\mathcal{D}\wedge T) + \tfrac{1}{4}\kappa^2\Big(4i\bar{\mathcal{S}}[i\mathcal{S}(b)] \\ - 2b\,\mathcal{S}(\partial_a)\wedge\mathcal{S}(a) + 2b\wedge\mathcal{S}(t) - 2\mathcal{S}(b)\wedge t + 3b\,T\wedge t\Big).\end{aligned} \tag{C.6}$$

The protraction of the Ricci tensor (equation (4.12)) generalises to

$$\partial_a\wedge\mathcal{R}(a) = -\kappa\dot{\mathcal{S}}(\dot{\mathcal{D}}) + \kappa^2\partial_a\cdot(i\bar{\mathcal{S}}[i\mathcal{S}(a)]) - \tfrac{1}{2}\kappa^2\mathcal{S}(t). \tag{C.7}$$

For general torsion, the Bianchi identity is still given by the compact expression (4.22). The contracted Bianchi identity now becomes.

$$\dot{\bar{\mathcal{G}}}(\dot{\mathcal{D}}) = \kappa[\bar{\mathcal{G}}(t) - \partial_a\cdot\bar{\mathcal{R}}(\mathcal{S}(a)) + \partial_a\cdot\mathcal{S}(\bar{\mathcal{R}}(a))]. \tag{C.8}$$

Taking the inner product with a vector a gives the adjoint relation

$$\dot{\mathcal{D}}\cdot\dot{\mathcal{G}}(a) = \kappa[t\cdot\mathcal{G}(a) + \mathcal{S}(\partial_b)\cdot\mathcal{R}(b\wedge a) - (a\cdot\mathcal{S}(\partial_b))\cdot\mathcal{R}(b)]. \tag{C.9}$$

D The Relation with Tensor Calculus

In this appendix we relate the frame-free approach of GTG to the standard approaches of tensor calculus. These results extend those given in Appendix C of [15]. Throughout this appendix we follow the conventions of [9].

We start with a set of coordinates $\{x^\mu\}$ and introduce the vectors

$$e_\mu \equiv \frac{\partial x}{\partial x^\mu}, \qquad e^\mu \equiv \nabla x^\mu. \tag{D.1}$$

From these we construct the vectors

$$g_\mu \equiv \underline{h}^{-1}(e_\mu), \qquad g^\mu \equiv \bar{h}(e^\mu). \tag{D.2}$$

These vectors satisfy the relation

$$g_\mu \cdot g^\nu = e_\mu \cdot e^\nu = \delta^\nu_\mu. \tag{D.3}$$

The metric is given by the 4×4 matrix

$$g_{\mu\nu} \equiv g_\mu \cdot g_\nu. \tag{D.4}$$

If the x-dependence in $g_{\mu\nu}$ is replaced by dependence on the coordinates $\{x^\mu\}$ alone then Riemann-Cartan geometry is recovered.

The connection is defined by

$$\mathcal{D}_\mu g_\nu = \Gamma^\alpha_{\mu\nu} g_\alpha, \tag{D.5}$$

so that

$$\Gamma^\lambda_{\mu\nu} = g^\lambda \cdot (\mathcal{D}_\mu g_\nu). \tag{D.6}$$

Unlike the coordinate-free object $\Omega(a)$, the connection contains artifacts from the chosen coordinate frame. Since

$$\partial_\mu g_{\nu\lambda} = (\mathcal{D}_\mu g_\nu) \cdot g_\lambda + g_\nu \cdot (\mathcal{D}_\mu g_\lambda), \tag{D.7}$$

we find that

$$\partial_\mu g_{\nu\lambda} = \Gamma^\alpha_{\mu\nu} g_{\alpha\lambda} + \Gamma^\alpha_{\mu\lambda} g_{\alpha\nu}, \tag{D.8}$$

which recovers 'metric compatibility' of the connection. The covariant derivative of a covariant vector $\mathcal{A} = A^\alpha g_\alpha = A_\alpha g^\alpha$ is

$$\begin{aligned} \mathcal{D}_\mu \mathcal{A} &= \mathcal{D}_\mu (A^\alpha g_\alpha) \\ &= (\partial_\mu A^\alpha) g_\alpha + A^\alpha \Gamma^\beta_{\mu\alpha} g_\beta \\ &= (\partial_\mu A^\alpha + \Gamma^\alpha_{\mu\beta} A^\beta) g_\alpha, \end{aligned} \tag{D.9}$$

as expected.

Equation (D.8) inverts to show that the connection contains a component given by the standard Christoffel symbol and a 'contorsion' term

$$\Gamma^\nu_{\lambda\mu} = \left\{ {\nu \atop \lambda\mu} \right\} - K_{\lambda\mu}{}^\nu. \tag{D.10}$$

The components of the contorsion tensor are given by

$$\begin{aligned} K_{\lambda\mu}{}^\nu &= \left\{ {\nu \atop \lambda\mu} \right\} - g^\nu \cdot (\mathcal{D}_\lambda g_\mu) \\ &= g^\nu \cdot [(\omega'(g_\lambda) - \omega(g_\lambda)) \cdot g_\mu] \\ &= \kappa (g^\nu \wedge g_\mu) \cdot [\mathcal{S}(g_\lambda) - \tfrac{3}{2} g_\lambda \cdot T - \tfrac{1}{2} g_\lambda \wedge t]. \end{aligned} \tag{D.11}$$

The torsion tensor is defined by the antisymmetric part of the connection:

$$\begin{aligned}
S_{\lambda\mu}{}^{\nu} &= \tfrac{1}{2}(\Gamma^{\nu}_{\lambda\mu} - \Gamma^{\nu}_{\mu\lambda}) \\
&= \tfrac{1}{2} g^{\nu}\cdot(\mathcal{D}_{\lambda} g_{\mu} - \mathcal{D}_{\mu} g_{\lambda}) \\
&= -\tfrac{1}{2}[g_{\mu}\cdot(\mathcal{D}_{\lambda} g^{\nu}) - g_{\lambda}\cdot(\mathcal{D}_{\mu} g^{\nu})] \\
&= -\tfrac{1}{2}(g_{\mu}\wedge g_{\lambda})\cdot(\mathcal{D}\wedge g^{\nu}) \\
&= \tfrac{1}{2}(g_{\lambda}\wedge g_{\mu})\cdot[\mathcal{S}(g^{\nu}) + \tfrac{1}{2} t\wedge g^{\nu}],
\end{aligned} \tag{D.12}$$

and the modified torsion tensor is defined by

$$\begin{aligned}
T_{\lambda\mu}{}^{\nu} &= S_{\lambda\mu}{}^{\nu} + \delta^{\nu}_{\lambda} S_{\mu\alpha}{}^{\alpha} - \delta^{\nu}_{\mu} S_{\lambda\alpha}{}^{\alpha} \\
&= \tfrac{1}{2}\kappa(g_{\lambda}\wedge g_{\mu})\cdot\mathcal{S}(g^{\nu}).
\end{aligned} \tag{D.13}$$

For covariant quantities such as the Riemann tensor the translation to tensor calculus is straightforward:

$$R^{\mu}{}_{\nu\rho\sigma} = (g^{\mu}\wedge g_{\nu})\cdot\mathcal{R}(g_{\sigma}\wedge g_{\rho}). \tag{D.14}$$

A vierbein $e_{\mu}{}^{i}$ (essentially an orthonormal tetrad) is given by

$$e_{\mu}{}^{i} = g_{\mu}\cdot\gamma^{i}, \qquad e^{\mu}{}_{i} = g^{\mu}\cdot\gamma_{i}, \tag{D.15}$$

where $\{\gamma^{i}\}$ is a fixed orthonormal frame. Any position dependence in the $\{\gamma^{i}\}$ is eliminated with a suitable rotor transformation.

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