Physical Applications of Geometric Algebra

Examples 2

1. With B a bivector and M and N general multivectors, prove that

$$B\times(MN) = (B\times M)N + M(B\times N).$$

Hence show that

$$B\times(a\wedge A_r) = (B\cdot a)\wedge A_r + a\wedge(B\times A_r).$$

Use this result to establish that the operation of commuting with a bivector is grade preserving.

2. Given rotors $R_1 = \mathrm{e}^{-\epsilon B_1/2}$ and $R_2 = \mathrm{e}^{-\epsilon B_2/2}$, the product is defined by

$$R_3 = \mathrm{e}^{-\epsilon B_3/2} = R_2 R_1.$$

Prove that $B_3 = B_1 + B_2 - \frac{1}{2}\epsilon B_2\times B_1 + \mathrm{O}(\epsilon^2)$.

3. Prove the Jacobi identity

$$(A\times B)\times C + (C\times A)\times B + (B\times C)\times A = 0.$$

4. Prove that for an orthonormal bivector basis, the structure constants C^i_{jk} of the Lie algebra of the 3-d rotation group are simply $-\epsilon_{ijk}$.

5. The bivector generators of the Unitary group are

$$\begin{array}{rcll} E_{ij} & = & e_i e_j + f_i f_j & (i < j = 1\ldots n) \\ F_{ij} & = & e_i f_j - f_i e_j & (i < j = 1\ldots n) \\ J_i & = & e_i f_i. & \end{array}$$

Establish that this algebra is closed under the commutator product. (A few examples should do.)

6. For a general Hamiltonian system with coordinates $\{x_i, p_i\}$, prove that the rescaling $x_i \mapsto \alpha x_i$, $p_i \mapsto p_i/\alpha$ is a canonical transformation. Given that J is dimensionless and H has units of energy, what are the units of the vector x?

7. Prove that the reflection $a \mapsto -nan^{-1}$ has determinant -1.

8. Prove that $\det(\mathsf{f}) = \det(\bar{\mathsf{f}})$.

9. Given a linear function $\mathsf{f}(a)$ and an orthonormal frame $\{e_k\}$, one can form the matrix

$$\mathsf{f}_{ij} = e_i\cdot\mathsf{f}(e_j).$$

Prove that the matrix for $\bar{\mathsf{f}}(a)$ is simply the transpose matrix. Also, prove that the matrix for the product transformation $\mathsf{h} = \mathsf{fg}$ is found by matrix multiplication of f_{ij} and g_{ij}.

10. In 2-d demonstrate explicitly that the frame-free definition of the determinant, $\mathsf{f}(I) = \det(\mathsf{f})I$ agrees with the matrix definition, with the matrix given by the formula in the preceding question. Now introduce the vectors $a_k = \mathsf{f}(e_k)$. Show that

$$\det(\mathsf{f}) = a_1 \wedge a_2 \wedge \cdots \wedge a_n \, I^{-1}.$$

What is the relation of the components of the $\{a_k\}$ vectors in the $\{e_k\}$ frame to the f_{ij} matrix? Hence prove that the determinant of a matrix changes sign when any two columns are interchanged. Also prove that any multiple of a column can be added to a different column without changing the value of the determinant.

11. What is the role of the 'Abelian' factor K in the bivector algebra of the general linear group? (Hint, consider the action of $\exp(\alpha K/2)$ on a general null vector.) What group is obtained when the subgroup generated by K is factored out?

12. Given points x_0, x_1 and x_2 on a 2-d surface, we define the function

$$m(x) = M_0 + \sum_{i=1}^{2} (x - x_0) \cdot e^i \, (M_i - M_0)$$

where $e_1 = x_1 - x_0$ and $e_2 = x_2 - x_0$. Prove that

$$\oint dS \, m(x) = \tfrac{1}{2} e_2 \wedge e_1 [e^1 (M_1 - M_0) + e^2 (M_2 - M_0)]$$

where the integral is taken in a positive sense around the triangle defined by x_0, x_1 and x_2. Does the result depend on the signature of the space? Can you see how this generalises?

13. In a 2-d spacetime with basis vectors γ_0 and γ_1, $\gamma_0^2 = -\gamma_1^2 = 1$, show that the curve satisfying $\langle x\gamma_0 x\gamma_0 \rangle = 1$ is a circle. The 'normal' to this curve is defined by $\nabla \langle x\gamma_0 x\gamma_0 \rangle$. Prove that this is $2\gamma_0 x\gamma_0$. Plot the direction of this normal vector around the curve. What do you notice about its direction when x is null?

14. Suppose that the spacetime bivector $\hat{B}$ satisfies $\hat{B}^2 = 1$. By writing $\hat{B} = \boldsymbol{a} + I\boldsymbol{b}$ in the γ_0-frame, show that we can write

$$\hat{B} = \cosh(u)\hat{\boldsymbol{a}} + \sinh(u) I \hat{\boldsymbol{b}} = \mathrm{e}^{uI\hat{\boldsymbol{b}}\hat{\boldsymbol{a}}} \, \hat{\boldsymbol{a}},$$

where $\hat{\boldsymbol{a}}^2 = \hat{\boldsymbol{b}}^2 = 1$. Hence explain why we can write $\hat{B} = R\boldsymbol{\sigma}_3\tilde{R}$. By considering the null vectors $\gamma_0 \pm \gamma_3$, prove that we can always find two null vectors satisfying

$$\hat{B} \cdot n_\pm = \pm n_\pm.$$

15. The logarithm of a multivector can be defined by

$$\ln X = 2[H + \frac{H^3}{3} + \frac{H^5}{5} + \cdots] \qquad \text{where} \quad H = \frac{X-1}{X+1}$$

Prove that $R = (\gamma_0 + \gamma_1 - \gamma_2)\gamma_2$ is a rotor and find its bivector generator. Can $-R$ be written as the exponential of a bivector?

16. *Desargues' theorem*

This is *non-examinable*, but can be used as further practice in some of the algebraic manipulations one can perform in 3-d.

The result we need to prove is

$$JJ'\langle a\wedge a'\, b\wedge b'\, c\wedge c'\rangle = \langle A\times A'\, B\times B'\, C\times C'\rangle,$$

following the notation of the Handout 6, Section 2.3. One way to proceed is as follows. First prove the two lemmas

$$(c\wedge a)\times(b\wedge c) = a\wedge b\wedge c\, c$$

and

$$\langle AB(c\wedge c')\rangle = (A\wedge c')\cdot(B\wedge c) - (A\wedge c)\cdot(B\wedge c').$$

(With A and B bivectors). Now introduce the dual vectors $A^* = AI$, $A^{*\prime} = A'I$ *etc.* and prove that

$$\begin{aligned}\langle A\times A'\, B\times B'\, C\times C'\rangle =& (A^*\wedge A^{*\prime}\wedge C^*)(B^*\wedge B^{*\prime}\wedge C^{*\prime}) \\ &- (A^*\wedge A^{*\prime}\wedge C^{*\prime})(B^*\wedge B^{*\prime}\wedge C^*).\end{aligned}$$

Next establish that

$$A^*\wedge B^* = Jc, \quad etc.$$

with the same holding for the primed quantities. From this, prove

$$\langle A\times A'\, B\times B'\, C\times C'\rangle = JJ'[A\wedge b'\, B'\wedge a - A'\wedge b\, B\wedge a']$$

Now combine the preceding results to prove the theorem.