Classical and Quantum Dynamics in a Black Hole Background

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Thanks etc.

- Work in collaboration with
  - Anthony Lasenby
  - Steve Gull
  - Jonathan Pritchard
  - Alejandro Caceres
  - Anthony Challinor
  - Ian Hinder
- Papers on www.mrao.cam.ac.uk/~Clifford
  - gr-qc/0106039
  - gr-qc/0209090

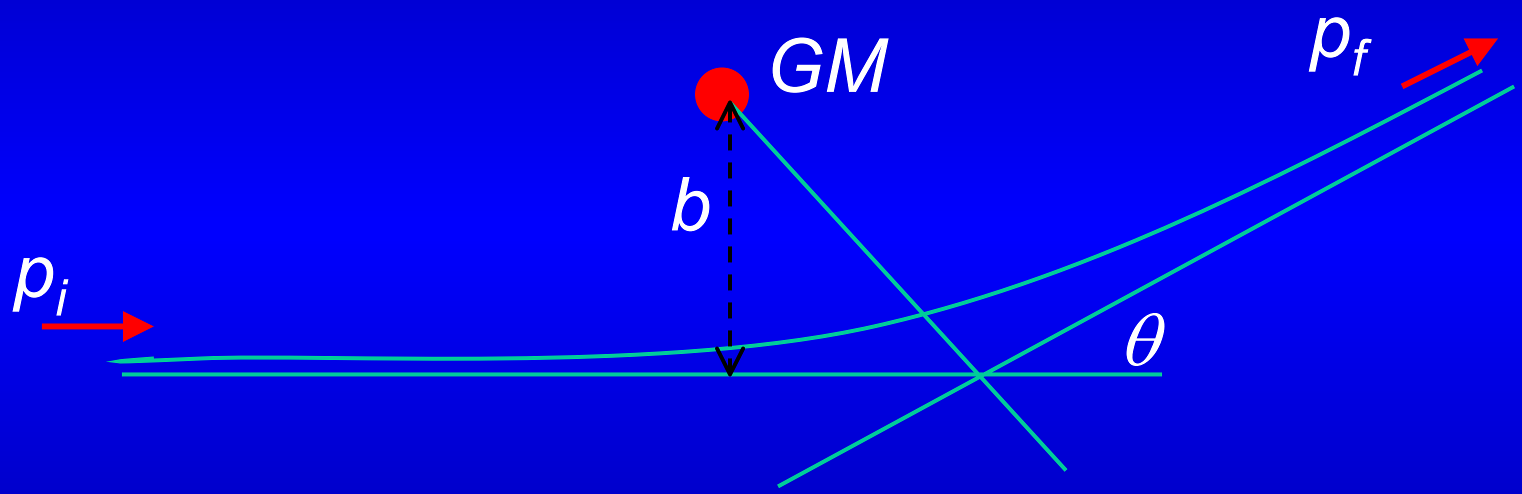
Outline

- 4 phenomena to give a classical and quantum description for

	Classical	Quantum
Scattering	✓	✓
Absorption	✓	✓
Bound states	✓	✓
Emission	x	✓

Classical Scattering

- Main method of comparison is the differential cross section



For r^{-1} potential get Rutherford formula

$$\frac{d\sigma}{d\Omega} = \frac{4(GMm)^2}{\boldsymbol{q}^4} = \frac{(GM)^2}{4v^4 \sin^4(\theta/2)}$$

Classical Dynamics

- The Schwarzschild line element contains all relativistic information ($c=1$)

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

- The geodesic equation for a radially infalling particle is essentially Newtonian

$$\ddot{r} = \frac{-GM}{r^2}$$

Painlevé Coordinates

- Necessary for later calculations to remove the singularity at the horizon
- Convert to time as measured by infalling observers

$$dt = d\bar{t} + \frac{(2GMr)^{1/2}}{r - 2GM} dr$$

- Find metric is now (no problem at horizon)

$$ds^2 = dt^2 - \left(dr + \left(\frac{2GM}{r} \right)^{1/2} dt \right)^2 - r^2 d\Omega^2$$

Geodesic Equation

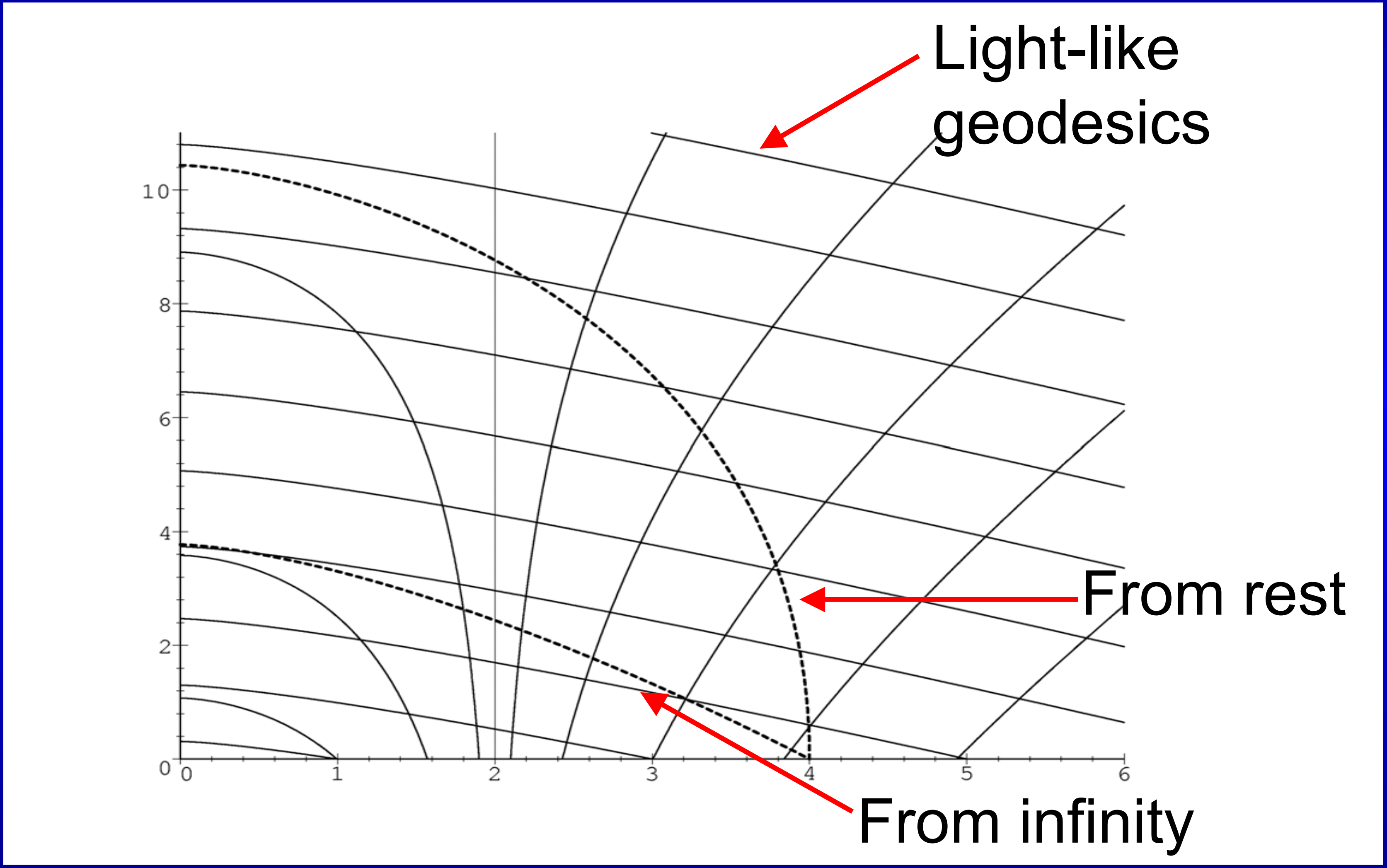
- The geodesic equation can be written

$$\ddot{\boldsymbol{x}} = -\frac{GM}{r^2}\left(1 + \frac{3L^2}{m^2c^2r^2}\right)\hat{\boldsymbol{x}}$$

- Vectors in 3-space
- Overdots denote proper time derivatives
- r is a *local observable* obtained from the strength of the tidal force – not just a coordinate
- Summarise in effective potential (per unit mass)

$$V_{\text{eff}} = -\frac{GM}{r} + \frac{l^2}{2r^2}\left(1 - \frac{2GM}{r}\right)$$

Radial geodesics



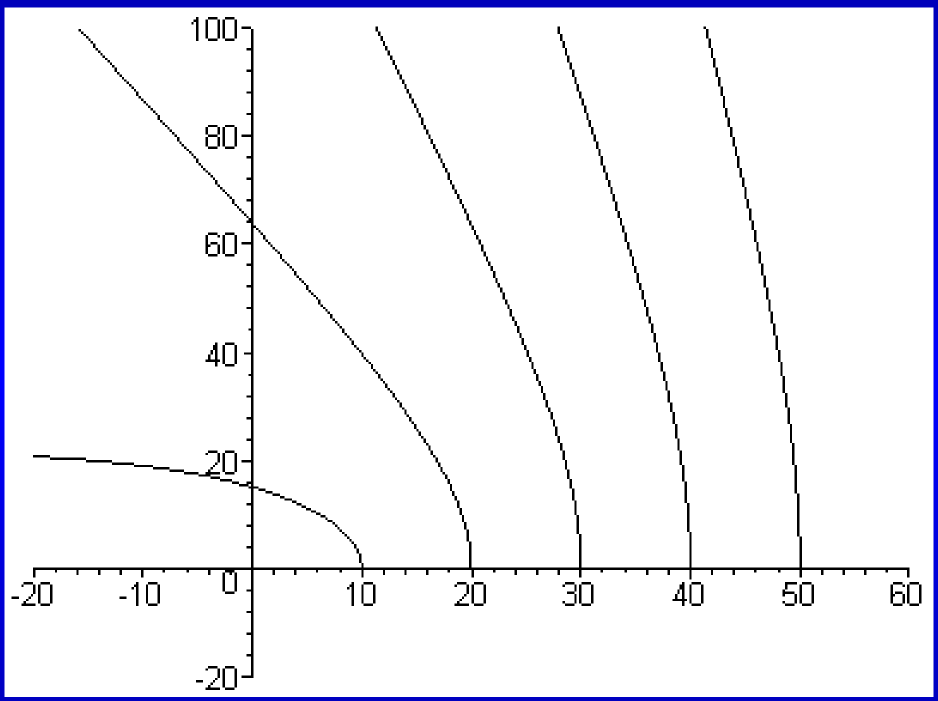
Geodesic Motion

- Geodesics can be quite complicated
- Write the geodesic equation in form (u=1/r)

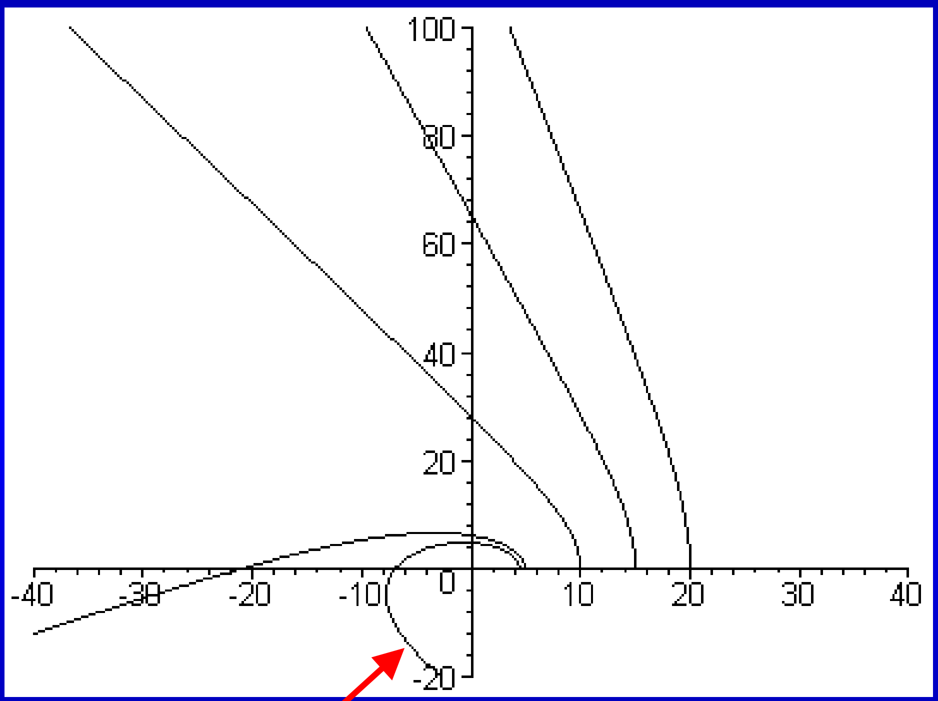
$$\left(\frac{du}{d\phi}\right)^2 = 2GMu^3 - u^2 + 2\frac{m}{L}u + \frac{E^2 - m^2}{L^2}$$

- A cubic equation, so solution is an elliptic function
- For intermediate angular velocities, get spiralling
- Complicates the calculation of the cross section

Sample Geodesics



$v=0.5c$



$v=0.9c$

Spiralling

Cross-section

- Analytic formula for the motion involves an elliptic integral
- Best evaluated numerically, for a range of velocities
- Collins et al. J. Phys A 6 (161), 1973
- Result in a series of cross-section graphs
- Can do *small angle* case analytically

$$\frac{d\sigma}{d\Omega} = \frac{4(GM)^2(2\beta^2 - 1)^2}{\theta^4(\beta^2 - 1)^2} \qquad \beta = \frac{E}{mc^2}$$

Numerical Results

Rutherford at small θ

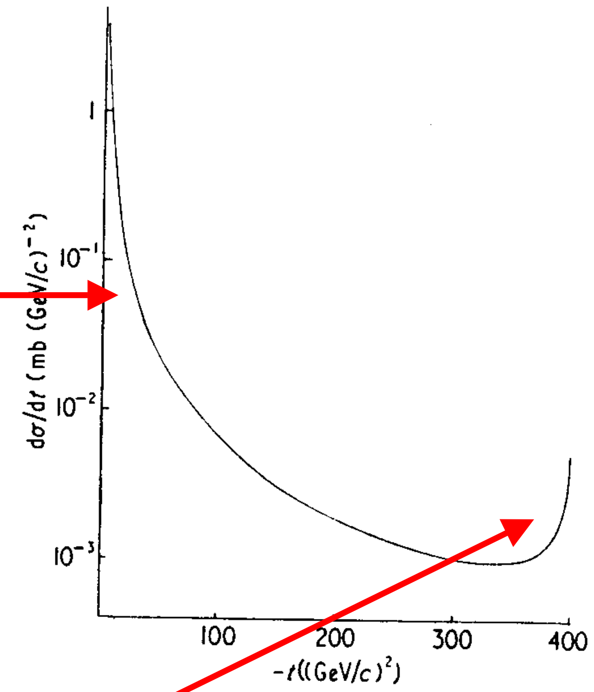


Figure 2. Differential cross section for a particle of mass 1 GeV/c^2 and momentum 10 GeV/c, with a scattering centre of mass 10^{-15} cm in gravitational units.

Additional scattering as $\theta \approx \pi$

Corresponds to v=0.995c

Quantum Treatment

- Concentrate on fermions.
- These are described by the *Dirac equation*
- Uses apparatus of spinors, Dirac matrices, tetrads and spin connections
- Typically neglected in black hole treatments – favour massless scalar fields
- But in fact, Dirac theory is *easier*
  - First order
  - Simple, Hamiltonian form

Dirac Equation

- Standard notation, in full gruesome detail

$$ig^{\mu}(\partial_{\mu} + \frac{i}{2}\Gamma_{\mu}^{\alpha\beta}\Sigma_{\alpha\beta})\psi = m\psi$$

$$\{g_{\mu}, g_{\nu}\} = 2g_{\mu\nu}I$$

$$\{g_{\mu}, g^{\nu}\} = 2\delta_{\mu}^{\nu}I$$

Spin Connection

Dirac spinor

$$\Sigma_{\alpha\beta} = \frac{i}{4}[\gamma_{\alpha}, \gamma_{\beta}]$$

- Of course, much easier using *geometric algebra* – which is how we do it!

Hamiltonian Form

- Return to the metric

$$ds^2 = dt^2 - \left(dr + \left(\frac{2GM}{r}\right)^{1/2} dt\right)^2 - r^2 d\Omega^2$$

- Convert to Cartesians

Hamiltonian Form

- Return to the metric

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu - \frac{2GM}{r} dt^2 - \frac{2}{r}\left(\frac{2GM}{r}\right)^{1/2} x^i dx^i \, dt$$

- Now introduce the matrices / vectors

'Flat' Minkowski vectors

$$g^0 = \gamma^0$$

$$g^i = \gamma^i - \boxed{\left(\frac{2GM}{r}\right)^{1/2} \frac{x^i}{r}\gamma^0}$$

Gravitational interaction

Hamiltonian Form II

- Now insert matrices into Dirac equation

$$i\partial\!\!\!/\psi - i\gamma^0\left(\frac{2GM}{r}\right)^{1/2}\left(\frac{\partial}{\partial r}+\frac{3}{4r}\right)\psi = m\psi$$

Flat space

Interaction

- Convert to *Hamiltonian* form
- All interactions contained in the interaction Hamiltonian

$$\widehat{H}_I\psi = i\hbar(2GM/r)^{1/2}r^{-3/4}\partial_r(r^{3/4}\psi)$$

The Interaction Hamiltonian

$$\hat{H}_I \psi = i\hbar (2GM/r)^{1/2} r^{-3/4} \partial_r (r^{3/4} \psi)$$

- All gravitational effects in a *single* term
- This is *gauge dependent*
- In all gauge theories, trick is to
  1. Find a sensible gauge
  2. Ensure that all physical predictions are gauge invariant
- Hamiltonian is scalar (no spin effects)
- Independent of particle mass
- Independent of c

Non-relativistic limit

- The non-relativistic limit of the Dirac equation is the Pauli equation
- No spin effects - insert directly into Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + i\hbar(2GM/r)^{1/2} r^{-3/4}\partial_r(r^{3/4}\psi) = E_{\mathrm{NR}}\psi$$

- Substitution $\Psi = \psi \exp\left(-i(8r/a_G)^{1/2}\right)$

$$a_G = \hbar^2/(2GMm^2)$$

Implications

- Recovered Newtonian potential
- With a Hamiltonian independent of mass!
- Solutions are confluent hypergeometrics
- Phase factor irrelevant to density, hence to cross-section
- Non-relativistic limit of cross-section must be *Rutherford* formula (exact)
- Also expect a bound state spectrum equivalent to Hydrogen atom (later)

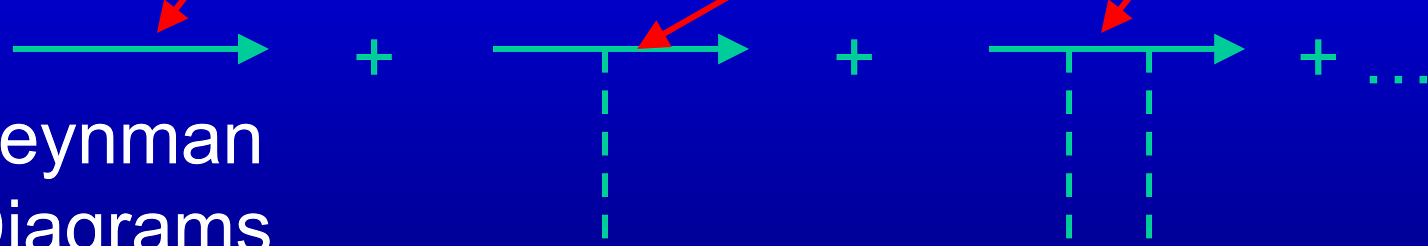
Iterative Solution

- Borrow technique from quantum field theory

$$[i\not{\partial}_2 - B(x_2) - m]S_G(x_2, x_1) = \delta^4(x_2 - x_1)$$

- Has an iterative solution

$$S_G(x_f, x_i) = S_F(x_f, x_i) + \int d^4x_1\, S_F(x_f, x_1)B(x_1)S_F(x_1, x_i)$$
$$+ \iint d^4x_1\, d^4x_2\, S_F(x_f, x_1)B(x_1)S_F(x_1, x_2)B(x_2)S_F(x_2, x_i) + \cdots$$



Feynman Diagrams

Amplitude

- Convert to momentum space

$$\mathcal{M} = \bar{u}_s(\boldsymbol{p}_f)V u_r(\boldsymbol{p}_i)$$

Amplitude

Plane wave spin states

$$V = B(\boldsymbol{p}_f,\boldsymbol{p}_i) + \int \frac{d^3k}{(2\pi)^3} B(\boldsymbol{p}_f,\boldsymbol{k}) \frac{\not{k}+m}{k^2-m^2+i\epsilon} B(\boldsymbol{k},\boldsymbol{p}_i) + \cdots$$

Use amplitude to compute differential cross section

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi}\right)^2 |\mathcal{M}|^2$$

Vertex Factor

- Fourier transform of interaction term is

$$B(\boldsymbol{p}_2, \boldsymbol{p}_1) = (2GM)^{1/2} i\gamma^0 \int d^3x\, e^{-i\boldsymbol{p}_2\cdot\boldsymbol{x}} \frac{1}{r^{1/2}} \left(\frac{\partial}{\partial r} + \frac{3}{4r}\right) e^{i\boldsymbol{p}_1\cdot\boldsymbol{x}}$$

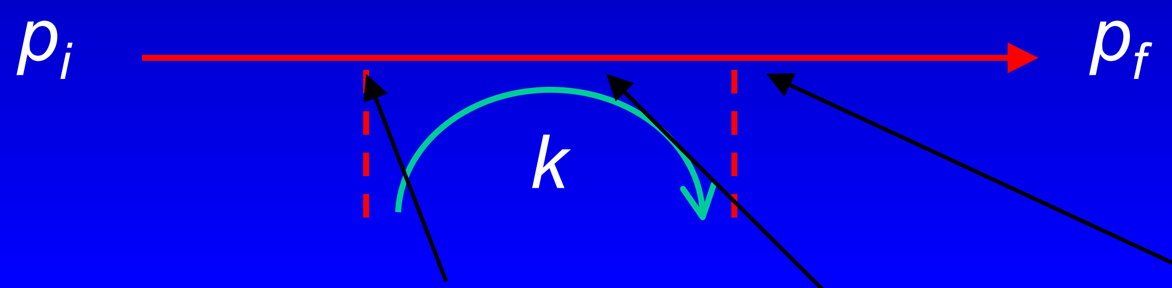
- Evaluates to

$$B(\boldsymbol{p}_2, \boldsymbol{p}_1) = 3\pi^{3/2} i (GM)^{1/2} \frac{\boxed{\boldsymbol{p}_2{}^2 - \boldsymbol{p}_1{}^2}}{|\boldsymbol{p}_2 - \boldsymbol{p}_1|^{7/2}} \gamma^0$$

Energy conserved so this vanishes on shell
Process must be second order

Vertex Factor II

- Evaluate the second order diagram



$$I_1 = \int \frac{d^3k}{(2\pi)^3} \frac{\boldsymbol{p}_f{}^2 - \boldsymbol{k}^2}{|\boldsymbol{p}_f - \boldsymbol{k}|^{7/2}} \frac{\not{k} + m}{k^2 - m^2 + i\epsilon} \frac{\boldsymbol{k}^2 - \boldsymbol{p}_i{}^2}{|\boldsymbol{k} - \boldsymbol{p}_i|^{7/2}}$$

Result is

$$= \frac{1}{9\pi^2 \boldsymbol{q}^2}\left(2m + 3(\not{p}_f + \not{p}_i) - 4E\gamma^0\right)$$

Cross-section

- Reinsert the asymptotic spinors. Get differential cross-section

$$\frac{d\sigma}{d\Omega} = \frac{(2GMm)^2}{\boldsymbol{q}^4} \left| \bar{u}_s(\boldsymbol{p}_f)(2E\gamma^0 - m)u_r(\boldsymbol{p}_i) \right|^2$$

- q is the momentum transfer $p_f - p_i$
- Unpolarised version, after spin sums, is

$$\frac{d\sigma}{d\Omega} = \frac{(GM)^2}{4v^4 \sin^4(\theta/2)} \left(1 + 2v^2 - 3v^2 \sin^2(\theta/2) + v^4 - v^4 \sin^2(\theta/2)\right)$$

Velocity $v = |\boldsymbol{p}|/E$

Scattering angle θ

Comments

$$\frac{d\sigma}{d\Omega} = \frac{(GM)^2}{4v^4 \sin^4(\theta/2)}\left(1+2v^2-3v^2\sin^2(\theta/2)+v^4-v^4\sin^2(\theta/2)\right)$$

- Result is independent of particle mass
- *Equivalence principle* holds to lowest order in quantum theory
- Small angle approximation agrees with point particle dynamics
- No boundary conditions specified at horizon
- Can extend to higher order and include radiation
- Get terms violating equivalence principle

Comments II

- Massless limit well defined ($v = 1$)

$$\frac{d\sigma}{d\Omega} = \frac{(GM)^2 \cos^2(\theta/2)}{\sin^4(\theta/2)}$$

- Reproduces photon deflection formula at small angles
- Zero in backward direction – a neutrino diffraction effect
- Can apply to scalar fields as well

$$\frac{d\sigma}{d\Omega} = \frac{(GM)^2}{4v^4 \sin^4(\theta/2)}(1 + v^2)^2$$

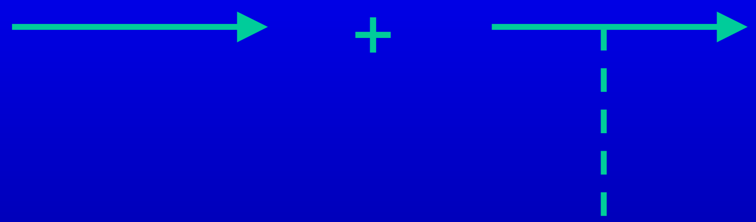
Gauge Invariance

- Important issue to address
- Do not have a general proof, but can reproduce calculation in another gauge
- In *Kerr-Schild* gauge set

$$g^0 = \gamma^0 + \frac{GM}{r}(\gamma^0 - \gamma_r)$$

$$g^i = \gamma^i - \frac{GM}{r}\frac{x^i}{r}(\gamma^0 - \gamma_r)$$

First-order in *M*



- Calculation is a different order
- But result is unchanged – a physical prediction

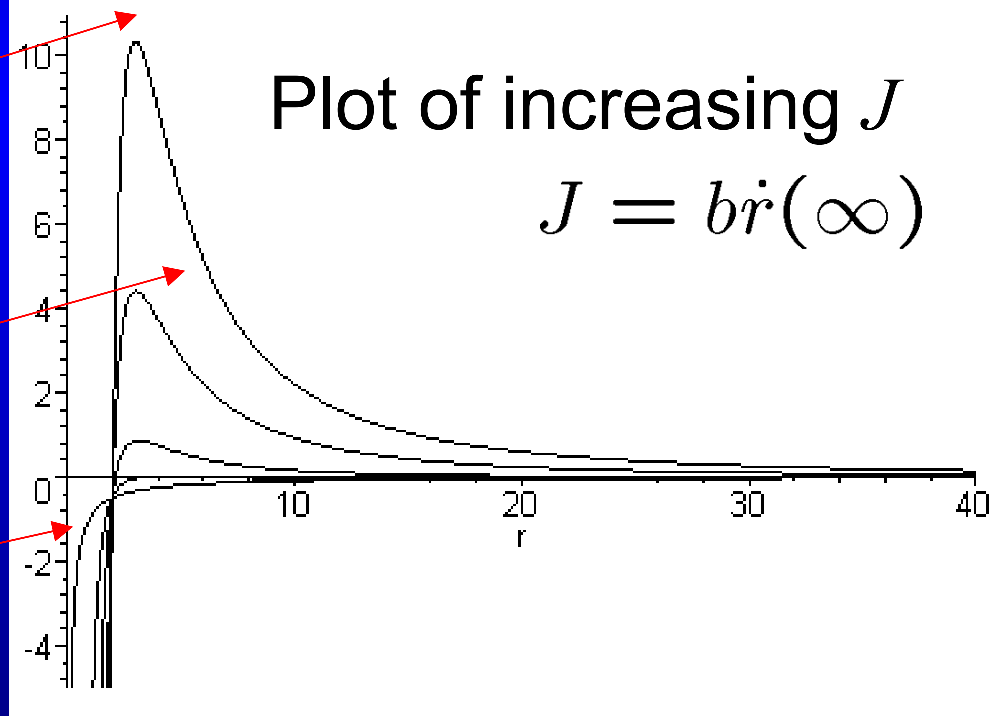
Absorption

- Particles too close to the horizon end up captured
- See this from the effective potential

E too high get absorbed

Higher J values are scattered

Low J are absorbed



Plot of increasing J

$$J = b\dot{r}(\infty)$$

Absorption Cross-section

- Impact parameter b is critical distance from hole for fixed velocity and angular momentum
- Total absorption cross-section is

$$\sigma_{\mathrm{abs}} = \pi b^2$$

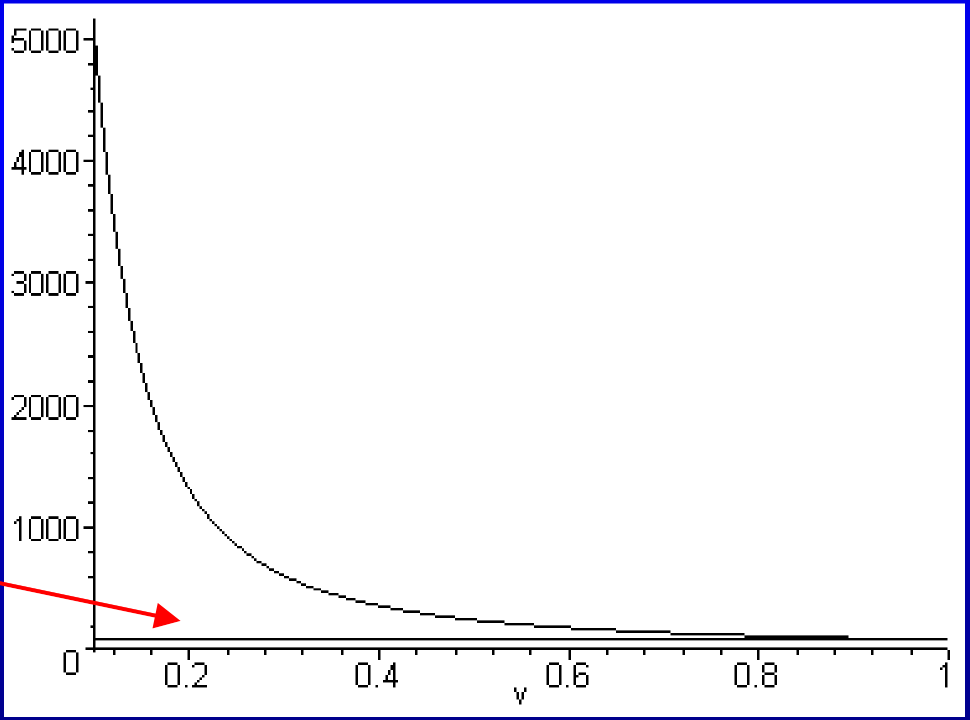
- For photons find that $b^2=27(GM)^2$
- Hole appears of a disk of radius b

$$\sigma_{\mathrm{abs}} = \pi b^2 = 27\pi(GM)^2 = \frac{27\pi(GM)^2}{c^4}$$

Absorption Cross-section II

- Slightly more complicated calculation gives

$$\sigma_{\text{abs}} = \frac{\pi(GM)^2}{2v^4}\left(8v^4 + 20v^2 - 1 + (1+8v^2)^{3/2}\right)$$



Photon limit

Quantum Equations

- Radial Schrodinger equation is

$$\frac{1}{r}\frac{d^2}{dr^2}(r\psi) - \frac{l(l+1)}{r^2}\psi = -(E-m)(E+m)\psi$$

- Convert to first-order form ($r\psi = u_1$)

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} \kappa/r & i(m+E+\hat{H}_I) \\ -i(m-E-\hat{H}_I) & -\kappa/r \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

- With $|\kappa| = l+1$ recover the correct Dirac radial separation
- Energy term tells us how to add in interaction

Black Hole Case

- Black hole Hamiltonian includes derivative terms. Find that radial equations are (G=1)

$$\left(1-2M/r\right)\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 1 & (2M/r)^{1/2} \\ (2M/r)^{1/2} & 1 \end{pmatrix} A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$A = \begin{pmatrix} \kappa/r & i(E+m)-(2M/r)^{1/2}/(4r) \\ i(E-m)-(2M/r)^{1/2}/(4r) & -\kappa/r \end{pmatrix}$$

- See that singular points exist at the origin ($r^{-3/4}$) horizon, and at infinity (irregular)
- Special function theory underdeveloped for this problem

Units and Dimensions

- Convert to dimensionless form by introducing distance function $x=2r/r_0$
- Dirac equation controlled by dimensionless coupling constant α and energy ε

$$x = \frac{rc^2}{GM} \qquad \alpha = \frac{GMm}{\hbar c} = \frac{Mm}{m_p^2} \qquad \varepsilon = \frac{EM}{c^2 m_p^2}$$

- α also ratio $\pi r_0/\lambda$ – horizon/Compton w/length
- $\alpha \approx 1$ corresponds to primordial black holes
- Also have $$\frac{\varepsilon}{\alpha} = \frac{E}{mc^2}$$

Horizon

- Series expansion about horizon $\eta = (r-2M)$

$$u_1 = \eta^s \sum_{k=0}^{\infty} \alpha_k \eta^k \qquad u_2 = \eta^s \sum_{k=0}^{\infty} \beta_k \eta^k$$

- Get indicial equation

$$\det \left[\begin{pmatrix} 1 & (2M/r)^{1/2} \\ (2M/r)^{1/2} & 1 \end{pmatrix} A - \frac{s}{r} I \right]_{r=2M} = 0$$

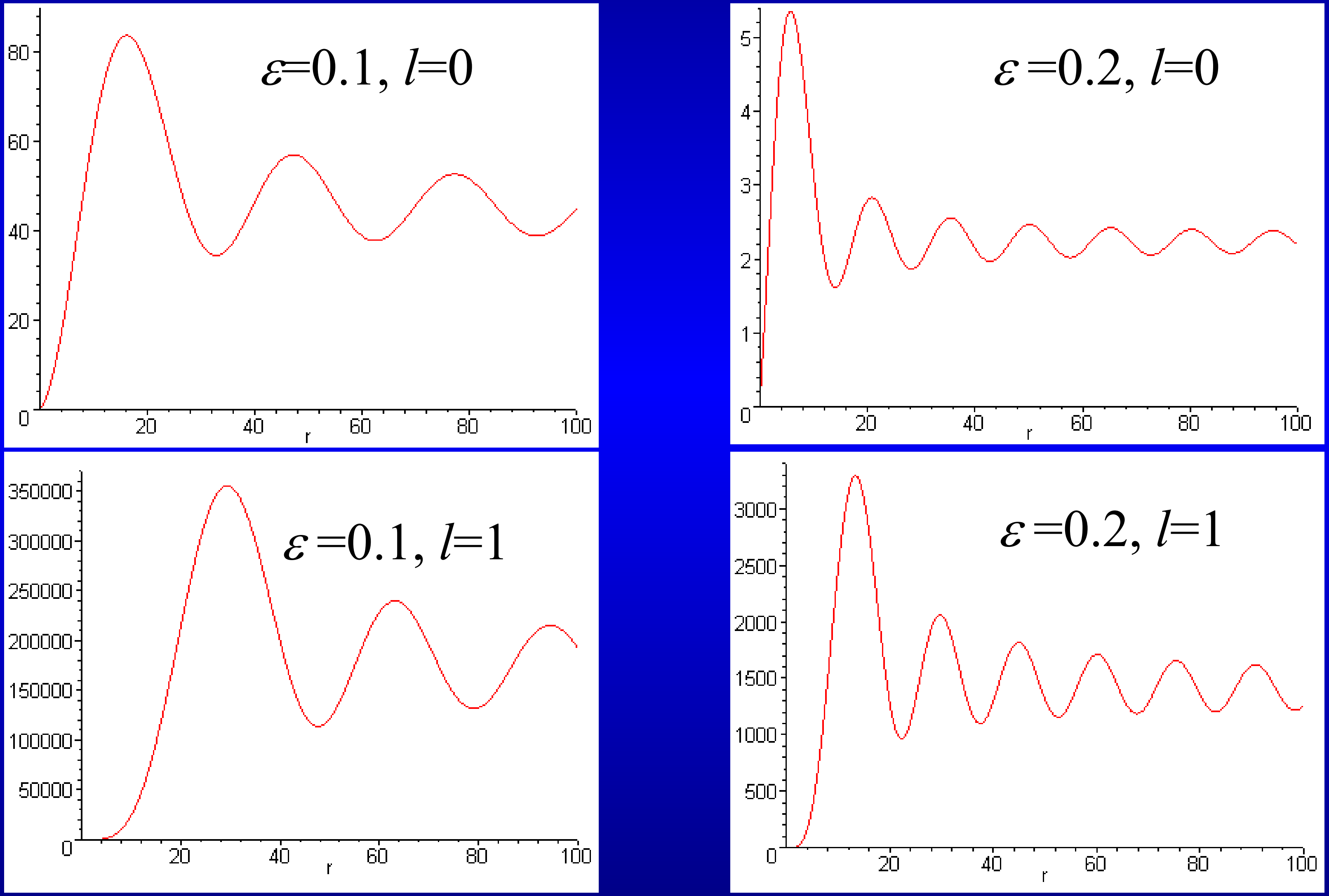
- Roots are $s = 0, -\frac{1}{2} + 4iME$

Gauge invariant

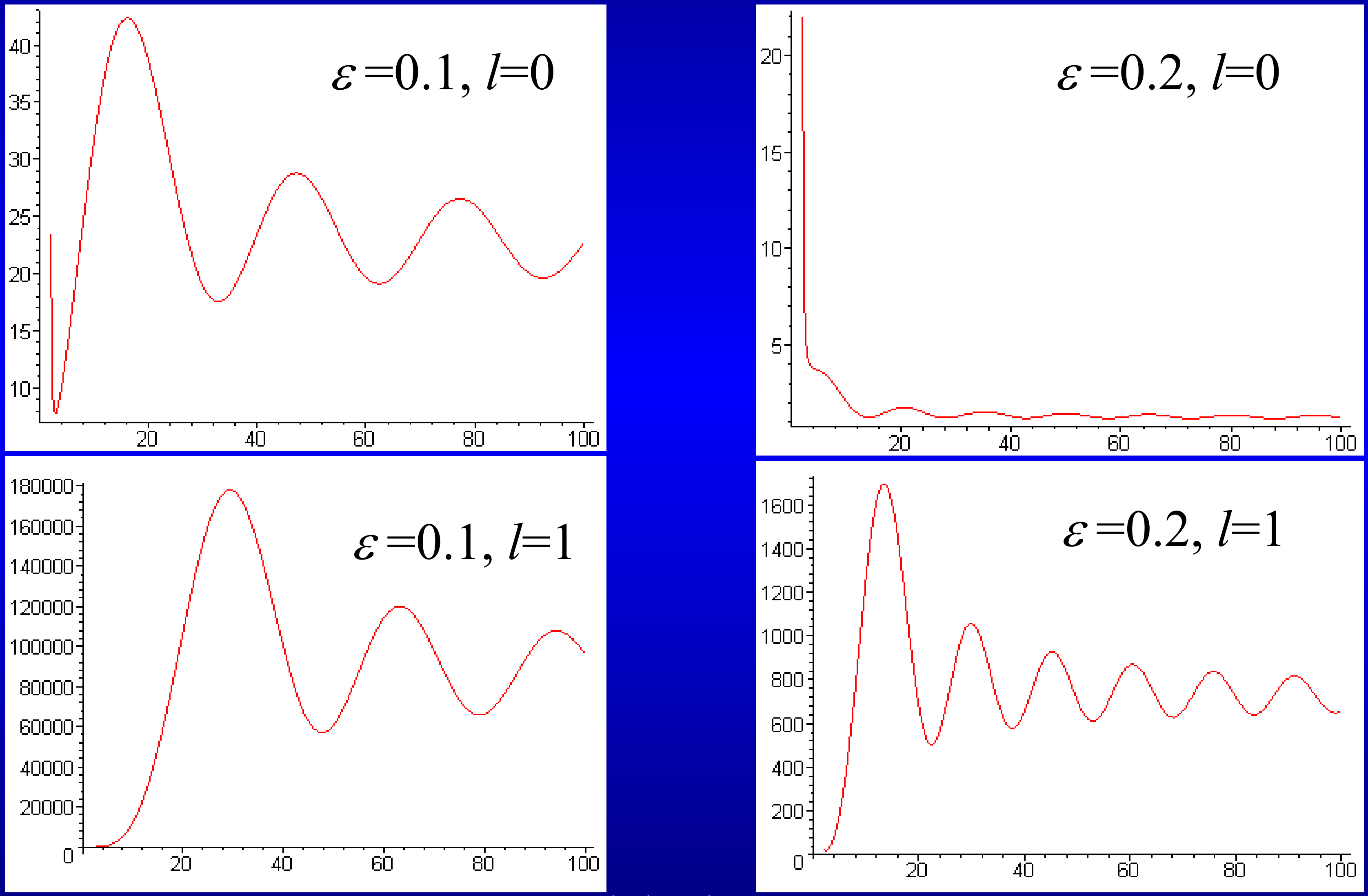
Regular branch - physical

Singular branch - unphysical

Regular Solutions (α=0.01)



Singular modes (α=0.01)



Asymptotic Behaviour

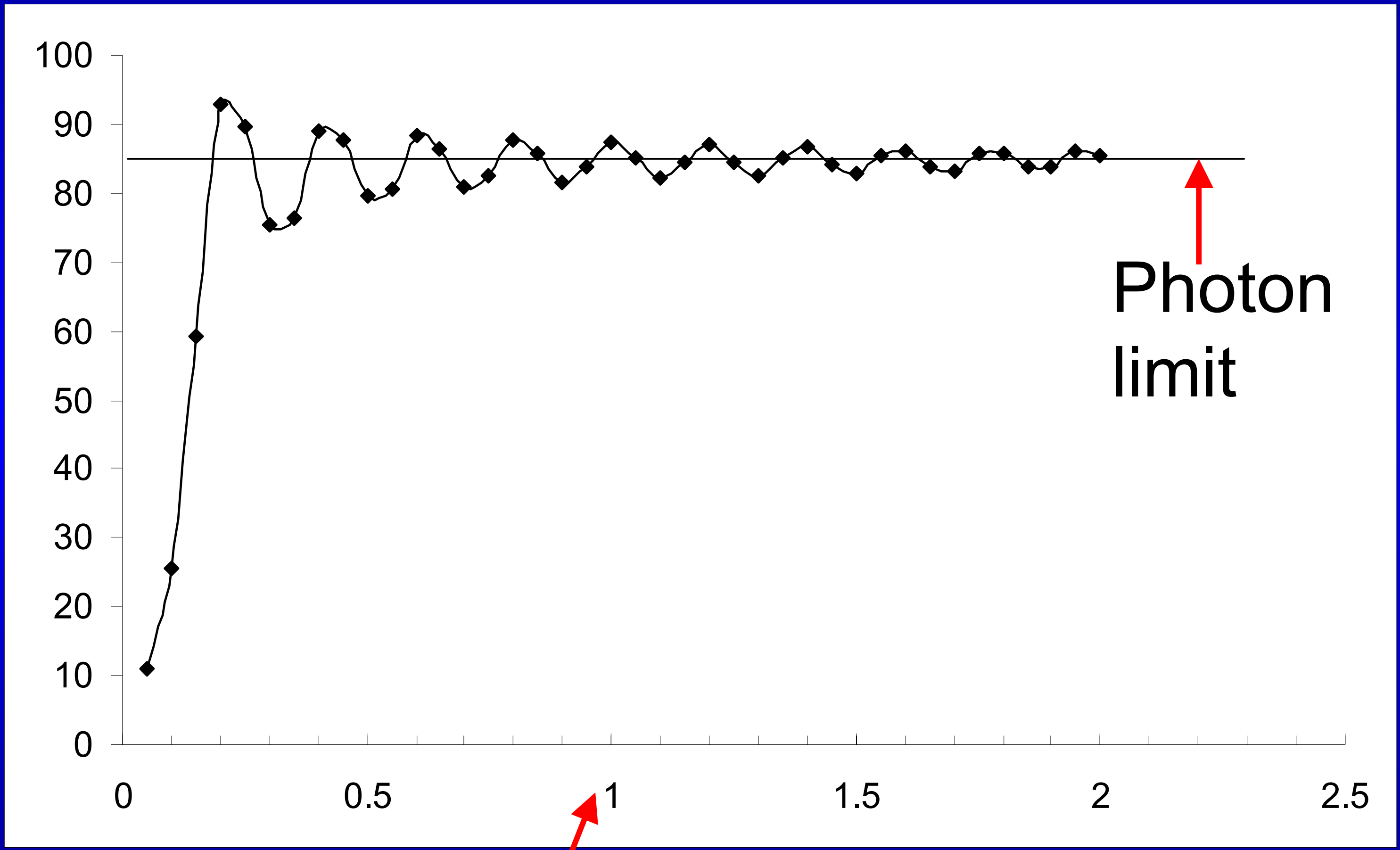
- At large r have

$$u_1 = \beta_\kappa \exp i\left(pr + \frac{M}{p}(m^2 + 2p^2)\ln(pr)\right) e^{2iE(2Mr)^{1/2}}$$
$$+ \alpha_\kappa \exp -i\left(pr + \frac{M}{p}(m^2 + 2p^2)\ln(pr)\right) e^{2iE(2Mr)^{1/2}}$$

- Similar for u_2
- Normalise such that $W = -\frac{2p}{E+m}(|\alpha_\kappa|^2 - |\beta_\kappa|^2) = -1$
- Absorption cross section is

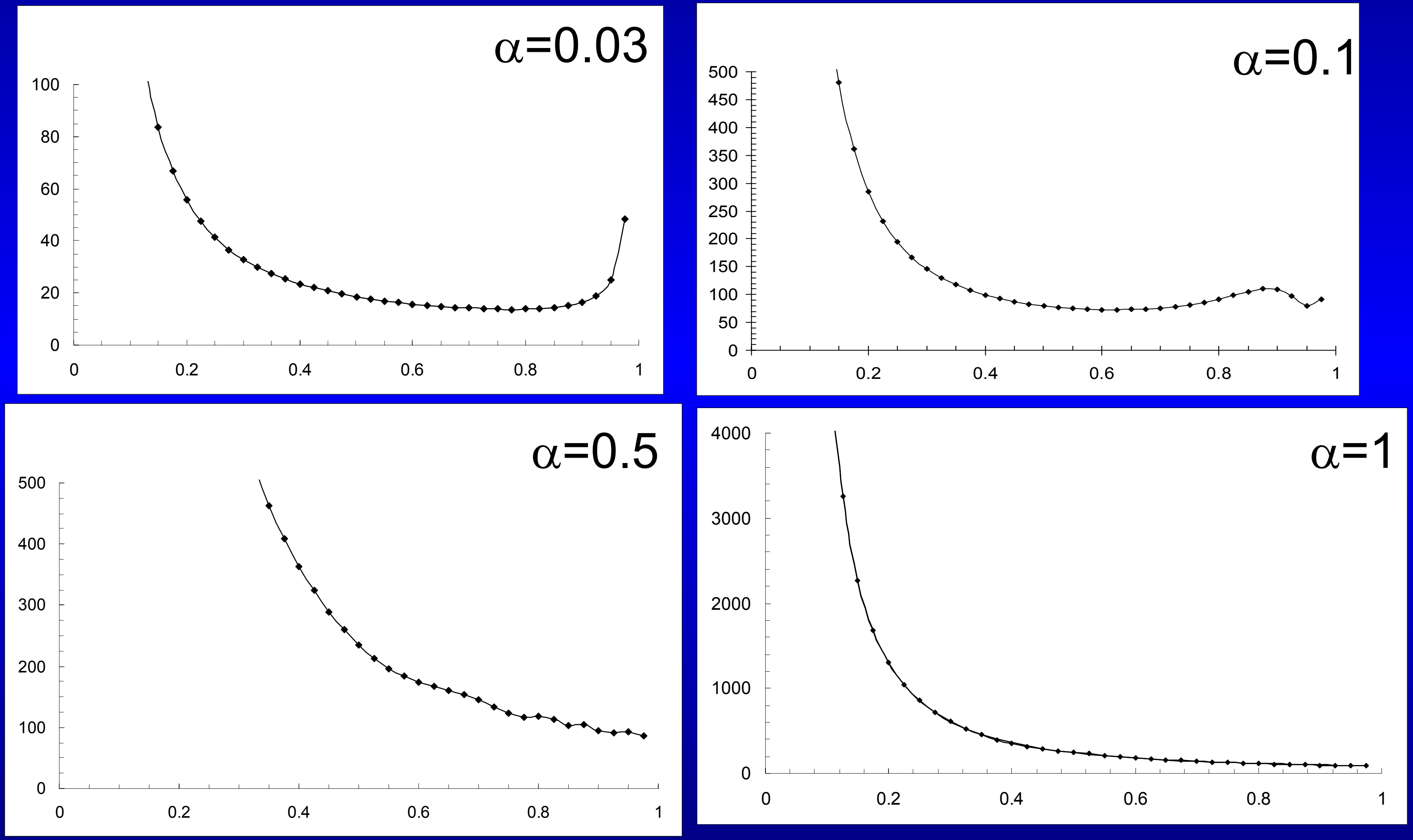
$$\sigma_{\text{abs}} = \frac{\pi}{2p(E-m)} \sum_{\kappa \neq 0} \frac{|\kappa|}{|\alpha_\kappa|^2}$$

Massless Case



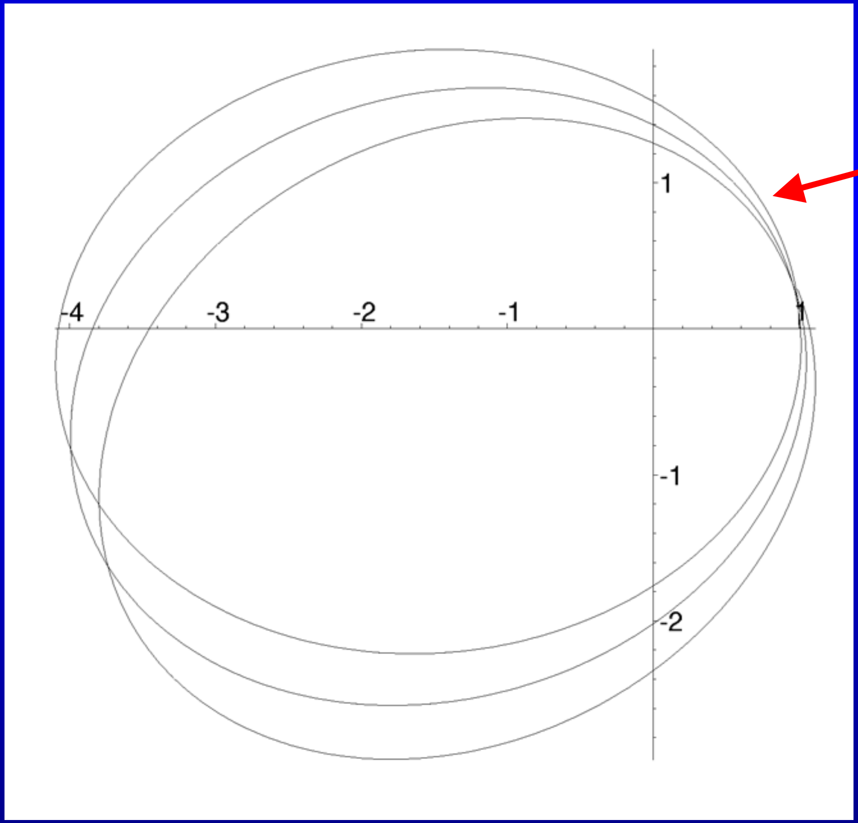
Momentum

Massive Case



Classical Bound States

- Can have stable, classical orbits outside a black hole



Precessing ellipse

Find minimum bound state energy $0.95mc^2$

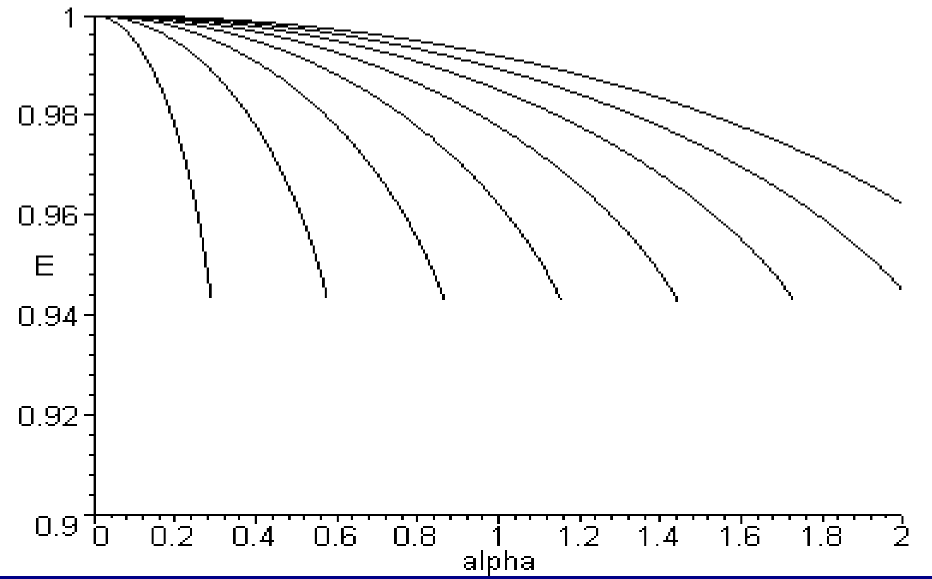
No stable orbits within $6M$

Semi-Classical Model

- Carry out a 'Bohr' quantisation $L=n\hbar$
- Find that energy is

$$\frac{E}{mc^2} = \frac{x-2}{\left(x(x-3)\right)^{1/2}}$$

$$x = \frac{n^2}{2\alpha^2}\left(1+\left(1-\frac{12\alpha^2}{n^2}\right)^{1/2}\right)$$



Dimensionless coupling

$$\alpha = \frac{Mm}{m_p^2}$$

Angular momentum of ground state increases with coupling

Quantum Bound States

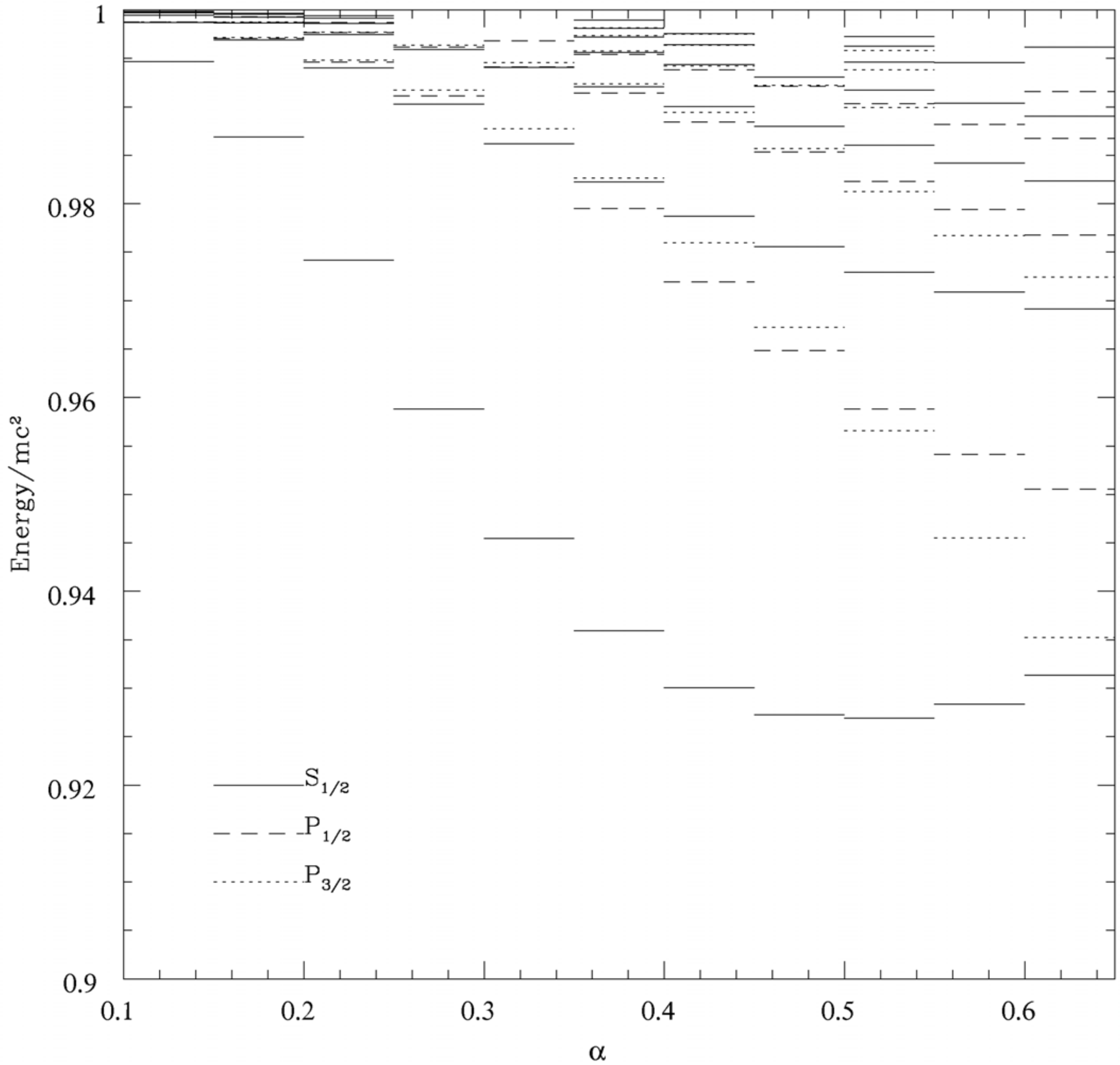
- Hamiltonian is *not* Hermitian

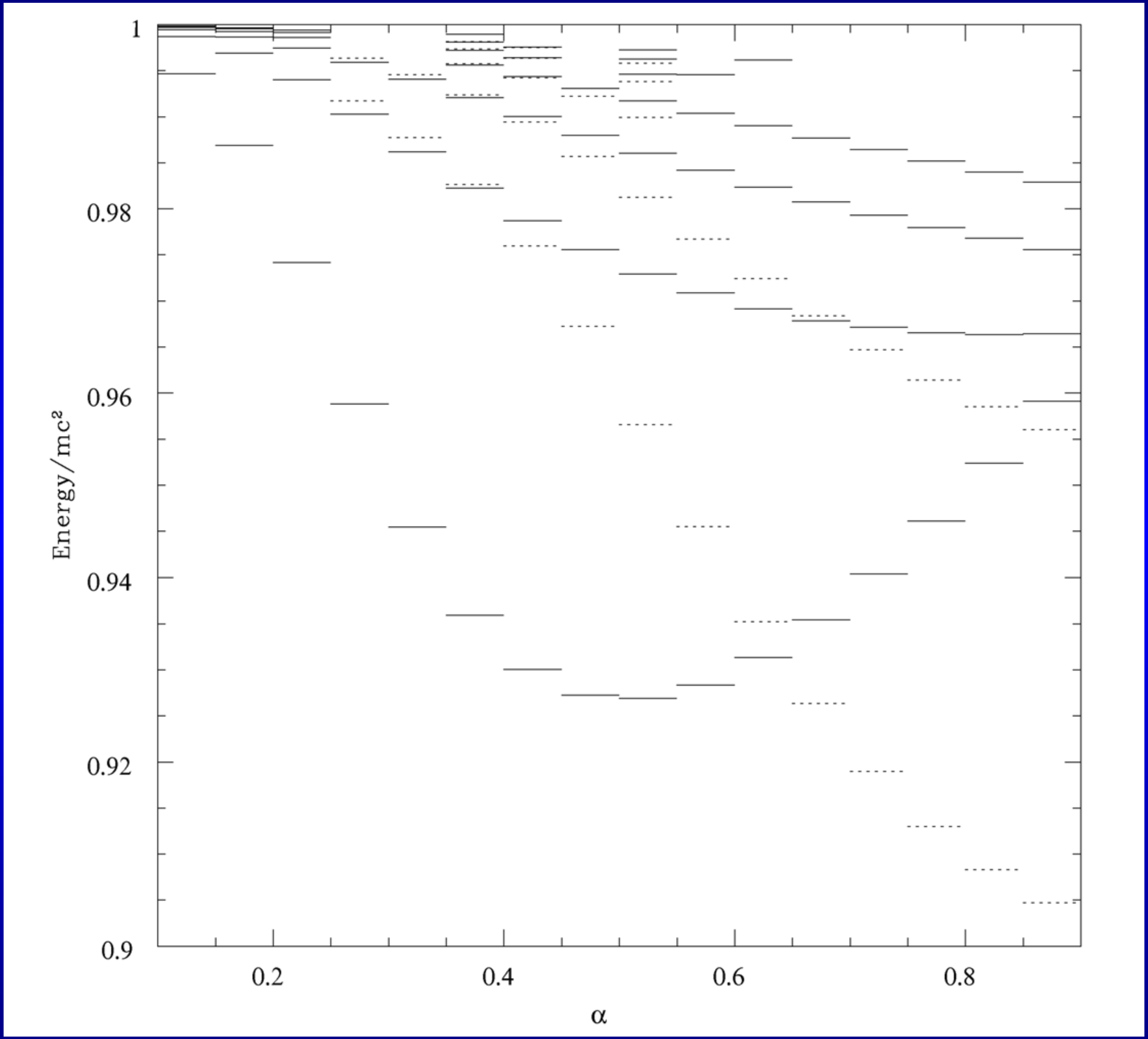
$$\hat{H}_{\mathrm{I}} - \hat{H}_{\mathrm{I}}^{\dagger} = -i\hbar(2GMr^3)^{1/2}\delta(\boldsymbol{x})$$

- Origin acts as a *sink*
- Dirac current is future-pointing, timelike
- Inside horizon, all current streamlines are swept onto the singularity
- Any normalizable states must have an imaginary component to E – resonance mode

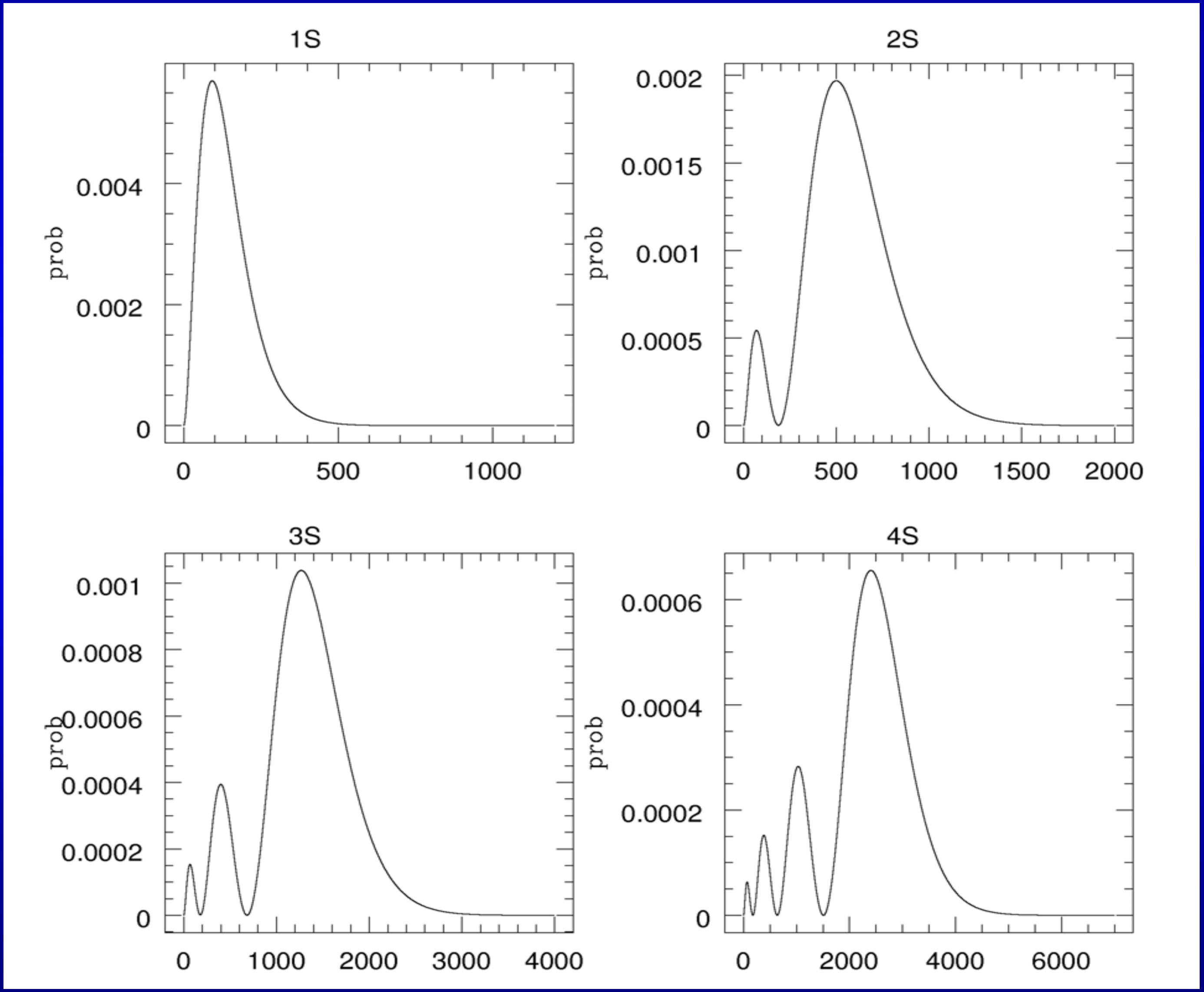
Method

- Start with regular solution at horizon and integrate outwards
- Simultaneously, integrate in from infinity, assuming exponential fall-off
- If both u_1 and u_2 meet at a fixed distance, have a solution
- Four terms to vary – real and imaginary energy and normalisation
- Four terms to set to zero – use a Newton-Raphson method

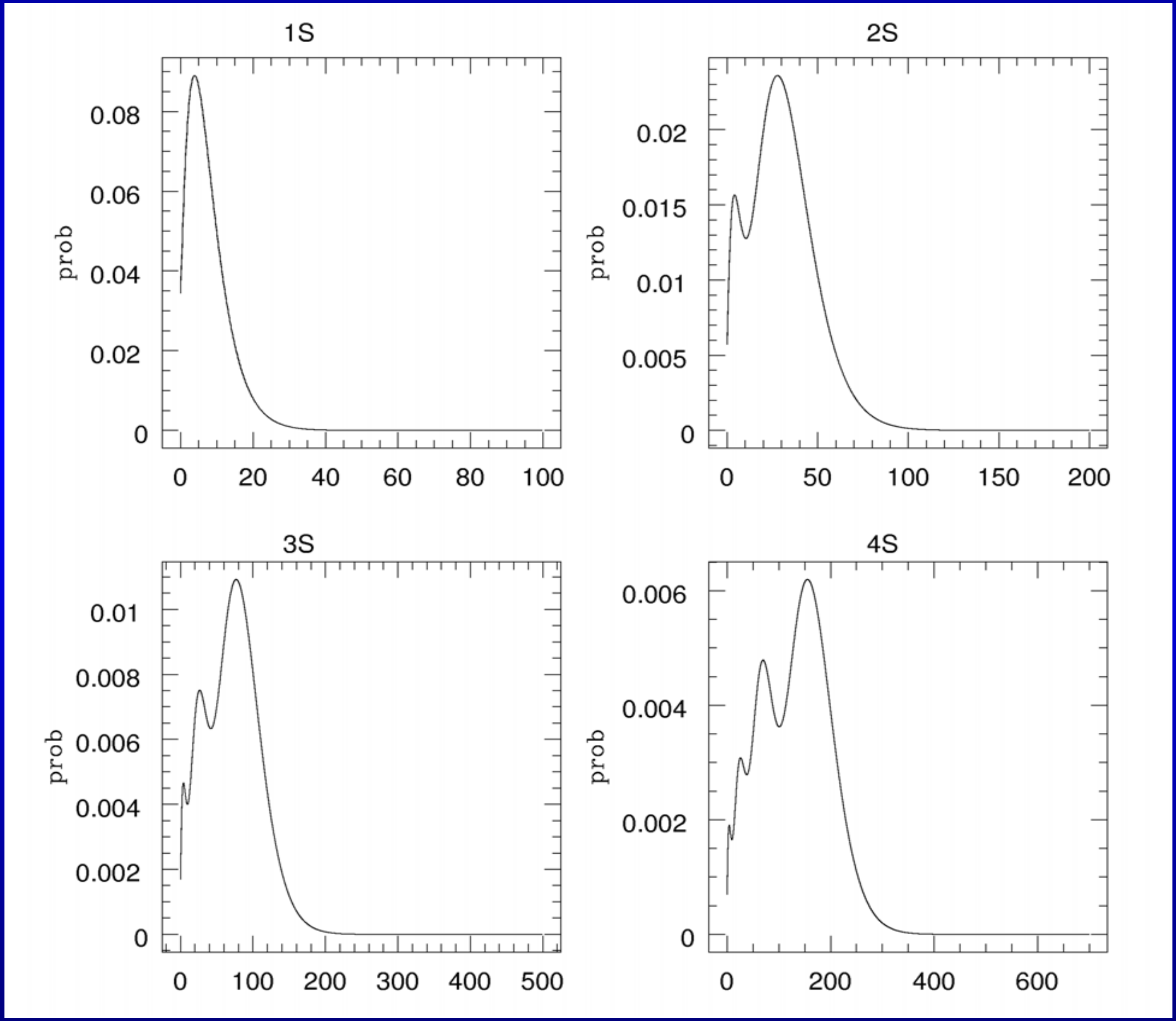

Energy/mc²
1
0.98
0.96
0.94
0.92
0.9
0.1
0.2
0.3
0.4
0.5
0.6
α
$S_{1/2}$
$P_{1/2}$
$P_{3/2}$


1
0.98
0.96
0.94
0.92
0.9
Energy/mc²
0.2
0.4
0.6
0.8
α

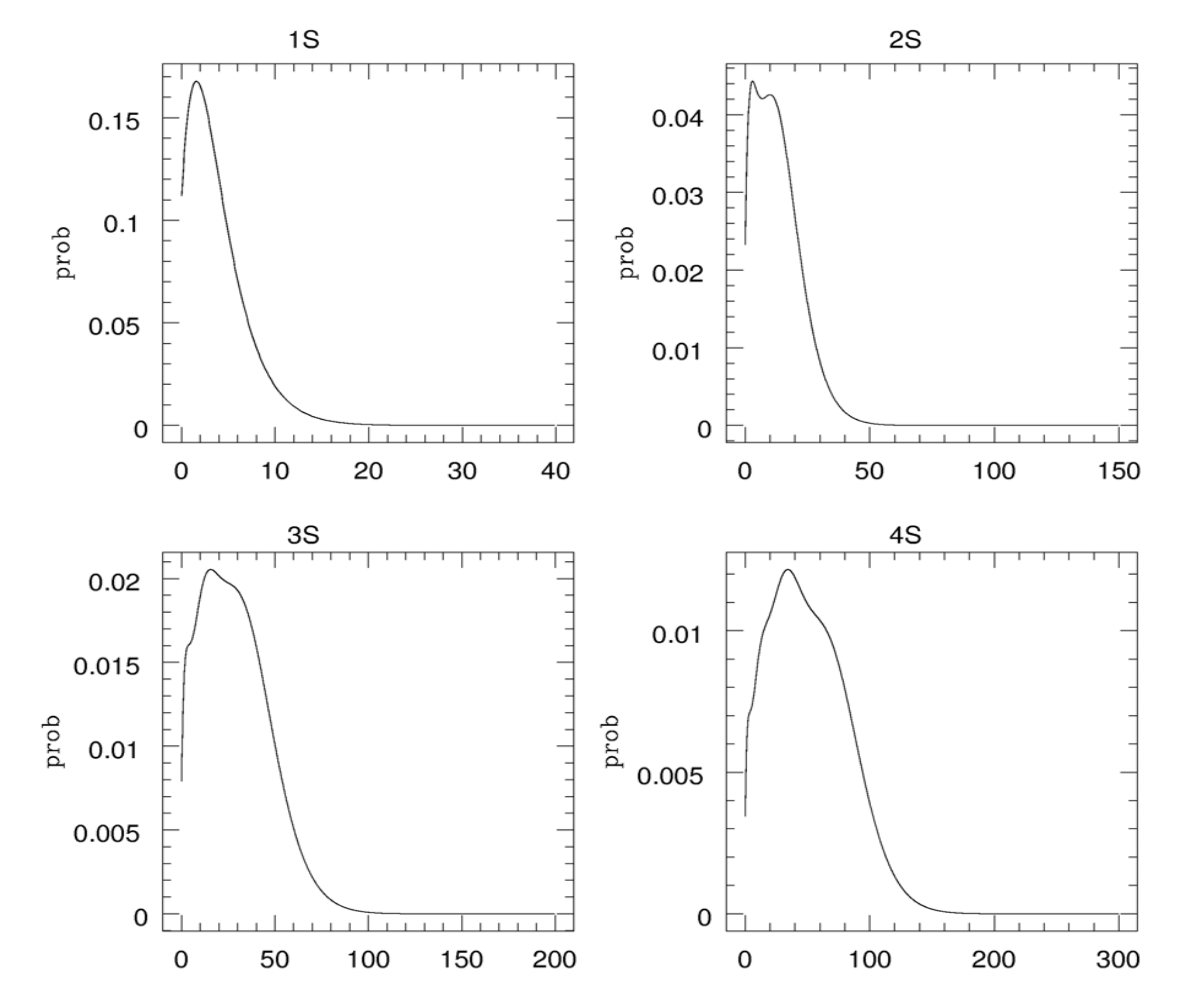
Probability Density α=0.1



Probability Density α=0.35



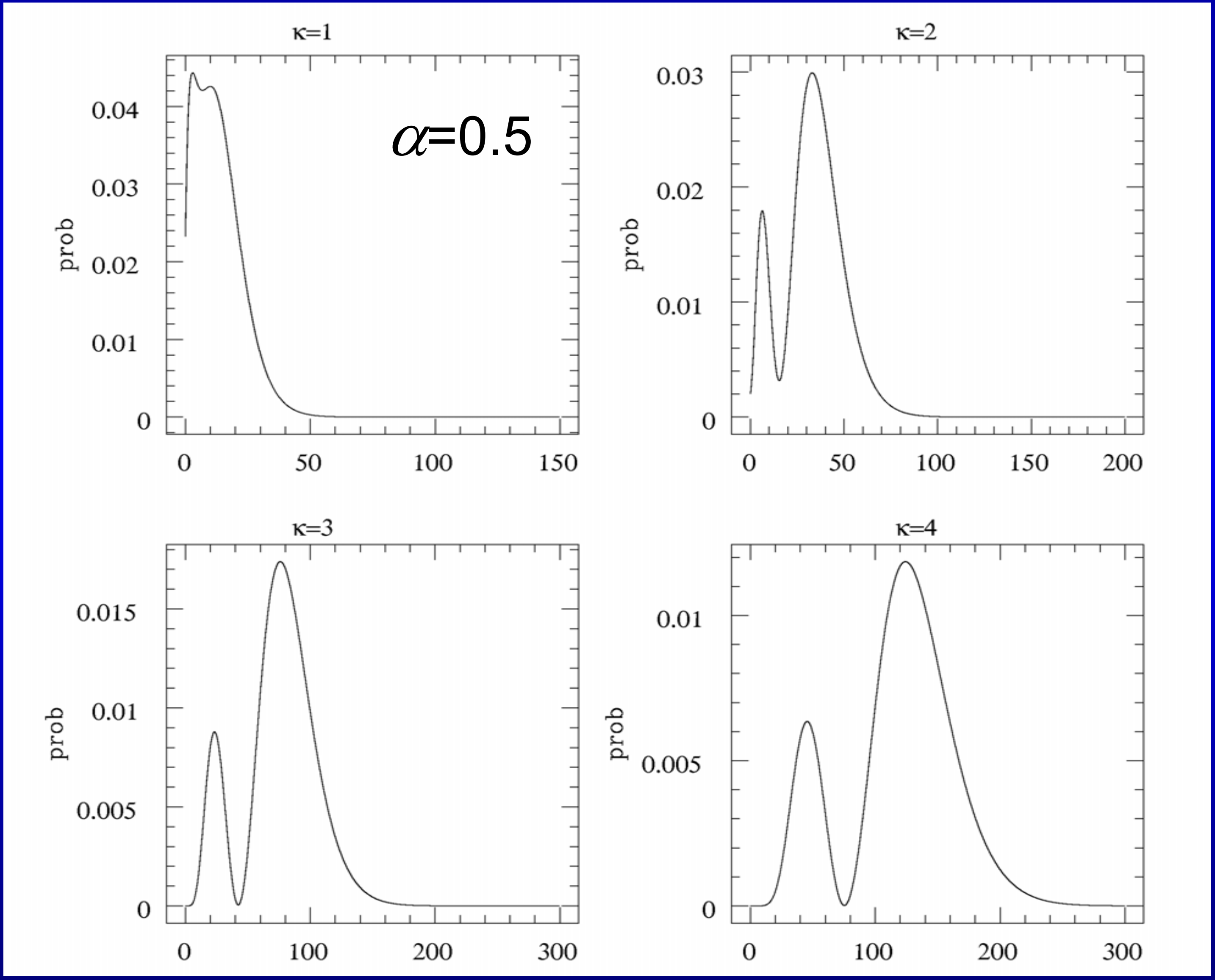
Probability Density $\alpha=0.5$



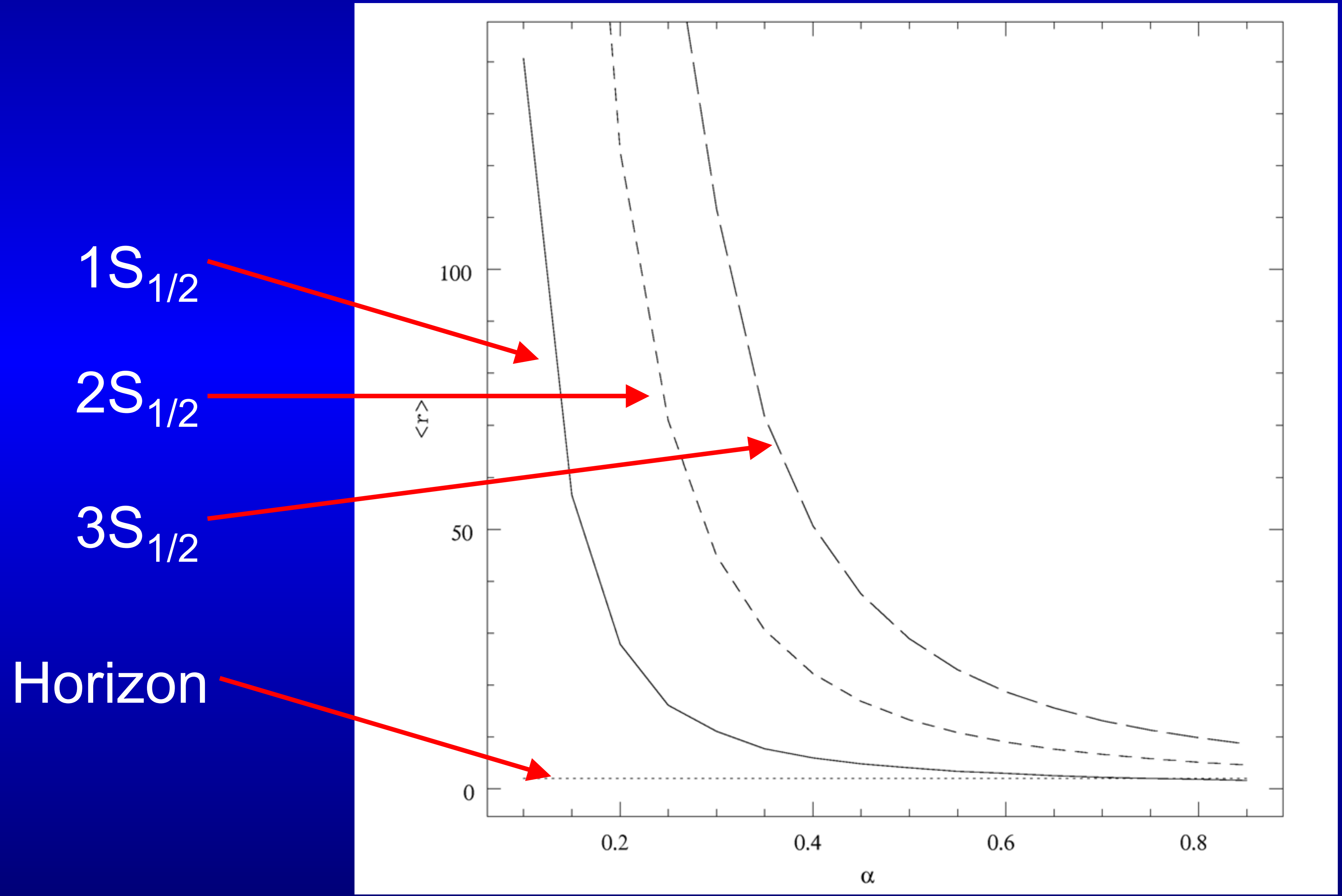
Variation with κ

First excited states with Increasing angular momentum

Further out, become Hydrogen-like

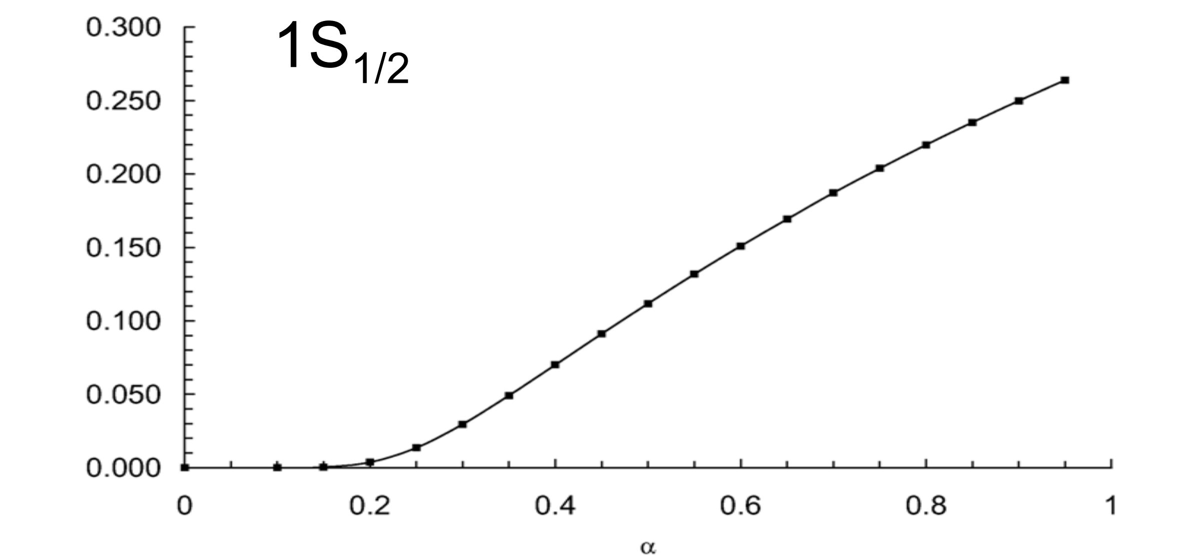


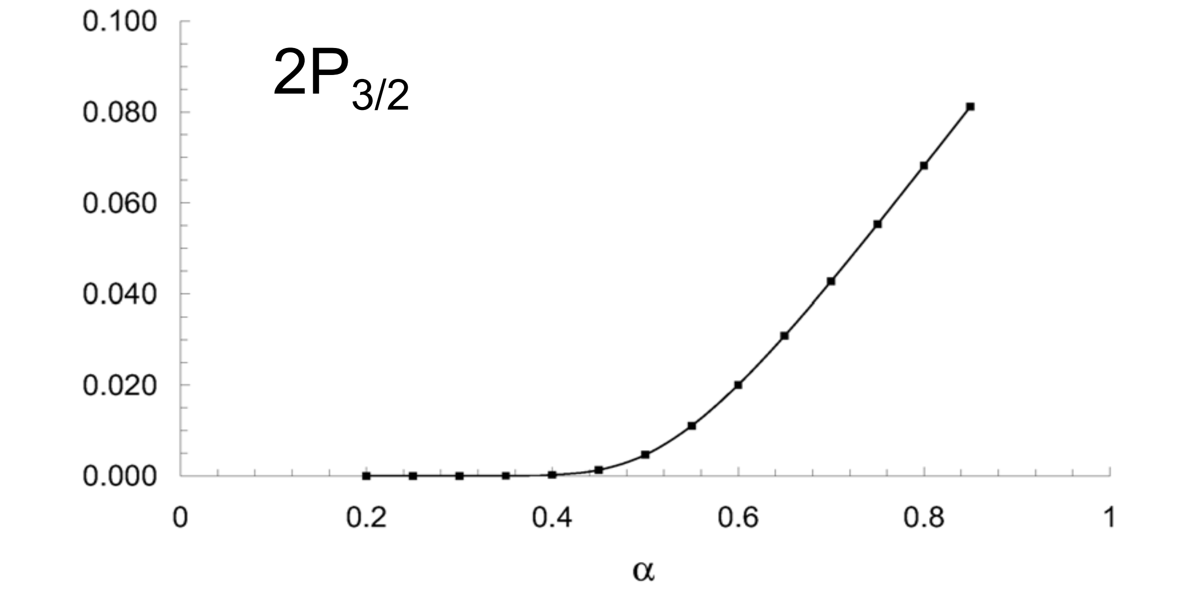
Expectation value $\langle r \rangle$



Imaginary Energy

Decay rate increases with coupling constant α and decreases with κ





Comments

- $\alpha \approx 1$ is the scale appropriate to primordial black holes
- Solar mass black holes have $\alpha \approx 1{,}000$
- Corresponding spectrum of *antiparticle* states also all have *decay* factors
- Decay rates can be extremely slow for orbits a long way from horizon
- Binding energies much larger than classical predictions

Emission

- Return to singular branch at horizon and compute radial currents

$$J_+ = \frac{A(\theta,\phi)}{4M} e^{-2\epsilon t} |r - 2M|^{8M\epsilon}$$

Outgoing

$$J_- = -\frac{A(\theta,\phi)}{4M} e^{-2\epsilon t} |r - 2M|^{8M\epsilon} e^{8\pi M E}$$

Ingoing

- Form ratio of outgoing to total current

$$\frac{J_+}{J_+ - J_-} = \frac{1}{e^{8\pi M E} + 1}$$

Fermi-Dirac distribution at the Hawking temperature

$$T = \frac{1}{8\pi M k_B}$$

Future Work

- Carry our scattering work to higher order
- Include radiation effects
- Partial wave analysis of cross-section
- Find bound state spectrum for larger coupling
- Repeat analysis for Kerr states
- Investigate QFT description of unstable states (quasi-normal modes)
- Contribution to Hawking radiation?