# Physical Applications of Geometric Algebra

## Handout 4

## Groups, Bivectors and Hamiltonian Mechanics

Lie groups are fundamental to much of our understanding of theoretical physics. They are groups of continuous transformations, such as rotations and phase transformations, and have a very natural expression in geometric algebra in terms of rotors. The elements of a Lie group do not form a linear space (you cannot add two rotations) instead they can be viewed as points on a curved surface called the *group manifold*. There is a natural linear space associated with a Lie group, however, called its Lie algebra. In geometric algebra the role of this is played by the bivector algebra. In studying Lie groups in terms of rotors we are giving a non-standard presentation, but this will enable us to quickly reach many of the more significant topics. In the course of this analysis we will uncover how more general complex structures fit into a real geometric algebra. This in turn paves the way for an alternative treatment of Hamiltonian mechanics, which turns out to be extremely powerful.

# 1 General Properties of Rotors

So far we have introduced rotors as the product of two unit vectors, and written

$$R = nm = \mathrm{e}^{-\hat{B}\theta/2}, \quad \text{where} \quad \cos(\theta/2) = n\cdot m, \quad \hat{B} = m\wedge n/\sin(\theta/2). \tag{1.1}$$

But the result of combining two rotations is a third rotation — they form a group — so we must first establish the group product rule for rotors. Suppose that $R_1$ and $R_2$ generate two distinct rotations. What does the product rotation look like? We find

$$a \mapsto R_2(R_1 a \tilde{R}_1)\tilde{R}_2 = R_2 R_1 a \tilde{R}_1 \tilde{R}_2. \tag{1.2}$$

We therefore define the product rotor

$$R = R_2 R_1 \tag{1.3}$$

so that the composite rotation is described by $Ra\tilde{R}$, as usual. The product $R$ is a new rotor, and in general it will consist of geometric products of an even number of unit vectors,

$$R = kl \cdots mn. \tag{1.4}$$

We will adopt this as our *definition* of a rotor. The reversed rotor is

$$\tilde{R} = nm \cdots lk. \tag{1.5}$$

The result of the map $a \mapsto Ra\tilde{R}$ returns a vector for any vector $a$, since

$$Ra\tilde{R} = kl \cdots [m(nan)m] \cdots lk \tag{1.6}$$

and each successive sandwich between a vector returns a new vector.

We can immediately establish the normalisation condition

$$R\tilde{R} = kl \cdots mnnm \cdots lk = 1 = \tilde{R}R. \tag{1.7}$$

In Euclidean spaces, where every vector has a positive square, this normalisation is automatic. In mixed signature spaces, like Minkowski spacetime, unit vectors can have $n^2 = \pm 1$. In this case the condition $R\tilde{R} = 1$ is taken as a further condition satisfied by a rotor. In the case where $R$ is the product of two rotors we can easily confirm that

$$R\tilde{R} = R_2 R_1 (R_2 R_1)\tilde{\ } = R_2 R_1 \tilde{R}_1 \tilde{R}_2 = 1. \tag{1.8}$$

The set of rotors therefore form a *group*, called a rotor group. This is similar to the group of rotation matrices, but not quite the same.

## 1.1 Multivector Transformations

A general multivector can be decomposed into a sum of blades, and each blade can be written as a product of orthogonal vectors. Suppose that the blade $A_r$ is written

$$A_r = a_1 a_2 \cdots a_r. \tag{1.9}$$

If we rotate each of the generating vectors to $a_i' = Ra_i\tilde{R}$ then resulting blade is

$$\begin{aligned} A_r' &= a_1' a_2' \cdots a_r' \\ &= Ra_1\tilde{R}\, Ra_2\tilde{R} \cdots Ra_r\tilde{R} \\ &= Ra_1 a_2 \cdots a_r \tilde{R}. \\ &= RA_r\tilde{R}. \end{aligned} \tag{1.10}$$

We recover precisely the same law as for vectors! All multivectors share the same transformation law regardless of grade when each component vector is rotated. This is one reason why the rotor formulation is so powerful. The alternative, tensor form would require an extra matrix for each additional vector.

### 1.2 Spin-1/2

Suppose now that we have an initial rotor $R$, and that this is composed with a second rotor $R_\theta = \exp(-\hat{B}\theta/2)$, where $\hat{B}^2 = -1$. The resulting rotor is

$$R' = R_\theta R = \mathrm{e}^{-\hat{B}\theta/2}\, R. \tag{1.11}$$

Now suppose that we start to increase $\theta$ from 0 through to $2\pi$. $\theta = 2\pi$ corresponds to a $360^{\circ}$ rotation, *i.e.* the identity. But under this we see that $R$ transforms as

$$R \mapsto R' = \mathrm{e}^{-\hat{B}\pi} R = (\cos\pi - \hat{B}\sin\pi) R = -R. \tag{1.12}$$

So rotors change sign under $360^{\circ}$ rotations. This is precisely the property of spin-1/2 fermions in quantum theory. This double-valued behaviour under rotations is often viewed as being something rather mysterious and quantum mechanical, but we have not said a word about quantum mechanics anywhere in the preceding derivation! You might suspect that no 'classical' phenomena could see the distinction between $R$ and $-R$ since both rotors encode the same rotation. But for systems of linked rotations one can see this distinction. This is the explanation of the $4\pi$ symmetry observed when rotating an arm holding a tray.

## 2 Lie Groups

We have already seen that rotors form a group, in the same way that rotations do. These groups are *continuous* and have an infinite number of elements. However, like vector spaces, the elements in the group can usually be written in terms of a finite number of parameters, such as the Euler angles for 3-d rotations. Groups of this type are called *Lie groups*, after the mathematician Sophus Lie.

While the group of rotors looks like it has a sort of vector space structure, there is an important subtlety: the space of rotors is not flat - it is a curved surface called a *group manifold*. For example, in 2-d all rotors are phase factors, and the group manifold is the unit circle. Every point on the circle corresponds to a distinct rotor.

What about in 3D? All rotors are built from the scalars and three bivectors. The only condition they have to satisfy is that $R\tilde{R} = 1$. Suppose that we write

$$R = x_0 + x_1 I e_1 + x_2 I e_2 + x_3 I e_3. \tag{2.1}$$

Then

$$R\tilde{R} = {x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2 = 1. \tag{2.2}$$

This defines a unit vector in the 4-d space spanned by $\{x_0, x_i\}$. The group manifold is therefore the set of unit vectors in 4-d space. This is called a 3-sphere; it is the 4-d analog of the surface of a ball in 3-d. In higher dimensions the rotor group manifolds get increasingly more complicated.

Since all rotations are given by the double-sided formula $Ra\tilde{R}$, both $R$ and $-R$ correspond to the same rotation. The group manifold for rotations, rather than for the rotors themselves, is therefore a bit more complicated. It involves taking a 3-sphere, or its higher dimensional analog, and projectively identifying opposite points.

This might all seem a bit esoteric, but it has many applications. For example, if the orientation of a rigid body is described by a rotor, the configuration space for the dynamics of the rigid body is a 3-sphere. This is important when looking for best-fit rotations in computer vision, or extrapolating between two rotations to find their mid-point. The group manifold is also the appropriate setting for a Lagrangian treatment. This has implications for constructing conjugate momenta, which are essential for the transition to a quantum theory. Applications of this include the rotational energy levels of molecules, many of which can be viewed as rigid bodies.

## 2.1 Aside — Formal Definitions

*Non-examinable*

The preceding considerations enable us to give an abstract definition of a Lie group. This is the one employed by mathematicians when discussing their general properties. The idea is to define a Lie group as a manifold, $\mathcal{M}$, together with a product $\phi(x, y)$. Points on the manifold can be labeled with vectors $\{x, y\}$, which are often viewed as lying in a higher-dimensional embedding space (as with the 3-sphere). The product $\phi$ takes as its argument two points in the manifold, and returns a third. This encodes the group product. The final set of conditions apply to $\phi(x, y)$ and ensure that the product has the correct group properties. These are

1. *Closure.* $\phi(x, y) \in \mathcal{M} \quad \forall x, y \in \mathcal{M}$.
2. *Identity.* There exists an element $e \in \mathcal{M}$ such that $\phi(e, x) = \phi(x, e) = x$, $\forall x \in \mathcal{M}$.
3. *Inverse.* For every element $x \in \mathcal{M}$ there exists a unique element $\bar{x}$ such that $\phi(x, \bar{x}) = \phi(\bar{x}, x) = e$.
4. *Associativity.* $\phi[\phi(x, y), z] = \phi[x, \phi(y, z)], \quad \forall x, y, z \in \mathcal{M}$.

Any manifold with a product defined on it with the preceding properties is called a Lie group manifold. Many of the group properties can be uncovered by examining the properties near the identity element. The product then induces a *Lie bracket* structure on elements of the tangent space at the identity. The tangent space is a linear space and the vectors in this space, together with their bracket, form a Lie algebra. Much of this is too abstract for our purposes, however, and we will adopt a different approach to uncovering the properties of a Lie algebra.

# 3 Lie Algebras and the Bivector Algebra

So far we have seen that simple rotations in the $\hat{B}$ plane are described by the rotor $R$ which can be written either as the exponential of a bivector, or as a product of two unit vectors. We have also established that a general rotor is the product of an even number of unit vectors. The natural question now is, can any rotor be written as the exponential of a bivector?

## 3.1 Families of Curves

To answer this question, choose a rotor $R_1$ and imagine a family of rotors $R(\lambda)$ for which

$$R(0) = 1, \qquad R(1) = R_1. \tag{3.1}$$

This implies that the rotor can be obtained from the identity by a continuous set of transformations. There are many possible ways to connect $R_1$ to the identity, but there is one unique path which has the additional property that

$$R(\lambda + \mu) = R(\lambda)R(\mu). \tag{3.2}$$

These form a one-parameter subgroup of the rotor group, and the interpretation in terms of 3-d rotations is clear — it is the subgroup of all rotations in a fixed plane.

Now introduce the family of vectors $a(\lambda) = Ra_0\tilde{R}$, where $a_0$ is some fixed initial vector. Differentiating this expression, and recalling that

$$\frac{d}{d\lambda}(R\tilde{R}) = 0 = R'\tilde{R} + R\tilde{R}' \tag{3.3}$$

where the dash denotes differentiation with respect to $\lambda$, we see that

$$\frac{d}{d\lambda}a(\lambda) = R'a_0\tilde{R} + Ra_0\tilde{R}' = (R'\tilde{R})a(\lambda) - a(\lambda)(R'\tilde{R}) \tag{3.4}$$

The quantity $R'\tilde{R}$ reverses to minus itself, so can only contain terms of grade 2, 6, 10 *etc.* But the commutator of $R'\tilde{R}$ with any vector must return another vector, otherwise the derivative of $a(\lambda)$ would grow non-vector terms. If follows that $R'\tilde{R}$ can only contain a bivector component. We can therefore write

$$\frac{d}{d\lambda}R(\lambda) = -\tfrac{1}{2}B(\lambda)R(\lambda). \tag{3.5}$$

This is true for any parameterised set of rotors. We now restrict ourselves to the curve defined by Eq. (3.2). For this curve we find that

$$\begin{aligned}\frac{d}{d\lambda}R(\lambda+\mu) &= -\tfrac{1}{2}B(\lambda+\mu)R(\lambda+\mu) = -\tfrac{1}{2}B(\lambda+\mu)R(\lambda)R(\mu) \\ &= \frac{d}{d\lambda}[R(\lambda)R(\mu)] = -\tfrac{1}{2}B(\lambda)R(\lambda)R(\mu).\end{aligned} \tag{3.6}$$

It follows that $B$ is constant along this curve. We can therefore integrate Eq. (3.5) to get

$$R(\lambda) = \mathrm{e}^{-\lambda B/2}, \tag{3.7}$$

and setting $\lambda = 1$ expresses $R_1$ as the exponential of a bivector. For Euclidean space it turns out that all rotors lie on a path described by Eq. (3.2) and so can be written as the exponential of a bivector. This is not the case in mixed signature spaces, but it does turn out that every rotor can be written as

$$R(\lambda) = \pm\,\mathrm{e}^{-\lambda B/2}. \tag{3.8}$$

Rotors suitably 'close' to the identity can always be written as the exponential of a bivector.

It is instructive to establish the inverse result, that the exponential of a bivector always returns a rotor. To see this, return to the one-parameter the family of vectors

$$a(\lambda) = \mathrm{e}^{-\lambda B/2}\, a_0\, \mathrm{e}^{\lambda B/2}. \tag{3.9}$$

To establish that these are the result of rotations we need only establish that $a$ is a vector, as the remaining properties follow automatically. Differentiating with respect to $\lambda$, we find that

$$\frac{da}{d\lambda} = \mathrm{e}^{-\lambda B/2}\, a_0 \cdot B\, \mathrm{e}^{\lambda B/2} \tag{3.10}$$

$$\frac{d^2a}{d\lambda^2} = \mathrm{e}^{-\lambda B/2}\, (a_0 \cdot B) \cdot B\, \mathrm{e}^{-\lambda B/2}, \quad etc. \tag{3.11}$$

For every extra derivative we pick up a further factor of $B$. But this operation of dotting a vector with $B$ is *grade-preserving*. Thus every term in the Taylor series of $a(\lambda)$ is a vector, and the overall operation is grade preserving, as it must be. We have also proved the following useful Taylor expansion

$$\mathrm{e}^{-B/2}\, a\, \mathrm{e}^{B/2} = a + a \cdot B + \frac{1}{2!}(a \cdot B) \cdot B + \cdots \tag{3.12}$$

## 3.2 The Bivector Algebra

The operation of commuting a multivector with a bivector is always grade-preserving (exercise). In particular, the commutator of a bivector with a second bivector produces a third bivector. *The space of bivectors is closed under the commutator product.* This closed algebra is called a *Lie algebra*, and encodes most of the properties of the associated Lie group of rotors. The group is formed from the algebra by the act of exponentiation.

The commutator of two bivectors expresses the fact that rotations do not commute. If we apply a pair of rotations, and then perform the back rotations in the incorrect order, the result is a new rotation:

$$Ra\tilde{R} = \tilde{R}_2\tilde{R}_1(R_2R_1a\tilde{R}_1\tilde{R}_2)R_1R_2. \tag{3.13}$$

The resulting rotor is given by

$$R = \mathrm{e}^{-B/2} = \mathrm{e}^{B_2/2}\,\mathrm{e}^{B_1/2}\,\mathrm{e}^{-B_2/2}\,\mathrm{e}^{-B_1/2} \tag{3.14}$$

Expanding the exponentials we find (exercise) that

$$B = B_1 \times B_2 + \text{higher order terms} \tag{3.15}$$

This result is known as the Baker-Campbell-Hausdorff formula. The formula guarantees that all rotors sufficiently close to the identity can be written as the exponential of a bivector.

## 3.3 The Jacobi Identity

In the abstract theory of Lie groups, the Lie algebra elements are acted on by the Lie bracket, which is antisymmetric and satisfies the *Jacobi identity*. For our purposes the Lie bracket is just the commutator product for bivectors. The Jacobi identity for three bivectors $A, B, C$ is then

$$(A\times B)\times C + (C\times A)\times B + (B\times C)\times A = 0. \tag{3.16}$$

The proof is simple and just involves expanding out each product in terms of geometric products. There is nothing special about the grade of the multivectors in this proof, so the identity is true for any set of three multivectors. This has some useful consequences. For example, given vectors $a$ and $b$, and a bivector $B$, we have

$$(a\wedge b)\times B = (a\cdot B)\wedge b - (b\cdot B)\wedge a. \tag{3.17}$$

### 3.4 The Structure Constants

Suppose now that we introduce a basis set of bivectors $\{B_i\}$. The commutator of any pair of these returns a third bivector, which can also be expanded in terms of this basis set. We can therefore write

$$B_j \times B_k = C^i_{jk} B_i \tag{3.18}$$

The set $C^i_{jk}$ are called the *structure constants* of the Lie algebra. They provide one of the most compact encodings of the group properties, since knowledge of the full bracket structure is sufficient to recover most of the properties of the group. The structure constants also provide a route through to solving the problem of classifying all possible Lie algebras. This solution of this problem was an important achievement and was finally completed by the mathematician E. Cartan.

## 4 Complex Structures and Unitary Groups

So far we have only dealt with the properties of rotation groups, but it will turn out that this is sufficient for us to uncover the properties of all Lie groups. We can start to see how this works by studying how complex groups fit into our *real* geometric algebra.

### 4.1 Complex Structures

We have seen that the geometric algebra of 2-d space naturally gives rise to complex numbers, with one axis $e_1$ singled out as the real axis. This suggests that a $n$-d complex space could have a natural realisation in a $2n$-dimensional real space. Suppose that the set $\{e_i\}$ form a basis for an $n$-dimensional space. We expand this to a $2n$-dimensional space by introducing a second set of vectors $\{f_i\}$ with the properties

$$f_i \cdot f_j = e_i \cdot e_j = \delta_{ij} \qquad e_i \cdot f_j = 0, \quad \forall i, j. \tag{4.1}$$

A complex structure is introduced through the *doubling bivector*

$$J = e_1 f_1 + e_2 f_2 + \cdots + e_n f_n = e_i \wedge f_i. \tag{4.2}$$

This is a sum of $n$ commuting blades, each playing the role of an imaginary in its own plane. The doubling bivector satisfies

$$\begin{aligned} J \cdot f_i &= (e_j \wedge f_j) \cdot f_i = e_j \delta_{ij} = e_i \\ J \cdot e_i &= (e_j \wedge f_j) \cdot e_i = -f_i, \end{aligned} \tag{4.3}$$

which illustrates the role of $J$ in relating one half of the vector space to the other. It follows that

$$J\cdot(J\cdot e_i) = -J\cdot f_i = -e_i, \quad \text{and} \quad J\cdot(J\cdot f_i) = J\cdot e_i = -f_i, \tag{4.4}$$

and hence that

$$J\cdot(J\cdot a) = (a\cdot J)\cdot J = -a \quad \forall a \tag{4.5}$$

where $a$ is any vector in the $2n$-dimensional space. One can now start to see how $J$ can generate a complex structure. For example, the analog of phase rotations must be generated by the bivector $J$, which describes a series of coupled rotations in each of the $e_i \wedge f_i$ planes. A Taylor expansion then yields the expected form

$$\begin{aligned}
\mathrm{e}^{-J\phi/2}\, a\, \mathrm{e}^{J\phi/2} &= a + \phi\, a\cdot J + \frac{\phi^2}{2!}(a\cdot J)\cdot J \cdots \\
&= (1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \cdots)a + (\phi - \frac{\phi^3}{3} - \cdots)a\cdot J \\
&= \cos\phi\, a + \sin\phi\, a\cdot J.
\end{aligned} \tag{4.6}$$

## 4.2 Hermitian Norms and Unitary Groups

The key to understanding unitary groups is the Hermitian inner product. Suppose that we have a pair of complex vectors $Z$ and $W$ with components

$$Z_i = x_i + iy_i, \quad \text{and} \quad W_i = u_i + iv_i. \tag{4.7}$$

The Hermitian inner product, familiar from quantum mechanics, is

$$\langle W|Z\rangle = W_i^* Z_i = u_i x_i + v_i y_i + i(u_i y_i - v_i x_i). \tag{4.8}$$

We seek the analog of this in our $2n$-dimensional space. We start by introducing vectors

$$x = x_i e_i + y_i f_i, \quad \text{and} \quad w = u_i e_i + v_i f_i. \tag{4.9}$$

The real part of the inner product is then simply $x\cdot w$. The imaginary component is

$$\begin{aligned}
w\cdot e_i\, x\cdot f_i - w\cdot f_i\, x\cdot e_i &= (w\cdot e_i\, x - x\cdot e_i\, w)\cdot f_i \\
&= [(x\wedge w)\cdot e_i]\cdot f_i \\
&= (x\wedge w)\cdot(e_i\wedge f_i) \\
&= (x\wedge w)\cdot J.
\end{aligned} \tag{4.10}$$

This nicely brings out the antisymmetry of the term. We can now write the Hermitian inner product compactly as

$$\langle a|b\rangle = a\cdot b - i\,(a\wedge b)\cdot J. \tag{4.11}$$

This maps from our $2n$-dimensional space onto the complex numbers. It is also immediately clear why $\langle a|a\rangle$ is real.

The next step is to establish the invariance group of this inner product. The group must leave the inner product invariant, so is built from reflections and rotations. But the group must also leave the antisymmetric term invariant as well. It is not hard to establish that this rules out reflections. We are therefore left with rotations, for which we require that

$$(a'\wedge b')\cdot J = (a\wedge b)\cdot J \tag{4.12}$$

where $a' = Ra\tilde{R}$, $b' = Rb\tilde{R}$. We find that

$$\begin{aligned}(a'\wedge b')\cdot J &= \langle a'b'J\rangle \\ &= \langle Ra\tilde{R}Rb\tilde{R}J\rangle \\ &= \langle ab\tilde{R}JR\rangle \\ &= (a\wedge b)\cdot(\tilde{R}JR).\end{aligned} \tag{4.13}$$

This must hold for all $a$ and $b$, so we must have

$$\tilde{R}JR = J. \tag{4.14}$$

That is, we are interested in the subgroup of the rotor group which leaves $J$ invariant. This defines the unitary group, denoted $\mathrm{U}(n)$. Such groups arise in a natural manner in real geometric algebra. We are expressing complex groups as sub-groups of real rotation groups in spaces of dimension $2n$. This is an unusual approach, but has a number of advantages.

Writing $R = \exp(-B/2)$ we see that bivector generators of the unitary group must satisfy

$$B\times J = 0. \tag{4.15}$$

This defines a bivector realisation of the Lie algebra of the unitary group, written $\mathrm{u}(n)$. We can construct bivectors satisfying this relation by first using the Jacobi identity to prove that

$$\begin{aligned}[(a\cdot J)\wedge(b\cdot J)]\times J &= -(a\cdot J)\wedge b + (b\cdot J)\wedge a \\ &= -(a\wedge b)\times J.\end{aligned} \tag{4.16}$$

It follows that

$$[a\wedge b + (a\cdot J)\wedge(b\cdot J)]\times J = 0. \tag{4.17}$$

A bivector of the form on the left-hand side will commute with $J$. Working through all combinations of the $\{e_i, f_i\}$ we can write down the following Lie algebra basis for u($n$):

$$\begin{aligned} E_{ij} &= e_ie_j + f_if_j \qquad (i < j = 1 \ldots n) \\ F_{ij} &= e_if_j - f_ie_j \qquad (i < j = 1 \ldots n) \\ J_i &= e_if_i. \end{aligned} \tag{4.18}$$

Establishing the closure of this algebra under the commutator product is left as an exercise. This algebra contains $J$, which commutes with all other elements and is responsible for a global phase term. Removing this term defines the special unitary group, denoted SU($n$).

# 5 Hamiltonian Mechanics

We now possess the necessary tools to reformulate Hamiltonian dynamics in a more geometric setting. We start by revising the basic ideas. Most dynamical systems can be described in terms of a Lagrangian $L(q_i, \dot{q}_i, t)$, where the $\{q_i\}$ are some set of $n$ coordinates. The Lagrange equations of motion are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}. \tag{5.1}$$

These equations are equivalent to the set of $2n$ first order equations (Hamilton's equations)

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{5.2}$$

The Hamiltonian $H(q_i, p_i, t)$ is given by

$$H(q_i, p_i, t) = p_i\dot{q}_i - L(q_i, \dot{q}_i, t) \tag{5.3}$$

in which the $\dot{q}_i$ are expressed in terms of the $p_i$ via

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{5.4}$$

The natural setting for Hamilton's equations is our $2n$-dimensional 'doubled' space generated by the $\{e_i, f_i\}$. We therefore define a point in *phase space* by the vector

$$x = p_ie_i + q_if_i. \tag{5.5}$$

The Hamiltonian can then be viewed as a function of this vector, $H = H(x, t)$. With this definition we find that

$$\nabla H = e_i\frac{\partial H}{\partial p_i} + f_i\frac{\partial H}{\partial q_i} = \dot{q}_ie_i - \dot{p}_if_i, \tag{5.6}$$

where $\nabla$ is the gradient operator

$$\nabla = e_i \frac{\partial}{\partial p_i} + f_i \frac{\partial}{\partial q_i}. \tag{5.7}$$

Hamilton's equations now specify a phase space trajectory $x(t)$ via

$$\begin{aligned} \dot{x} &= \dot{p}_i e_i + \dot{q}_i f_i \\ &= -\frac{\partial H}{\partial q_i} e_i + \frac{\partial H}{\partial p_i} f_i \\ &= (\frac{\partial H}{\partial q_i} f_i)\cdot(e_j \wedge f_j) + (\frac{\partial H}{\partial p_i} e_i)\cdot(e_j \wedge f_j). \end{aligned} \tag{5.8}$$

We again see the need for the bivector $J$. In terms of this Hamilton's equations take the simple form

$$\dot{x} = \nabla H \cdot J. \tag{5.9}$$

This new, geometric, version of Hamilton's equations has a number of advantages. It is easy to prove consequences such as conservation theorems and Liouville's theorem, and the formulation is well-suited to studying canonical transformations. Most importantly, the equations extend naturally to more complicated systems, such as constrained systems, where the dynamics takes place on a manifold. In this case the bivector $J$ varies from point to point on the manifold (it is called a "symplectic 2-form" in the more mathematical literature), but the basic equation structure is unchanged. This provides the natural setting for studies of instability and chaos in dynamical systems.

## 5.1 Conservation Theorems and Flows

We now restrict to the case where $H$ is independent of time $t$. Suppose a scalar function $f(x)$ is defined over phase space. The evolution of this along a phase space trajectory $x(t)$ is determined by

$$\dot{f} == \dot{x}\cdot\nabla f = (\nabla f \wedge \nabla H)\cdot J. \tag{5.10}$$

It follows immediately that $\dot{H} = 0$. A further consequence follows if $H$ is invariant along some direction $a$ in phase space. That is,

$$a\cdot\nabla\, H = 0 = -[(a\cdot J)\cdot J]\cdot\nabla H = (a' \wedge \nabla H)\cdot J, \tag{5.11}$$

where $a' = a\cdot J$. Comparing with above we see immediately that

$$\frac{d}{dt}(a'\cdot x) = 0 = \frac{d}{dt}[(x\wedge a)\cdot J], \tag{5.12}$$

extracting the conserved quantity $(x\wedge a)\cdot J$.