JOURNAL OF MATHEMATICAL PHYSICS VOLUME 40, NUMBER 7 JULY 1999

Geometric algebra and the causal approach to multiparticle quantum mechanics

Shyamal Somaroo, Anthony Lasenby, and Chris Doran[a)]
Astrophysics Group, Cavendish Laboratory, Madingley Road, Cambridge CB3 0HE, United Kingdom

(Received 28 January 1998; accepted 14 February 1999)

It is argued that geometric algebra, in the form of the multiparticle spacetime algebra, is well suited to the study of multiparticle quantum theory, with advantages over conventional techniques both in ease of calculation and in providing an intuitive geometric understanding of the results. This is illustrated by comparing the geometric algebra approach for a system of two spin-1/2 particles with the nonrelativistic approach of Holland [Phys. Rep. **169**, 294 (1988)]. © *1999 American Institute of Physics.* [S0022-2488(99)00907-X]

I. INTRODUCTION

Geometric (Clifford) algebra is a powerful algebraic tool with applications throughout the fields of physics and engineering. The geometric algebra of space–time—the spacetime algebra or STA—is well suited to describing many aspects of classical and quantum relativistic physics[1–4] including gravitation.[5] In Refs. 2 and 6 the multiparticle spacetime algebra (MSTA) was introduced and applied to relativistic multiparticle quantum theory. In the present paper the algebraic advantages of the MSTA approach are demonstrated through a comparison with work on a causal approach to nonrelativistic multiparticle quantum theory based on the Pauli equation.[7,8] We show that the MSTA elucidates a number of features of the multiparticle causal theory and, in particular, clarifies its geometric content.

The causal, or Bohmian, approach to quantum mechanics is an interpretation in which the statistical results of quantum theory are recovered from an ensemble of deterministically evolving systems. The approach is based on establishing a connection between the wave equation and a deterministic model that is supposed to underlie the quantum process. In the case of one spin-1/2 particle, this model consists of a classical spinning rigid body under the additional influence of a quantum potential.[7] In this way physical properties can be associated with the quantum particle, and equations for their evolution obtained from the conventional wave equation. Furthermore, the variables (including spin) on which the wave function depends are consistently interpreted as the spatial position and orientation of the particle through this model.

In n-particle nonrelativistic quantum theory the wave function depends on a dynamical configuration space of dimension $3n$, as well as on a temporal parameter t. To apply a causal approach to this system one must first associate the wave function in configuration space with a set of physical properties. These are then interpreted as the properties of the individual particles in the ensemble making up the system under consideration. Equations describing the evolution of these properties are then derived from the conventional n-particle Pauli wave equation.

Holland[7,8] has addressed the problem of extracting a set of physical properties from the n-particle wave function. His method is to construct a set of tensor variables from quadratic combinations of the spinorial wave function. These tensor variables are more easily associated with a set physical properties than the underlying spinorial degrees of freedom.

Here we show that the MSTA formulation of multiparticle quantum theory considerably simplifies the task of extracting these physical variables. Its lack of redundant mathematical

[a)]Electronic mail: C.Doran@mrao.cam.ac.uk

complications considerably simplifies calculations, as well as clarifying the relation between the spinorial and tensorial degrees of freedom. Furthermore, the method of construction readily lends itself to an interpretation in terms of an underlying physical model, though such considerations are not pursued here. In fact, since we are only concerned with nonrelativistic quantum theory, only a fraction of the full power of the MSTA is brought into play here. An introduction to the MSTA approach to relativistic multiparticle quantum theory is contained in Ref. 2 (see also Ref. 6).

We start with an outline of the MSTA and introduce our conventions and notations. We then give a schematic overview of Holland's method of relating spinorial and tensorial variables. In Sec. IV we study the one-particle case, giving a detailed account of the correspondence scheme between the MSTA and Holland's approach. In Sec. V we turn to the two-particle case, as discussed first in Ref. 8. We show how the various physical variables are found more easily in the MSTA approach, and reveal their simple geometric origins. We end with a brief look at how the MSTA approach generalizes to the n-particle case.

II. MULTIPARTICLE SPACE–TIME ALGEBRA

Spacetime algebra is the geometric, or Clifford, algebra of Minkowski space–time. Both geometric and spacetime algebra have been widely discussed by many authors (see Refs. 1, 2, 9, and 10 for further material). The multiparticle spacetime algebra (MSTA) was introduced to tackle the problem of formulating relativistic multiparticle mechanics within geometric algebra.[2,6] It is the geometric algebra of n-particle configuration space which, for relativistic systems, consists of n copies of Minkowski space–time. We usually refer to each copy as a ''one-particle space.''

An appropriate orthonormal basis for the MSTA is provided by the set $\{\gamma_\mu^a\}$, where $\mu=0,...,3$ labels the space–time vector, and $a=1,...,n$ labels the particle space. For cases where only one particle is present this index is often omitted. These vectors have an associative (geometric) product denoted by juxtaposition. The symmetrized product is denoted by a dot and satisfies

$$\gamma_\mu^a \cdot \gamma_\nu^b = \tfrac{1}{2}(\gamma_\mu^a \gamma_\nu^b + \gamma_\nu^a \gamma_\mu^b) = \eta_{\mu\nu}\delta^{ab}, \tag{2.1}$$

where $\eta_{\mu\nu}=\mathrm{diag}(+,-,-,-)$. Vectors from different particle spaces anticommute as a consequence of their orthogonality. The remaining, antisymmetrized product is denoted by a wedge and generates a bivector,

$$\gamma_\mu^i \wedge \gamma_\nu^j \equiv \tfrac{1}{2}(\gamma_\mu^i \gamma_\nu^j - \gamma_\nu^i \gamma_\mu^j). \tag{2.2}$$

In this manner a basis for the entire MSTA can be constructed. This has 2^{4n} degrees of freedom. A general element of the MSTA is termed a *multivector*.

In this paper we deal only with nonrelativistic quantum mechanics. We therefore need to pick out a preferred timelike vector for each of the particle spaces. We take this vector to be γ_0^a for each a. Spatial vectors relative to these timelike vectors are modeled as bivectors through a ''space–time split.''[3,4] For these we introduce the notation (with no sum over a)

$$\sigma_j^a \equiv \gamma_j^a \gamma_0^a, \quad j=1,...,3, \quad a=1,...,n. \tag{2.3}$$

For each particle space the set $\{\sigma_j^a\}$ generates the geometric algebra of relative space, which we denote $\mathcal{G}_3$. Each has a basis of the form

$$1, \quad \{\sigma_j\}, \quad \{i\sigma_j\}, \quad i, \tag{2.4}$$

where the volume element i is defined by

$$i \equiv \sigma_1\sigma_2\sigma_3 \tag{2.5}$$

and we have suppressed the particle-space indices. The notation reflects the fact that the geometric algebra of (relative) space is isomorphic to the Pauli algebra, though we stress that the $\{\sigma_j^a\}$ are a basis set of vectors for three-space, and not abstract operators in spin-space. The even subalgebra of the basis (2.4) is spanned by $\{1, i\sigma_j\}$ and is isomorphic to the quaternion algebra.

An important property of the $\{\sigma_j^a\}$ is that, unlike space–time basis vectors, relative vectors from separate particle spaces commute. This follows immediately from their definition:

$$\sigma_i^a\sigma_j^b=\gamma_i^a\gamma_0^a\gamma_j^b\gamma_0^b=\gamma_i^a\gamma_j^b\gamma_0^b\gamma_0^a=\gamma_j^b\gamma_0^b\gamma_i^a\gamma_0^a=\sigma_j^b\sigma_i^a \quad (a\neq b). \tag{2.6}$$

It follows that the $\{\sigma_j^a\}$ generate the direct product space $\mathcal{G}_3^n\equiv\mathcal{G}_3\otimes\cdots\otimes\mathcal{G}_3$ of n copies of the geometric algebra of three-dimensional space $\mathcal{G}_3$. All properties of this space follow from the properties of the fully relativistic MSTA.

Within the Pauli algebra of space, an important role is played by *rotors*. These are elements of the even subalgebra of the Pauli algebra satisfying the relation

$$R\tilde{R}=1, \tag{2.7}$$

where the tilde denotes the operation of reversing the order of the vectors in any geometric product in the MSTA. The operation of rotating a multivector is performed by

$$A\mapsto A'=RA\tilde{R}, \tag{2.8}$$

which is easily shown to keep lengths and angles unchanged.

A spinor transforms single sidedly under the action of a rotor, and can be defined as an element of a linear space that is closed under left multiplication by the rotor group. Traditionally, Pauli spinors are either taken as complex column vectors acted on by the 2×2 Pauli matrices, or as elements of a minimal left ideal of the Pauli algebra.[11] A third approach, which turns out to be very powerful in applications, is to represent spinors as elements of the even subalgebra of the Pauli algebra. This space has four real dimensions, and is closed under the action of the rotor group. It is a straightforward matter to establish a 1↔1 map between Pauli column spinors and elements of the even subalgebra.[2,12] We start with the Pauli spin matrices in the form

$$\hat{\sigma}_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \hat{\sigma}_2=\begin{pmatrix}0 & -j\\ j & 0\end{pmatrix},\quad \hat{\sigma}_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{2.9}$$

where the carets denote that the $\{\hat{\sigma}_i\}$ are explicitly matrices, and j is used for the scalar unit imaginary of quantum mechanics since the symbol i is already employed for the spatial volume element. A column spinor ψ^a is then placed into a 1↔1 correspondence with an element of the even subalgebra as follows:

$$\psi^a=\begin{pmatrix}a^0+ja^3\\ -a^2+ja^1\end{pmatrix}\leftrightarrow\psi=a^0+a^k i\sigma_k. \tag{2.10}$$

The action of the quantum operators $\{\hat{\sigma}_k\}$ and j is now replaced by the operations

$$\hat{\sigma}_k|\psi\rangle\leftrightarrow\sigma_k\psi\sigma_3 \quad (k=1,...,3), \tag{2.11}$$

$$j|\psi\rangle\leftrightarrow\psi i\sigma_3. \tag{2.12}$$

Every calculation that can be performed with the column spinor ψ^a can also be performed with the even element ψ, and in practice the latter approach is usually easier. One reason for this is the natural decomposition of ψ into a density term and a rotor:

$$\psi=\rho^{1/2}R, \tag{2.13}$$

where

$$\rho \equiv \psi \tilde{\psi}. \tag{2.14}$$

The rotor R is an instruction to rotate the fixed $\{\sigma_i\}$ frame onto the frame of observables. This establishes a natural link with the description of a rotating rigid body.[9,12]

Nonrelativistic multiparticle spinors are formed from direct products of single particle spinors. If we denote the even subalgebra of $\mathcal{G}_3$ by $\mathcal{G}_3^+$, we see that nonrelativistic MSTA spinors belong to $(\mathcal{G}_3^+)^n = \mathcal{G}_3^+ \otimes \cdots \otimes \mathcal{G}_3^+$. An advantage of the MSTA approach is that this direct product coincides with the geometric product already defined. At this point it is useful to introduce the notation

$$i\sigma_j^a = i^a \sigma_j^a, \tag{2.15}$$

which removes some superscripts without introducing any ambiguity. Multiparticle spinors therefore belong to the space generated by the elements $\{1,\ i\sigma_j^a\}$. This space is closed under the left-sided action of the group of rotors of the form $R^1R^2,\ldots,R^n$, where each R^a denotes a copy of the same rotor for each particle space.

In Eq. (2.12) we saw that the role of the unit imaginary of traditional quantum theory is played by right multiplication by $i\sigma_3$. For the n-particle case there will be n copies of $i\sigma_3$, and right-multiplication by all of these must yield the same result. This is achieved by introducing the n-particle ''correlator''[2]

$$E_n \equiv \tfrac{1}{2}(1 - i\sigma_3^1 i\sigma_3^2)\cdots\tfrac{1}{2}(1 - i\sigma_3^1 i\sigma_3^n), \tag{2.16}$$

which locks the various one-particle complex structures;

$$E_n i\sigma_3^1 = E_n i\sigma_3^2 = \cdots = E_n i\sigma_3^n \equiv J_n. \tag{2.17}$$

The E_n and J_n satisfy

$$E_n E_n = E_n, \quad J_n J_n = -E_n. \tag{2.18}$$

Correlating all n-particle states $\psi \in (\mathcal{G}_3^+)^n$ by right-multiplying by the idempotent E_n ensures that the conventional complex structure is reproduced by the operation of right multiplication by any of the $i\sigma_3^a$ or J_n. The correlator also reduces the degrees of freedom in an n-particle spinor from 4^n to the expected 2^{n+1}. It is worth noting that one effect of the correlator is to ''phase lock'' all one-particle phase-factors:

$$e^{\alpha i\sigma_3^1} E_n = e^{\alpha i\sigma_3^2} E_n = \cdots = e^{\alpha J_n} E_n. \tag{2.19}$$

This suggests an interesting substructure to the theory, which could prove useful in constructing a suitable particle model for a causal interpretation.

As an example of the above scheme, consider the MSTA analog of the spin singlet state

$$|\epsilon\rangle = \frac{1}{\sqrt{2}}\left\{\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\right\}. \tag{2.20}$$

This is represented in the two-particle MSTA by the multivector

$$\epsilon = \frac{1}{\sqrt{2}}(i\sigma_2^1 - i\sigma_2^2)\frac{1}{2}(1 - i\sigma_3^1 i\sigma_3^2). \tag{2.21}$$

It can be shown that ϵ satisfies[2]

$$M^1 \epsilon = \tilde{M}^2 \epsilon \tag{2.22}$$

for an arbitrary Pauli-even multivector M. This result quickly establishes the rotation invariance of ϵ, since under a rotation in two-particle space, ϵ transforms as

$$\epsilon \mapsto R^1 R^2 \epsilon = R^1 \tilde{R}^1 \epsilon = \epsilon. \tag{2.23}$$

Throughout this paper superscripts on one-particle multivectors denote the one-particle space being referred to. For example, ϕ^k denotes a copy of the one-particle spinor ϕ in the particle-k space. Information regarding the kth particle is extracted from an arbitrary MSTA multivector by projecting it onto the particle-k space. Following Holland, Pauli spin indices are denoted by lower case letters $a-h$, and matrices are denoted by ''caretted'' versions of symbols denoting their algebraic analogs. When considered as a matrix algebra generated by (2.9) over the complex field, the Pauli algebra is denoted as $\mathcal{P}$, with $\mathcal{P}^n$ being the corresponding n-copy direct product. The range $i-l$ is used to denote spatial tensor indices, and the Einstein summation convention also applies throughout unless stated otherwise.

III. THE CAUSAL APPROACH TO MULTIPARTICLE STATES

Conventionally, a nonrelativistic spin-1/2 particle is described by a Pauli spinor ψ^a, usually viewed as a complex linear combination of the spin basis states $|\uparrow\rangle$ and $|\downarrow\rangle$, or more explicitly as a 2×1 complex column matrix. Two spin-1/2 particles are described by a rank-2 spinor, or spin tensor, ψ^{ab}. A basis for this is taken to be a complex linear combination of the direct product of two copies of the spin basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. We may also view ψ^{ab} as a 4×1 complex column matrix by considering the two separate spin indices a and b in ψ^{ab} as one compounded index $\Delta=[ab]$, $\Delta=1,...,4$. This extends to a rank-n spinor $\psi^{abc\cdots}$, which describes a system of n spin-1/2 particles. $\psi^{abc\cdots}$ has 2^n complex degrees of freedom and can be viewed as a $2^n\times 1$ complex column matrix ψ^Δ, where Δ is the compounded index $\Delta=[abc\ldots]$.

Following Holland,[8] a tensor can be constructed from a pair of spinors $\psi^\Delta\equiv\psi^{abc\cdots}$ and $\xi^{efg\cdots}\equiv\xi^\Theta$ of the same rank (2^n) by first constructing the $2^n\times 2^n$ complex matrix

$$A^\Delta{}_\Theta = A^{abc\ldots}{}_{efg\ldots} \equiv \psi^{abc\cdots}(\xi_{efg\ldots})^T = \psi^\Delta(\xi_\Theta)^T, \tag{3.1}$$

where the superscript T denotes matrix transposition. Spinor indices are raised and lowered by the spin metric[13]

$$\epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = j\hat{\sigma}_2. \tag{3.2}$$

(Conventionally, raised or lowered indices on an object indicate different coordinate representations of the same object. Here however, its significance is to indicate a distinct object which carries the same information as the original but in a different form.)

Any $2^n\times 2^n$ (complex) matrix may be expanded in terms of a set of independent basis matrices $\{(\hat{e}_{klm\ldots})^\Delta{}_\Theta\}$ for the direct product of n Pauli algebras $\mathcal{P}^n$,

$$A^\Delta{}_\Theta = \sum_{k,l,m,\ldots} c_{klm\ldots}(\hat{e}_{klm\ldots})^\Delta{}_\Theta. \tag{3.3}$$

The basis matrices $\{(\hat{e}_{klm\ldots})^\Delta{}_\Theta\}$ carry both spinor indices (Δ and Θ) and tensor indices ($\{k,l,m,\ldots\}$) which they inherit from the Pauli matrices. The complex expansion coefficients $c_{klm\ldots}$ are spatial tensors and may be determined from A via

$$c_{klm\ldots} = \frac{1}{2^n}\mathrm{Tr}[\psi^\Delta\xi_\Theta(\hat{e}_{klm\ldots})^\Theta{}_\Omega] = \frac{1}{2^n}\xi_\Theta(\hat{e}_{klm\ldots})^\Theta{}_\Delta\psi^\Delta, \tag{3.4}$$

where Tr denotes the matrix trace.

If ξ_Θ is obtained directly from ψ^Δ then $c_{klm\ldots}$ will characterize some of the information in ψ^Δ. This is the basis of Holland's approach. Note that, since A satisfies the relation

$$A^2 = A^\Delta{}_\Theta A^\Theta{}_\Omega = \psi^\Delta(\xi_\Theta)^T \psi^\Theta(\xi_\Omega)^T = A^\Theta{}_\Theta A^\Delta{}_\Omega = \mathrm{Tr}(A)A, \tag{3.5}$$

A therefore belongs to an ideal of the algebra $\mathcal{P}^n$. It is via this ''ideal'' characterization of ψ^Δ that Holland associates the tensors $c_{klm\ldots}$ with spinor degrees of freedom. We now study how geometric algebra both simplifies this scheme, and reveals much of the hidden geometry relating spinors to tensors. We start with an analysis of the one-particle setup.

IV. THE ONE-PARTICLE CASE

In the one-particle setup we expand the 2×2 complex matrix $A^a{}_b$ in terms of the Pauli matrices:

$$A^a{}_b \equiv \psi^a(\xi_b)^T \equiv s + u_k\hat{\sigma}_k = \Re(s) + j\Im(s) + [\Re(u_k) + j\Im(u_k)]\hat{\sigma}_k. \tag{4.1}$$

We can construct a MSTA version of this by first writing out $A^a{}_b$ explicitly in terms of the spinor components

$$\psi^a = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}, \quad \xi^a = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}. \tag{4.2}$$

The matrix $A^a{}_b$ then has components

$$A^a{}_b = \begin{pmatrix} \psi^1\xi^2 & -\psi^1\xi^1 \\ \psi^2\xi^2 & -\psi^2\xi^1 \end{pmatrix} = \begin{pmatrix} \psi^1 & 0 \\ \psi^2 & 0 \end{pmatrix} \begin{pmatrix} \xi^2 & 0 \\ -\xi^1 & 0 \end{pmatrix}^T. \tag{4.3}$$

By writing this as the product of two matrices, we can easily establish an equivalent expression within the one-particle geometric algebra of space. First, we note the following equivalence:

$$\begin{pmatrix} \psi^1 & 0 \\ \psi^2 & 0 \end{pmatrix} \leftrightarrow \psi\tfrac{1}{2}(1+\sigma_3), \tag{4.4}$$

where ψ is the Pauli-even multivector formed from ψ^a according to Eq. (2.10). Second, we need the analog of matrix transposition for multivectors. It is easily confirmed that this is performed by

$$M^T = \sigma_2 \tilde{M} \sigma_2. \tag{4.5}$$

If we now denote the multivector equivalent of $A^a{}_b$ by A, we find that

$$A = \psi\tfrac{1}{2}(1+\sigma_3)[i\sigma_2\xi\tfrac{1}{2}(1+\sigma_3)]^T = \psi\tfrac{1}{2}(1+\sigma_3)\sigma_2\tfrac{1}{2}(1-\sigma_3)\tilde{\xi}(-i\sigma_2)\sigma_2 = -\psi\tfrac{1}{2}(\sigma_1 + i\sigma_2)\tilde{\xi}, \tag{4.6}$$

which immediately gives all of the components s and $\{u_k\}$ by simply reading off the terms of different grades. Each grade returns a genuine geometric object, since under the rotation $\psi \mapsto R\psi$, $\xi \mapsto R\xi$, we find that A transforms as

$$A \mapsto RA\tilde{R}, \tag{4.7}$$

which is the correct transformation law for geometric objects. The same approach extends easily to the case where the rotor R includes Lorentz transformations. This is not such a surprise when one considers that the algebraic manipulations described here closely resemble those of the 2-spinor calculus,[13,14] which are designed to be fully relativistic. We can now give the following explicit formulas for the s and $\{u_k\}$:

$$\Re(s)=-\tfrac{1}{2}\langle\psi i\sigma_2\tilde{\xi}\rangle,\quad \Im(s)=\tfrac{1}{2}\langle\psi i\sigma_1\tilde{\xi}\rangle,$$
$$\Re(u_k)=-\tfrac{1}{2}\langle\psi\sigma_1\tilde{\xi}\sigma_k\rangle,\quad \Im(u_k)=-\tfrac{1}{2}\langle\psi\sigma_2\tilde{\xi}\sigma_k\rangle, \tag{4.8}$$

where $\langle M\rangle$ denotes the result of projecting out the scalar part of the multivector M.

Having established the correspondence between the matrix formalism and geometric algebra, it is now straightforward to consider the choices Holland makes for ξ_b and the resulting tensors obtained.[8] The first choice is

$$\xi_b=\psi^{a*} \tag{4.9}$$

$$\Rightarrow\xi^a=\epsilon^{ab}\xi_b=-\epsilon_{ab}\psi^{a*}. \tag{4.10}$$

The multivector analog of complex conjugation is defined by

$$\psi^*=\sigma_2\psi\sigma_2, \tag{4.11}$$

so this choice corresponds to setting

$$\xi=(-i\sigma_2)\sigma_2\psi\sigma_2=-\psi i\sigma_2. \tag{4.12}$$

It follows that A is given by

$$A=\psi\tfrac{1}{2}(1+\sigma_3)\tilde{\psi}=\tfrac{1}{2}\psi\tilde{\psi}+\tfrac{1}{2}\psi\sigma_3\tilde{\psi}, \tag{4.13}$$

and from (4.8) we find

$$\rho\equiv 2s=\psi\tilde{\psi}, \tag{4.14}$$

$$S\equiv 2u_k\sigma_k=\psi\sigma_3\tilde{\psi}, \tag{4.15}$$

where ρ and S are the symbols used by Holland. As expected, we have isolated the scalar density $\psi\tilde{\psi}$ and the spin vector $\psi\sigma_3\tilde{\psi}$. On decomposing ψ in the form

$$\psi=\rho^{1/2}R=\rho^{1/2}e^{i\sigma_3\theta/2}e^{i\sigma_1\varphi/2}e^{i\sigma_3\chi/2}, \tag{4.16}$$

where we have written the rotor R in terms of the Euler angles, we see that the scalar ρ and the vector

$$S=\rho[\sin(\theta)\cos(\varphi)\sigma_1+\sin(\theta)\sin(\varphi)\sigma_2+\cos(\theta)\sigma_3] \tag{4.17}$$

are independent of the phase χ. This pair of tensors therefore only embodies three of the four degrees of freedom in ψ and consequently an additional characterization of ψ is required. From our geometric viewpoint it is clear that the only other tensors that can be obtained from ψ are the vectors $\psi\sigma_1\tilde{\psi}$ and $\psi\sigma_2\tilde{\psi}$, or their corresponding duals. To verify this, consider Holland's other choice:

$$\xi_b=\psi_b\Rightarrow\xi=\psi. \tag{4.18}$$

On substituting this into (4.13) we find that

$$A=-\psi\tfrac{1}{2}(1+\sigma_3)i\sigma_2\tilde{\psi}=-\tfrac{1}{2}(\psi i\sigma_2\tilde{\psi}+\psi\sigma_1\tilde{\psi}). \tag{4.19}$$

Clearly for this choice of ξ, $s=0$ and from (4.8)

$$M \equiv 2\Re(\mathbf{u}) = -\psi\sigma_1\tilde{\psi}, \quad N \equiv 2\Im(\mathbf{u}) = -\psi\sigma_2\tilde{\psi}, \tag{4.20}$$

where M and N are Holland's Cartan–Kramers vectors.

The mutual geometric relations of M, N, and S are transparent from the geometric point of view, since the set is obtained by a rotation and dilation of the orthonormal basis $\{\sigma_1, \sigma_2, \sigma_3\}$. It follows immediately that

$$\rho = |M| = |N| = |S| \tag{4.21}$$

and that M, N, and S are mutually orthogonal. Since the phase factor $\exp\{i\sigma_3\chi/2\}$ does not commute with either of σ_1 or σ_2, the pair $\{M,N\}$ contain information about all four degrees of freedom in ψ. Indeed, any pair from the set $\{S,M,N\}$ contains information about all four degrees of freedom in ψ, and can be used to reconstruct ψ (and hence ψ^a, if required) up to an arbitrary sign. The simple manner in which the triad $\{M,N,S\}$ is formed and understood in the geometric algebra approach fully demonstrates its advantages.

V. THE TWO-PARTICLE CASE

Two-particle states are conventionally represented by rank-2 spinors ψ^{ab}. From these we construct the matrix

$$A^{ab}{}_{cd} = \psi^{ab}(\xi_{cd})^T, \tag{5.1}$$

which is a 4×4 complex matrix, where the transpose is understood to be with respect to the compounded index $[cd]$. This matrix can be expanded in the 16-dimensional basis

$$\begin{gathered}\mathbf{1}^{ab}_{cd} = \delta^a{}_c\delta^b{}_d, \quad (\hat{e}_{1k})^{ab}{}_{cd} = \hat{\sigma}_k{}^a{}_c\delta^b{}_d \\ (\hat{e}_{2l})^{ab}{}_{cd} = \delta^a{}_c\hat{\sigma}_l{}^b{}_d, \quad (\hat{e}_{1k}\hat{e}_{2l})^{ab}{}_{cd} = \hat{\sigma}_k{}^a{}_c\hat{\sigma}_l{}^b{}_d\end{gathered} \tag{5.2}$$

belonging to $\mathcal{P}^2$. With a suppression of spinor indices, we can write (following Ref. 8)

$$A = s\mathbf{1} + c_k\hat{e}_{1k} + d_k\hat{e}_{2k} + f_{kl}\hat{e}_{1k}\hat{e}_{2l}. \tag{5.3}$$

These coefficients are not independent because of the relation $A^2 = \mathrm{Tr}(A)A$.

A complete basis for two-particle spin states, together with their MSTA analogs, is provided by

$$\begin{gathered}\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} \leftrightarrow E, \\ \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} \leftrightarrow -i\sigma_2^1E, \\ \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} \leftrightarrow -i\sigma_2^2E, \\ \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} \leftrightarrow i\sigma_2^1i\sigma_2^2E,\end{gathered} \tag{5.4}$$

where $E = \frac{1}{2}(1 - i\sigma_3^1i\sigma_3^2)$ is the two-particle correlator. A rank-2 spinor ψ^{ab} can be expanded in terms of this basis and hence mapped directly to the MSTA element $\psi = \psi E \in (\mathcal{G}_3^+)^2$, where

$$\psi = (\phi_1 - i\sigma_2^1\phi_2 - i\sigma_2^2\phi_3 + i\sigma_2^1i\sigma_2^2\phi_4)E \tag{5.5}$$

and ϕ_n, $n=1,...,4$ are the combinations of 1 and J appropriate to the expansion of ψ^{ab}. All MSTA spinors $\psi \in (\mathcal{G}_3^+)^2$ contain an implicit factor of E. This is only shown explicitly in cases where its presence increases clarity.

We can again construct the matrix $A^{ab}{}_{cd}$ by first introducing the MSTA analog of a 4×4 complex matrix in which ψ^{ab} is the first and only nonzero column. The multivector equivalent of this is

$$\psi\tfrac{1}{2}(1+\sigma_3^1)\tfrac{1}{2}(1+\sigma_3^2)E=\psi\tfrac{1}{2}(1+\sigma_3^1)\tfrac{1}{2}(1+\sigma_3^1)\tfrac{1}{2}(1-i^1 i^2). \tag{5.6}$$

The analog of the matrix transpose operation is now

$$M^T=\sigma_2^1\sigma_2^2\tilde{M}\sigma_2^1\sigma_2^2 \tag{5.7}$$

and we find that the information contained in $A^{ab}{}_{cd}$ can be encoded in the multivector

$$A=\psi\tfrac{1}{2}(\sigma_1^1+i\sigma_2^1)\tfrac{1}{2}(\sigma_1^2+i\sigma_2^2)E\tilde{\xi}=\psi\tfrac{1}{2}(\sigma_1^1+i\sigma_2^1)\tfrac{1}{2}(\sigma_1^2+i\sigma_2^2)\tilde{\xi}(1-i^1 i^2). \tag{5.8}$$

The fact that A satisfies $A=A\frac{1}{2}(1-i^1 i^2)$ is to be expected. The tensor product $\mathcal{G}_3\otimes\mathcal{G}_3$ defines a space of 64 real dimensions, whereas the product $\mathcal{P}\otimes\mathcal{P}$ defines a 16-dimensional complex space, with 32 real dimensions. The space $\mathcal{P}\otimes\mathcal{P}$ therefore fails to provide a matrix representation of the full algebra $\mathcal{G}_3\otimes\mathcal{G}_3$, the reason being that the two pseudoscalars i^1 and i^2 are given the same representation in terms of the unit imaginary j. The only multivectors in $\mathcal{G}_3\otimes\mathcal{G}_3$ which do correspond directly to matrices in $\mathcal{P}\otimes\mathcal{P}$ are therefore those which contain a factor of the idempotent $\frac{1}{2}(1-i^1 i^2)$, which links the pseudoscalars together.

The quantities of interest are the complex tensor coefficients s, c_k, d_l, and f_{kl} in (5.3). These can all be recovered by taking appropriate traces of the form

$$\tfrac{1}{4}\mathrm{Tr}(A^{ab}{}_{cd}\Gamma^{cd}{}_{ef})=\tfrac{1}{4}\xi_{cd}\Gamma^{cd}{}_{ab}\psi^{ab}, \tag{5.9}$$

where Γ is some combination of the basis elements (5.2). From the above scheme, a general matrix of the form $A^{ab}{}_{cd}\Gamma^{cd}{}_{ef}$ will have a multivector equivalent $M=M\frac{1}{2}(1-i^1 i^2)$. With this multivector we have the explicit relations

$$\begin{aligned}\Re(\tfrac{1}{4}\xi_{cd}\Gamma^{cd}{}_{ab}\psi^{ab})&=2\langle M\rangle,\\ \Im(\tfrac{1}{4}\xi_{cd}\Gamma^{cd}{}_{ab}\psi^{ab})&=-2\langle Mi^1\rangle.\end{aligned} \tag{5.10}$$

The extra factor of 2 is required because of the presence of the factor of $\frac{1}{2}(1-i^1 i^2)$ in M.

We now have analogs for most of the operations performed on rank-2 spinors. The one remaining operation is that of particle interchange

$$\psi^{ab}\mapsto\psi^{ba}. \tag{5.11}$$

The algebraic effect of this on the MSTA spinor ψ is

$$\psi\mapsto\psi^I=(\phi_1-i\sigma_2^2\phi_2-i\sigma_2^1\phi_3+i\sigma_2^1 i\sigma_2^2\phi_4)E. \tag{5.12}$$

This operation has no one-particle analog. As an algebraic operation it can be expressed as

$$\psi\to\psi^I=E\psi E-i\sigma_2^1 i\sigma_2^2\bar{E}\psi E=\tfrac{1}{2}(1-i\sigma_k^1 i\sigma_k^2)\psi, \tag{5.13}$$

where $\bar{E}\equiv\frac{1}{2}(1+i\sigma_3^1 i\sigma_3^2)$. If we recall the definition of the MSTA rotation singlet state from Sec. II,

$$\epsilon \equiv \frac{1}{\sqrt{2}}(i\sigma_2^1 - i\sigma_2^2)E, \tag{5.14}$$

we find that

$$\epsilon\tilde{\epsilon} = \tfrac{1}{2}(1 + i\sigma_k^1 i\sigma_k^2). \tag{5.15}$$

It follows that

$$\psi^I = (1 - \epsilon\tilde{\epsilon})\psi, \tag{5.16}$$

confirming that the antisymmetrized state $\psi - \psi^I = \epsilon\tilde{\epsilon}\psi$ is a rotation singlet. Equation (2.23) ensures that the interchange operation is rotationally covariant,

$$R^1R^2(1-\epsilon\tilde{\epsilon})\psi = R^1R^2\psi - \epsilon\tilde{\epsilon}\psi = (1-\epsilon\tilde{\epsilon})R^1R^2\psi, \tag{5.17}$$

as expected.

We are now in a position to give explicit MSTA formulas for the two-particle tensors constructed in Ref. 8. With ψ the direct map of ψ^{ab}, according to Eq. (5.5), we find that

$$\begin{aligned} \rho &\equiv \psi^{ab}\psi^{*ab} = 2\langle\psi\tilde{\psi}\rangle, \\ S_{1k} &\equiv \psi^{*ab}e_{1k}{}^{ab}{}_{cd}\psi^{cd} = -2(\psi J\tilde{\psi})\cdot(i\sigma_k^1), \\ S_{2k} &\equiv \psi^{*ab}e_{2k}{}^{ab}{}_{cd}\psi^{cd} = -2(\psi J\tilde{\psi})\cdot(i\sigma_k^2), \\ S_{kl} &\equiv \psi^{*ab}e_{1k}{}^{ab}{}_{cd}e_{2k}{}^{cd}{}_{ef}\psi^{ef} = -2(\psi\tilde{\psi})\cdot(i\sigma_k^1 i\sigma_l^2). \end{aligned} \tag{5.18}$$

The only terms involved in the MSTA approach are the scalar + four-vector quantities $\psi\tilde{\psi} = \psi E\tilde{\psi}$ and the bivector $\psi J\tilde{\psi}$. This information is summarized in the single multivector

$$A = \psi\tfrac{1}{2}(1+\sigma_3^1)\tfrac{1}{2}(1+\sigma_3^2)\tilde{\psi}, \tag{5.19}$$

as expected.

Holland interprets S_{kl} as a spin correlation tensor, an observation justified by that fact that S_{kl} encodes the four-vector component of the ''expectation,'' $\psi E\tilde{\psi}$, of the correlator E in the MSTA formulation. The quantity $\psi J\tilde{\psi}$ is a bivector on account of its even grade and reversion asymmetry. Since we work in the closed algebra $(\mathcal{G}_3^+)^2$, it can only have $i\sigma_k^1$ and $i\sigma_k^2$ parts. We can therefore write

$$\psi J\tilde{\psi} = \tfrac{1}{2}(S_1^1 + S_2^2), \tag{5.20}$$

where S_1^1 and S_2^2 are one-particle bivectors. The subscripts indicate that S_1 and S_2 are separate variables, while the superscripts denote the particle spaces these quantities are expressed in. The quantity $\psi J\tilde{\psi}$ is the two-particle spin bivector whose one-particle projections S_1 and S_2 are the spin bivectors of particle-1 and particle-2, respectively.[2]

One surprising result proved in Ref. 8 is that the spin bivectors S_1 and S_2 have the same magnitude, $|S_1| = |S_2|$. This result is obviously true for direct-product states, but it is not intuitively obvious why it should hold for general superpositions. It is therefore instructive to see how to prove the result in the MSTA. We start by noting that the components of S_1 are given by

$$-\tfrac{1}{2}S_{1k} = \tfrac{1}{2}S_1^1\cdot(i\sigma_k^1) = (\psi J\tilde{\psi})\cdot(i\sigma_k^1) = (\tilde{\psi} i\sigma_k^1\psi)\cdot(i\sigma_3^1). \tag{5.21}$$

The implicit idempotent E on either side of the bivector $\tilde{\psi} i\sigma_k^1 \psi = E\tilde{\psi} i\sigma_k^1 \psi E$, acts as a projection and implies that $\tilde{\psi} i\sigma_k^1 \psi$ can only contain an equal linear combination of $i\sigma_3^1$ and $i\sigma_3^2$. We can therefore write

$$\tilde{\psi} i\sigma_k^1 \psi = S_{1k} J, \quad \tilde{\psi} i\sigma_k^2 \psi = S_{2k} J. \tag{5.22}$$

From this we deduce that the magnitudes of S_1 and S_2 are given by

$$|S_a|^2 = S_{ak} S_{ak} = -2\langle S_{ak} J S_{ak} J\rangle = -2\langle \tilde{\psi} i\sigma_k^a \psi \tilde{\psi} i\sigma_k^a \psi\rangle, \tag{5.23}$$

where $a=1,2$ and no summation over a is implied. Since $\psi\tilde{\psi}$ is even and reversion symmetric, it contains only a scalar and four-vector part. The four-vector part necessarily has the form $B^1 C^2$, where B^1 and C^2 are bivectors in the separate one-particle spaces. Employing the one-particle identity

$$i\sigma_k^1 B^1 i\sigma_k^1 = B^1, \tag{5.24}$$

we see that (with no sum over a)

$$i\sigma_k^a \psi\tilde{\psi} i\sigma_k^a = i\sigma_k^a(\langle \psi\tilde{\psi}\rangle + \langle \psi\tilde{\psi}\rangle_4) i\sigma_k^a = -3\langle \psi\tilde{\psi}\rangle + \langle \psi\tilde{\psi}\rangle_4 = \psi\tilde{\psi} - 4\langle \psi\tilde{\psi}\rangle. \tag{5.25}$$

This result is independent of the label a, and upon substitution into (5.23) implies that $|S_1|^2 = |S_2|^2$. Specifically,

$$|S_a|^2 = -2\langle \tilde{\psi}(\psi\tilde{\psi} - 4\langle \psi\tilde{\psi}\rangle)\psi\rangle = -2\langle \tilde{\psi}\psi\tilde{\psi}\psi\rangle + 2\rho^2. \tag{5.26}$$

This result does not generalize to higher particle numbers.

In Ref. 8 the quantity Ω is defined as

$$|S_1|^2 = |S_2|^2 = 2\Omega - \rho^2, \tag{5.27}$$

and it follows from Eq. (5.26) that we can write

$$\Omega = \tfrac{3}{2}\rho^2 - \langle \tilde{\psi}\psi\tilde{\psi}\psi\rangle. \tag{5.28}$$

Since the elements S_{kl} are the components of $-2\langle \psi\tilde{\psi}\rangle_4$, we can substitute

$$\psi\tilde{\psi} = \tfrac{1}{2}(\rho - S_{kl} i\sigma_k^1 i\sigma_l^2) \tag{5.29}$$

into the preceding expression for Ω to obtain

$$\Omega = \tfrac{1}{4}(5\rho^2 - S_{kl} S_{kl}), \tag{5.30}$$

recovering Ω in terms of previously defined quantities. This result is not so easily deduced in the spin-tensor approach.

A second choice of ξ_{cd} considered in Ref. 8 is

$$\xi_{cd} = \psi^{*dc}. \tag{5.31}$$

The equivalent MSTA spinor is

$$\xi = (1 - \epsilon\tilde{\epsilon})\psi i\sigma_2^1 i\sigma_2^2, \tag{5.32}$$

leading to the new multivector

$$A' = \psi \tfrac{1}{2}(1+\sigma_3^1)\tfrac{1}{2}(1+\sigma_3^2)\tilde{\psi}(1 - \epsilon\tilde{\epsilon}) = A(1 - \epsilon\tilde{\epsilon}), \tag{5.33}$$

where A is as defined in Eq. (5.19). This expression makes it immediately clear that A' contains precisely the same information as A. This conclusion is much harder to reach in the tensor approach, where one is forced to consider each of the terms in the matrix $\psi^{ab}\psi^{*dc}$ and show that they can be written in terms of the $\psi^{ab}\psi^{cd}$.

It is clear from the expressions $\psi\tilde{\psi}$ and $\psi J\tilde{\psi}$ that $\{\rho, S_{1k}, S_{2l}, S_{kl}\}$ are invariant under an overall phase change of ψ. Consequently, this set only encodes seven of the eight degrees of freedom in ψ and some other form for ξ must be used to recover all of the information in ψ. The MSTA approach enables us to immediately write down two further geometric entities which pick up the phase information. These are

$$\tfrac{1}{2}W \equiv \psi i\sigma_2^1 i\sigma_2^2\tilde{\psi}, \quad \tfrac{1}{2}V \equiv \psi J i\sigma_2^1 i\sigma_2^2\tilde{\psi}. \tag{5.34}$$

With these we can proceed to give MSTA equivalents of the remaining tensors defined in Ref. 8. The first set are formed by taking $\xi_{ab} = \psi_{ab}$, yielding the two new quantities

$$\bar{\rho} \equiv \psi_{ab}\psi^{ab} = 2\langle i\sigma_2^1 i\sigma_2^2\tilde{\psi}\psi\rangle - 2\langle i\sigma_2^1 i\sigma_2^2\tilde{\psi}\psi J\rangle j = \langle W\rangle - j\langle V\rangle \tag{5.35}$$

and

$$\begin{aligned} T_{kl} \equiv \psi_{ab}e_{1k}{}^{ab}{}_{cd}e_{2k}{}^{cd}{}_{ef}\psi^{ef} &= -2\langle i\sigma_2^1 i\sigma_2^2\tilde{\psi} i\sigma_k^1 i\sigma_l^2\psi\rangle + 2j\langle i\sigma_2^1 i\sigma_2^2\tilde{\psi} i\sigma_k^1 i\sigma_l^2\psi J\rangle \\ &= -W\cdot(i\sigma_k^1 i\sigma_l^2) + jV\cdot(i\sigma_k^1 i\sigma_l^2). \end{aligned} \tag{5.36}$$

As expected, V and W are the only quantities necessary for the evaluation of these coefficients. Since $i\sigma_2^1 i\sigma_2^2$ and $i\sigma_1^1 i\sigma_2^2$ anticommute with J, the phase transformation $\psi \rightarrow \psi e^{\theta J}$ implies that

$$W \rightarrow \cos(2\theta)W + \sin(2\theta)V, \quad V \rightarrow \cos(2\theta)V - \sin(2\theta)W. \tag{5.37}$$

By taking scalar parts of these transformations we can deduce the behavior of $\bar{\rho}$ under phase changes. Similarly, taking the four-vector parts gives us the phase transformation properties for the real and imaginary parts of T_{kl}.

The remaining quantities defined in Ref. 8 are obtained from the choice $\xi_{ab} = \psi_{ba}$. Again, the MSTA equivalent for this choice,

$$\xi = (1 - \epsilon\tilde{\epsilon})\psi \tag{5.38}$$

makes it clear that this choice yields nothing new, and that all of the tensor coefficients derived by setting $\xi_{ab} = \psi_{ba}$ can be recovered from V and W.

One remaining MSTA construct is the scalar + four-vector quantity $\tilde{\psi}\psi$. By its construction this quantity is automatically invariant under rotations. The scalar term is just the density ρ already defined. The four-vector invariant is more interesting since it picks up phase information. In terms of the decomposition (5.5) we have explicitly

$$\tilde{\psi}\psi = [\rho + 2i\sigma_2^1 i\sigma_2^2(\phi_1\phi_4 - \phi_2\phi_3)]E, \tag{5.39}$$

which demonstrates that it is the complex quantity $\phi_1\phi_4 - \phi_2\phi_3$ which picks up the phase information. The same information is encoded in the trace T_{kk}, so $\tilde{\psi}\psi$ yields no new information.

In summary, we see that all of the information required to completely encode ψ in tensor form is contained in the set of multivectors

$$\{\psi\tilde{\psi},\ \psi J\tilde{\psi},\ \psi i\sigma_2^1 i\sigma_2^2\tilde{\psi},\ \psi i\sigma_2^1 i\sigma_2^2 J\tilde{\psi}\}. \tag{5.40}$$

This is the complete set of distinct objects obtainable by taking any basis element Γ of the direct product of two Pauli algebras and forming the bilinear construct $\psi\Gamma\tilde{\psi}$. The other objects one might try to construct would be of the form $\psi i\sigma_1^1\tilde{\psi}$, but the presence of the idempotent E ensures that all quantities of this form vanish. (If one looks to construct models beyond those suggested by quantum theory, however, one can contemplate not correlating the phases of the particles, in which case such quantities would come into consideration.)

The set (5.40) is directly analogous to the one-particle set $\{\rho, iM, iN, iS\}$ which can be written as

$$\{\psi\tilde{\psi},\ \ \psi J_1\tilde{\psi},\ \ \psi i\sigma_2\tilde{\psi},\ \ \psi i\sigma_2 J_1\tilde{\psi}\}. \tag{5.41}$$

We are now in a position to appreciate just how systematic and simple the MSTA approach is. The full set of distinct objects (5.40) could have been written down easily at the start of the analysis, and all terms calculated simply with the MSTA, without requiring laborious matrix and tensor manipulations. Furthermore, the MSTA approach is very amenable to generalization to higher dimensions, as discussed in Sec. VI.

VI. EXTENSIONS AND FURTHER WORK

As in the one-particle and two-particle cases, the n-particle spinor $\psi^{abc\cdots}$ can be mapped directly to an element in the direct product of n Pauli-even algebras $(\mathcal{G}_3^+)^n$. The ambiguity in the complex structure for each of the factors in $(\mathcal{G}_3^+)^n$ requires the introduction of the n-particle correlator E_n defined at Eq. (2.16). We find that the spinor ψ can be written as a combination of 2^n terms:

$$\psi=\left(\phi+\sum_{a=1}^{n} i\sigma_2^a\phi_a+\sum_{a<b} i\sigma_2^a i\sigma_2^b\phi_{ab}+\cdots+i\sigma_2^1 i\sigma_2^2\cdots i\sigma_2^n\phi_{12\ldots n}\right)E_n, \tag{6.1}$$

where the ϕ are complex combinations of 1 and J_n. The tensor observables formed from two n-particle spinors are summarized in the multivector

$$A=\psi\left(\tfrac{1}{2}(\sigma_1^1+i\sigma_2^1)\cdots\tfrac{1}{2}(\sigma_1^n+i\sigma_2^n)\right)\tilde{\xi}, \tag{6.2}$$

and it is clear that the various tensors one might construct correspond to the various multivector parts of bilinear constructs of the form $\psi\Gamma\tilde{\psi}$, where Γ is some fixed set of MSTA basis elements.

For example, one of the key objects to analyze is the multivector $\psi J\tilde{\psi}$. This has grade-2, grade-6,..., components, of which the grade-2 component is the multiparticle spin bivector[2]

$$S\equiv 2^{n-1}\langle\psi J_n\tilde{\psi}\rangle_2. \tag{6.3}$$

Interpretations for the other components of $\psi J\tilde{\psi}$ can be made in terms of spin correlations between particles. This approach to constructing tensors from spinors is clearly more economic than the matrix/tensor approach, which gets progressively worse with increasing particle number due to the large degree of redundancy in the tensor coefficients contained in the various bilinear constructs.

A further advantage of the MSTA approach is that it is easily generalized to the relativistic domain. A discussion of this is contained in Ref. 2 and further details are contained in Ref. 6. Such an extension is essential if this approach is to shed light on questions of nonlocality in Einstein–Podolski–Rosen- (EPR)-type experiments—in particular a full analysis of these must incorporate relativity, as this lies at the heart of the paradox. A simple model for two-particle relativistic spin correlations is contained in Ref. 2, though more work is needed to extend this work to model an EPR-type setup. Finally, it should be borne in mind that the MSTA is equally applicable to classical as well as quantum physics, and many of the techniques described here are useful for studying multiparticle classical relativistic dynamics, a notoriously difficult subject.

ACKNOWLEDGMENTS

S.S. is grateful to the Cambridge Livingstone Trust for funding during the course of this work. C.D. is funded by a grant from the Lloyd's of London Tercentenary Foundation.

[1] S. F. Gull, A. N. Lasenby, and C. J. L. Doran, ''Imaginary numbers are not real—the geometric algebra of spacetime,'' Found. Phys. **23**, 1175 (1993).
[2] C. J. L. Doran, A. N. Lasenby, S. F. Gull, S. S. Somaroo, and A. D. Challinor, ''Spacetime algebra and electron physics,'' Adv. Imaging Electron Phys. **95**, 271 (1996).
[3] D. Hestenes, *Space–Time Algebra* (Gordon and Breach, New York, 1966).
[4] D. Hestenes, ''Proper dynamics of a rigid point particle,'' J. Math. Phys. **15**, 1778 (1974).
[5] A. N. Lasenby, C. J. L. Doran, and S. F. Gull, ''Gravity, gauge theories and geometric algebra,'' Philos. Trans. R. Soc. London, Ser. A **356**, 487 (1998).
[6] C. J. L. Doran, ''Geometric algebra and its application to mathematical physics,'' Ph.D. thesis, Cambridge University, 1994.
[7] P. R. Holland, *The Quantum Theory of Motion* (Cambridge University Press, Cambridge, 1993).
[8] P. R. Holland, ''Causal interpretation of a system of two spin-1/2 particles,'' Phys. Rep. **169**, 294 (1988).
[9] C. J. L. Doran, A. N. Lasenby, S. F. Gull, and J. Lasenby, ''Lectures in geometric algebra,'' in *Clifford (Geometric) Algebras*, edited by W. E. Baylis (Birkhauser, Boston, 1996), pp. 65–236.
[10] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus* (Reidel, Dordrecht, 1984).
[11] I. W. Benn and R. W. Tucker, *An Introduction to Spinors and Geometry* (Hilger, Bristol, 1988).
[12] C. J. L. Doran, A. N. Lasenby, and S. F. Gull, ''States and operators in the spacetime algebra,'' Found. Phys. **23**, 1239 (1993).
[13] R. Penrose and W. Rindler, *Spinors and Space–time Volume 1: Two-spinor Calculus and Relativistic Fields* (Cambridge University Press, Cambridge, 1984).
[14] A. N. Lasenby, C. J. L. Doran, and S. F. Gull, ''2-spinors, twistors and supersymmetry in the spacetime algebra,'' in *Spinors, Twistors, Clifford Algebras and Quantum Deformations*, edited by Z. Oziewicz, B. Jancewicz, and A. Borowiec (Kluwer, Dordrecht, 1993), p. 233.