Physical Applications of Geometric Algebra

Handout 11

Quantum Theory

Both the Pauli and Dirac algebras arise naturally as matrix representations of geometric algebra in 3-d space and 4-d spacetime. It is no surprise, then, that much of quantum theory finds a natural expression within geometric algebra. To achieve this, however, one must reconsider the standard interpretation of the quantum spin operators. Like much discussion of the interpretation of quantum theory, the issues raised here are controversial. There is no question about the validity of our algebraic approach, however, and little doubt about its advantages. Whether the algebraic simplifications obtained here are indicative of a deeper structure embedded in quantum mechanics is an open question.

1 Non-Relativistic Quantum Spin

The Pauli matrices of non-relativistic quantum mechanics are

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.1}$$

The hats are used to record the fact that these are viewed explicitly as matrix operators, rather than as elements of a geometric algebra. It is easy to confirm that these matrices satisfy the same algebra as the $\{\boldsymbol{\sigma}_k\}$ vectors.

1.1 Pauli Spinors

The $\{\hat{\sigma}_k\}$ operators act on 2-component complex spinors

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{1.2}$$

where ψ_1 and ψ_2 are complex numbers. Quantum states are written with bras and kets to distinguish them from multivectors.

The set of $|\psi\rangle$'s form a two-dimensional complex vector space. We seek a way to represent these as multivectors. One way to achieve this is to add a column of zeros to the complex vector $|\psi\rangle$ and form the matrix

$$\begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix} = \begin{pmatrix} \psi_1 & \psi_3 \\ \psi_2 & \psi_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.3}$$

where ψ_3 and ψ_4 are two further arbitrary (irrelevant) coefficients. Any 2×2 matrix can be decomposed in terms of the identity and the Pauli matrices, with complex coefficients. These can be mapped directly to multivectors, with the imaginary i replaced by I (see below). The first term on the right-hand side of Eq. (1.3) therefore maps to a general 8-component multivector, ψ. The second term is the matrix $\frac{1}{2}(1+\hat{\sigma}_3)$, so our multivector equivalent of Eq. (1.3) is $\psi\frac{1}{2}(1+\boldsymbol{\sigma}_3)$. But we know that the term on the right here is a projection operator, and satisfies

$$\tfrac{1}{2}(1+\boldsymbol{\sigma}_3) = \boldsymbol{\sigma}_3\tfrac{1}{2}(1+\boldsymbol{\sigma}_3). \tag{1.4}$$

We can use this property to ensure that the multivector ψ in $\psi\frac{1}{2}(1+\boldsymbol{\sigma}_3)$ is placed in the *even subalgebra* of the 3-d geometric algebra . This algebra is 4 dimensional, so has precisely the number of degrees of freedom needed to encode a Pauli spinor. We can now strip off the projection operator and simply identify Pauli spinors with elements of the even subalgebra. Working through for each term establishes the map

$$|\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \leftrightarrow \psi = a^0 + a^k I\boldsymbol{\sigma}_k. \tag{1.5}$$

In particular, the spin-up and spin-down basis states $|+\rangle$ and $|-\rangle$ become

$$|+\rangle \leftrightarrow 1 \qquad |-\rangle \leftrightarrow -I\boldsymbol{\sigma}_2. \tag{1.6}$$

The preceding explains the process by which the map is established. Now that we have achieved this map, however, the details of the process are largely irrelevant. Of greater significance is the representation of operators acting on our new form of spinor.

1.2 Pauli Operators

The action of the quantum operators $\{\hat{\sigma}_k\}$ on states $|\psi\rangle$ has an analogous operation on the multivector ψ, which is

$$\hat{\sigma}_k|\psi\rangle \leftrightarrow \boldsymbol{\sigma}_k\psi\boldsymbol{\sigma}_3 \quad (k=1,2,3). \tag{1.7}$$

The $\boldsymbol{\sigma}_3$ on the right-hand side is a remnant of the $\frac{1}{2}(1+\boldsymbol{\sigma}_3)$ projector, and ensures that the result $\boldsymbol{\sigma}_k\psi\boldsymbol{\sigma}_3$ stays in the even subalgebra. One can explicitly verify that the translation procedure of Eq. (1.5) and Eq. (1.7) is consistent by routine computation; for example

$$\hat{\sigma}_1|\psi\rangle = \begin{pmatrix} -a^2 + ia^1 \\ a^0 + ia^3 \end{pmatrix} \leftrightarrow -a^2 + a^1 I\boldsymbol{\sigma}_3 - a^0 I\boldsymbol{\sigma}_2 + a^3 I\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1 \psi \boldsymbol{\sigma}_3. \tag{1.8}$$

As well as the action of the Pauli matrices, we also need a translation for the action of multiplication by the unit imaginary i. To find this we note that

$$\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}. \tag{1.9}$$

So multiplication of both components of $|\psi\rangle$ can be achieved by multiplying by the product of the three matrix operators. We therefore arrive at the translation

$$i|\psi\rangle \leftrightarrow \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3\psi(\boldsymbol{\sigma}_3)^3 = \psi I\boldsymbol{\sigma}_3. \tag{1.10}$$

So on this scheme the unit imaginary of quantum theory is replaced by right multiplication by the *bivector* $I\boldsymbol{\sigma}_3$. This is very suggestive, though it should be borne in mind that this conclusion is a feature of our chosen representation. Whether the quantum i really does represent a unit bivector is an open issue.

1.3 Pauli Observables

We next need to establish the quantum inner product for our multivector forms of spinors. Before this, we first introduce a new piece of notation in the STA by defining the Hermitian adjoint as

$$M^\dagger = \gamma_0 \tilde{M} \gamma_0. \tag{1.11}$$

For the $\boldsymbol{\sigma}_i$ we find that $\boldsymbol{\sigma}_i^\dagger = \boldsymbol{\sigma}_i$, whereas $I^\dagger = -I$. It follows that the dagger operation is equivalent to reversion in 3-d. We therefore employ the dagger symbol for the operation of 3-d reversion and reserve the tilde symbol for the spacetime reverse. We can then view all of our work on Pauli spinors as sitting naturally in the full STA.

If we first consider the real part of the quantum inner product, we have

$$\Re\langle\psi|\phi\rangle = \Re(\psi_1^\dagger\phi_1 + \psi_2^\dagger\phi_2). \tag{1.12}$$

This is reproduced by

$$\Re\langle\psi|\phi\rangle \leftrightarrow \langle\psi^\dagger\phi\rangle, \tag{1.13}$$

so that, for example,

$$\langle\psi|\psi\rangle \leftrightarrow \langle\psi^\dagger\psi\rangle = \langle(a^0 - a^j I\boldsymbol{\sigma}_j)(a^0 + a^k I\boldsymbol{\sigma}_k)\rangle = (a^0)^2 + a^k a^k. \tag{1.14}$$

(Note that no spatial integral is implied in our use of the bra-ket notation.) Since

$$\langle\psi|\phi\rangle = \Re\langle\psi|\phi\rangle - i\Re\langle\psi|i\phi\rangle, \tag{1.15}$$

the full inner product can be written

$$\langle\psi|\phi\rangle \leftrightarrow \langle\psi^\dagger\phi\rangle - \langle\psi^\dagger\phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3. \tag{1.16}$$

The right hand side projects out the 1 and $I\boldsymbol{\sigma}_3$ components from the geometric product $\psi^\dagger\phi$. The result of this projection on a multivector A is written $\langle A\rangle_q$. For even grade multivectors in 3-d this projection has the simple form

$$\langle A\rangle_q = \tfrac{1}{2}(A + \boldsymbol{\sigma}_3 A\boldsymbol{\sigma}_3). \tag{1.17}$$

If the result of an inner product is used to multiply a second multivector, one has to remember to keep the terms in $I\boldsymbol{\sigma}_3$ to the right of the multivector. This might appear a slightly clumsy procedure at first, but it is easy to establish conventions so that manipulations are just as efficient as in the standard treatment. Furthermore, the fact that all manipulations are now performed directly in the STA offers a number of new ways to simplify the analysis of many problems.

1.4 The Spin Vector

Now consider the expectation value of the spin in the k-direction,

$$\langle\psi|\hat{\sigma}_k|\psi\rangle \leftrightarrow \langle\psi^\dagger\boldsymbol{\sigma}_k\psi\boldsymbol{\sigma}_3\rangle - \langle\psi^\dagger\boldsymbol{\sigma}_k\psi I\rangle I\boldsymbol{\sigma}_3. \tag{1.18}$$

Since $\psi^\dagger I\boldsymbol{\sigma}_k\psi$ reverses to give minus itself, it has zero scalar part and the final term on the right-hand side vanishes. For the remaining term we note that in 3-d $\psi\boldsymbol{\sigma}_3\psi^\dagger$ is both odd grade and reverses to itself, so is a pure vector. We therefore define the *spin vector*

$$\boldsymbol{s} \equiv \psi\boldsymbol{\sigma}_3\psi^\dagger. \tag{1.19}$$

The quantum expectation now reduces to

$$\langle\psi|\hat{\sigma}_k|\psi\rangle = \boldsymbol{\sigma}_k\cdot\boldsymbol{s}. \tag{1.20}$$

This new expression has a rather different interpretation to that usually encountered in quantum theory. Rather than forming the expectation value of a quantum operator, we are simply projecting out the kth component of the vector $\boldsymbol{s}$.

1.5 Spinors and Rotations

The STA approach focuses attention on the vector $\boldsymbol{s}$, whereas the operator/matrix theory treats only its individual components. Further insight into the role of Pauli spinors is gained by defining the scalar

$$\rho \equiv \psi\psi^\dagger. \tag{1.21}$$

The spinor ψ then decomposes into

$$\psi = \rho^{1/2} R, \tag{1.22}$$

where $R = \rho^{-1/2}\psi$. The multivector R satisfies $RR^\dagger = 1$, so is a rotor. In this approach, Pauli spinors are simply unnormalised rotors!

The spin-vector $\boldsymbol{s}$ can now be written as

$$\boldsymbol{s} = \rho R \boldsymbol{\sigma}_3 R^\dagger. \tag{1.23}$$

The double-sided construction of the expectation value of Eq. (1.18) contains an instruction to rotate the fixed $\boldsymbol{\sigma}_3$ axis into the spin direction and dilate it. This view offers a number of insights. For example, suppose that the vector $\boldsymbol{s}$ is to be rotated to a new vector $R_0 \boldsymbol{s} R_0^\dagger$. The rotor group combination law tells us that R transforms to $R_0 R$. This induces the spinor transformation law

$$\psi \mapsto R_0 \psi. \tag{1.24}$$

This explains the 'spin-1/2' nature of spinor wave functions. We saw in Handout 4 that rotors are generated by half the rotation angle, so return to minus themselves under a 360^o rotation. This is the characteristic property of spinors.

In writing the spin vector as $\boldsymbol{s} = \psi\boldsymbol{\sigma}_3\psi^\dagger$ it might look like we are singling out some preferred direction in space. But in fact all we are doing is borrowing an idea from rigid-body dynamics. The $\boldsymbol{\sigma}_3$ on the right of ψ represents a vector in a 'reference' frame. All physical vectors, like $\boldsymbol{s}$, are obtained by rotating this frame onto the physical value (see Fig. 1). There is nothing special about $\boldsymbol{\sigma}_3$ — one can choose any (constant) reference frame and use the appropriate rotation onto $\boldsymbol{s}$, in the same way that there is nothing special about the orientation of the reference configuration of a rigid body. In rigid-body mechanics this freedom is usually employed to align the reference configuration with the initial state of the body. In quantum theory the convention is to work with the z-axis as the reference vector.

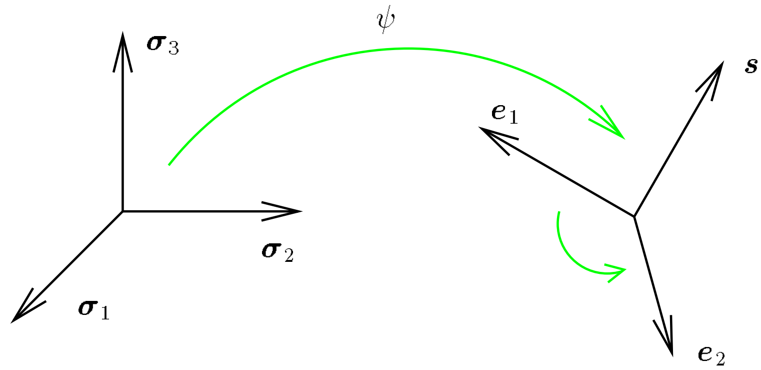

Figure 1: *The Spin Vector.* The normalised spinor ψ transforms the initial, reference frame onto the frame $\{e_k\}$. The vector e_3 is the spin vector. A phase transformation of ψ generates a rotation in the $e_1 e_2$ plane. Such a transformation is unobservable, so the e_1 and e_2 vectors are also unobservable.

2 Application — Magnetic Fields

Particles with non-zero spin also have a magnetic moment. This is conventionally expressed as the operator relation

$$\hat{\mu}_k = \gamma \hat{s}_k, \tag{2.1}$$

where $\hat{\mu}_k$ is the magnetic moment operator, γ is the gyromagnetic ratio and $\hat{s}_k$ is the spin operator. The gyromagnetic ratio is usually written in the form

$$\gamma = g\frac{-q}{2m}, \tag{2.2}$$

where m is the particle mass, q is the charge and g is the reduced gyromagnetic ratio. The latter are determined experimentally to be

electron	$g_e = 2$	(actually $2(1+\alpha/2\pi+\dots)$)
proton	$g_p = 5.587$	
neutron	$g_n = -3.826$	

(The value for the neutron is negative because its spin and magnetic moment are anti-parallel.) All of the above are spin-1/2 particles for which we would conventionally write

$$\hat{s}_k = \tfrac{1}{2}\hbar\hat{\sigma}_k. \tag{2.3}$$

(For clarity in this section we will keep in factors of $\hbar$.) The $\hat{\sigma}_k$ matrix operators are then viewed as the *components* of a single vector $\hat{\boldsymbol{\sigma}}$.

2.1 Particle in a Magnetic Field

Now suppose that the particle is in a magnetic field. We introduce the Hamiltonian operator

$$\hat{H} = -\tfrac{1}{2}\gamma\hbar B_k\hat{\sigma}_k = -\hat{\boldsymbol{\mu}}\cdot\boldsymbol{B}. \tag{2.4}$$

The spin state at time t is then written as

$$|\psi(t)\rangle = a_1(t)|+\rangle + a_2(t)|-\rangle, \tag{2.5}$$

with a_1 and a_2 general complex coefficients. The dynamical equation for these coefficients is given by the time-dependent Schrödinger equation

$$\hat{H}|\psi\rangle = i\hbar\frac{d|\psi\rangle}{dt}. \tag{2.6}$$

This is conventionally hard to analyse, because one ends up with a pair of coupled differential equations in a_1 and a_2.

Now let us see what the Schrödinger equation looks like in our new setup. We first write the equation in the form

$$\frac{d|\psi\rangle}{dt} = \tfrac{1}{2}\gamma i B_k\hat{\sigma}_k|\psi\rangle. \tag{2.7}$$

Now replacing $|\psi\rangle$ by the multivector ψ we see that the left-hand side is simply $\dot{\psi}$, where the dot denotes the time derivative. The right-hand side involves multiplication of the spinor $|\psi\rangle$ by $i\hat{\sigma}_k$, which we replace by

$$i\hat{\sigma}_k|\psi\rangle \leftrightarrow \boldsymbol{\sigma}_k\psi\boldsymbol{\sigma}_3(I\boldsymbol{\sigma}_3) = I\boldsymbol{\sigma}_k\psi. \tag{2.8}$$

Our STA version of the Schrödinger equation (2.6) is therefore simply

$$\dot{\psi} = \tfrac{1}{2}\gamma B_k I\boldsymbol{\sigma}_k\psi = \tfrac{1}{2}\gamma I\boldsymbol{B}\psi. \tag{2.9}$$

If we now decompose ψ into $\rho^{1/2}R$ we see that

$$\dot{\psi}\tilde{\psi} = \tfrac{1}{2}\dot{\rho} + \rho\dot{R}\tilde{R} = \tfrac{1}{2}\rho\gamma I\boldsymbol{B}. \tag{2.10}$$

The right-hand side is a bivector, so ρ must be constant and the dynamics reduces to

$$\dot{R} = \tfrac{1}{2}\gamma I\boldsymbol{B}R. \tag{2.11}$$

The quantum theory of a spin-1/2 particle in a magnetic field reduces to another rotor equation! This is very natural, if one thinks about the behaviour of particles in magnetic fields, and is the main justification for our approach.

Recovering a rotor equation explains the difficulty of the traditional analysis based on a pair of coupled equations for the components of $|\psi\rangle$. This approach fails to capture the fact that there is a rotor underlying the dynamics, and so carries along redundant degrees of freedom in the normalisation. Also, the separation of a rotor into a pair of components is far from natural. Two examples illustrate this. As a simple example, consider a constant field $\boldsymbol{B} = B_0\boldsymbol{\sigma}_3$. The rotor equation integrates immediately to give

$$\psi(t) = \mathrm{e}^{\gamma B_0 t I\boldsymbol{\sigma}_3/2}\,\psi_0. \tag{2.12}$$

The spin vector $\boldsymbol{s}$ therefore just precesses about the 3 axis at a rate $\omega_0 = \gamma B_0$. Even this almost trivial result is substantially more complicated when following traditional methods.

2.2 Magnetic Resonance Imaging

A more interesting example is to include an oscillatory $\boldsymbol{B}$ field $(B_1\cos(\omega t), B_1\sin(\omega t), 0)$ together with a constant field along the z-axis. This oscillatory field induces transitions (spin-flips) between the up and down states, which differ in energy because of the constant component of the field. This is a very interesting system of great practical importance. It is the basis of magnetic resonance imaging and Rabi molecular beam spectroscopy.

To study this system we first write the $\boldsymbol{B}$ field as

$$\begin{aligned} B_1(\cos(\omega t)\boldsymbol{\sigma}_1 + \sin(\omega t)\boldsymbol{\sigma}_2) + B_0\boldsymbol{\sigma}_3 &= \mathrm{e}^{-\omega t I\boldsymbol{\sigma}_3}\,B_1\boldsymbol{\sigma}_1 + B_0\boldsymbol{\sigma}_3 \\ &= \mathrm{e}^{-\omega t I\boldsymbol{\sigma}_3/2}\,(B_1\boldsymbol{\sigma}_1 + B_0\boldsymbol{\sigma}_3)\,\mathrm{e}^{\omega t I\boldsymbol{\sigma}_3/2}. \end{aligned} \tag{2.13}$$

We now define

$$S = \mathrm{e}^{-\omega t I\boldsymbol{\sigma}_3/2} \quad \text{and} \quad \boldsymbol{B}_c = B_1\boldsymbol{\sigma}_1 + B_0\boldsymbol{\sigma}_3 \tag{2.14}$$

so that we can write $\boldsymbol{B} = S\boldsymbol{B}_c\tilde{S}$. The rotor equation can now be written

$$\tilde{S}\dot{\psi} = \tfrac{1}{2}\gamma I\boldsymbol{B}_c\tilde{S}\psi, \tag{2.15}$$

where we have pre-multiplied by $\tilde{S}$. Now noting that

$$\dot{\tilde{S}} = \tfrac{1}{2}\omega I\boldsymbol{\sigma}_3\tilde{S} \tag{2.16}$$

we see that

$$\frac{d}{dt}(\tilde{S}\psi) = \tfrac{1}{2}(\gamma I\boldsymbol{B}_c + \omega I\boldsymbol{\sigma}_3)\tilde{S}\psi. \tag{2.17}$$

It is now $\tilde{S}\psi$ that satisfies a rotor equation with a constant field. The solution is straightforward,

$$\tilde{S}\psi(t) = \exp\left(\frac{t}{2}(\gamma I\boldsymbol{B}_c + \omega I\boldsymbol{\sigma}_3)\right)\psi_0, \tag{2.18}$$

and we arrive at

$$\psi(t) = \exp\left(-\frac{wt}{2}I\boldsymbol{\sigma}_3\right)\exp\left(\frac{t}{2}[(\omega_0+\omega)I\boldsymbol{\sigma}_3 + \omega_1 I\boldsymbol{\sigma}_1]\right)\psi_0, \tag{2.19}$$

where $\omega_1 = \gamma B_1$. There are three separate frequencies in this solution, which contains a wealth of interesting physics. Needless to say, this derivation is a vast improvement over standard methods!

To complete our analysis we must relate our solution to the results of experiments. Suppose that at time $t = 0$ we switch on the oscillating field. The particle is initially in a spin-up state, so $\psi_0 = 1$, which also ensures that the state is normalised. The probability that at time t the particle is in the spin-down state is

$$P_- = |\langle -|\psi(t)\rangle|^2 \tag{2.20}$$

We therefore need to form the inner product

$$\langle -|\psi(t)\rangle \leftrightarrow \langle I\boldsymbol{\sigma}_2\psi\rangle_q = \langle I\boldsymbol{\sigma}_2\psi\rangle - I\boldsymbol{\sigma}_3\langle I\boldsymbol{\sigma}_2\psi I\boldsymbol{\sigma}_3\rangle = \langle I\boldsymbol{\sigma}_2\psi\rangle - I\boldsymbol{\sigma}_3\langle I\boldsymbol{\sigma}_1\psi\rangle. \tag{2.21}$$

To find this inner product we write

$$\psi(t) = \mathrm{e}^{-wtI\boldsymbol{\sigma}_3/2}\left[\cos(\alpha t/2) + I\hat{B}\sin(\alpha t/2)\right] \tag{2.22}$$

where

$$\hat{B} = \frac{(\omega_0+\omega)\boldsymbol{\sigma}_3 + \omega_1\boldsymbol{\sigma}_1}{\alpha}, \quad \text{and} \quad \alpha = \sqrt{(w+w_0)^2 + \omega_1^2}. \tag{2.23}$$

The only term giving a contribution in the $I\boldsymbol{\sigma}_1$ and $I\boldsymbol{\sigma}_2$ planes is that in $\omega_1 I\boldsymbol{\sigma}_1/\alpha$. We therefore have

$$\langle I\boldsymbol{\sigma}_2\psi\rangle_q = \frac{\omega_1\sin(\alpha t/2)}{\alpha}\,\mathrm{e}^{-wtI\boldsymbol{\sigma}_3/2}\,I\boldsymbol{\sigma}_3 \tag{2.24}$$

and the probability is immediately

$$P_- = \left(\frac{\omega_1\sin(\alpha t/2)}{\alpha}\right)^2. \tag{2.25}$$

The maximum value is at $\alpha t = \pi$, and the probability at this time is maximised by choosing α as small as possible. This is achieved by setting $\omega = -\omega_0 = -\gamma B_0$. This is the *spin resonance condition.*

3 Relativistic Quantum Spin

The relativistic quantum mechanics of a spin-1/2 particle is described by the *Dirac theory*. The Dirac matrix operators are

$$\hat{\gamma}_0 = \begin{pmatrix} \mathsf{I} & 0 \\ 0 & -\mathsf{I} \end{pmatrix}, \quad \hat{\gamma}_k = \begin{pmatrix} 0 & -\hat{\sigma}_k \\ \hat{\sigma}_k & 0 \end{pmatrix} \quad \text{and} \quad \hat{\gamma}_5 = \begin{pmatrix} 0 & \mathsf{I} \\ \mathsf{I} & 0 \end{pmatrix}, \tag{3.1}$$

where $\hat{\gamma}_5 = -i\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$ and I is the 2×2 identity matrix. These matrices act on Dirac spinors, which have 4 complex components (8 real degrees of freedom). We follow an analogous procedure to the Pauli case and map these spinors onto elements of the 8-dimensional even subalgebra of the STA. Dirac spinors can be visualised as decomposing into 'upper' and 'lower' components,

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix}, \tag{3.2}$$

where $|\phi\rangle$ and $|\eta\rangle$ are a pair of 2-component spinors. We already know how to represent these as multivectors ϕ and η, which lie in the space of scalars + relative bivectors. Our map from the Dirac spinor onto an element of the full 8-dimensional subalgebra is simply

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix} \leftrightarrow \psi = \phi + \eta\boldsymbol{\sigma}_3. \tag{3.3}$$

The action of the Dirac matrix operators now becomes,

$$\begin{aligned} \hat{\gamma}_\mu|\psi\rangle &\leftrightarrow \gamma_\mu\psi\gamma_0 \quad (\mu = 0, \ldots, 3) \\ i|\psi\rangle &\leftrightarrow \psi\, I\boldsymbol{\sigma}_3 \\ \hat{\gamma}_5|\psi\rangle &\leftrightarrow \psi\boldsymbol{\sigma}_3. \end{aligned}$$

Again, verifying that this map is consistent is a matter of routine computation. One thing to note is that we now have two 'reference' vectors that can appear on the right-hand side of ψ: γ_0 and γ_3. That is, the relative vector $\boldsymbol{\sigma}_3$ used in the Pauli theory has been decomposed into a spacelike and timelike direction. Since $I\boldsymbol{\sigma}_3$ and γ_0 commute, our use of right multiplication by $I\boldsymbol{\sigma}_3$ for the complex structure remains consistent.

3.1 Dirac Observables

In Dirac theory one replaces the non-covariant Hermitian adjoint with the Dirac adjoint

$$\langle\bar{\psi}| = (\langle\psi_u|, -\langle\psi_l|) \tag{3.4}$$

where the subscript u and l refer to the upper and lower components. The inner product then decomposes into

$$\langle\bar{\psi}|\phi\rangle = \langle\psi_u|\phi_u\rangle - \langle\psi_l|\phi_l\rangle. \tag{3.5}$$

This has the STA equivalent

$$\langle\psi_u^\dagger\phi_u\rangle_q - \langle\psi_l^\dagger\phi_l\rangle_q = \langle(\tilde{\psi}_u - \boldsymbol{\sigma}_3\tilde{\psi}_l)(\phi_u + \phi_l\boldsymbol{\sigma}_3)\rangle_q = \langle\tilde{\psi}\phi\rangle_q. \tag{3.6}$$

So, unsurprisingly, the Dirac adjoint reversion is replaced by the covariant operation of spacetime reversion.

We can now look at the main observables formed from a Dirac spinor. The first is the current

$$J_\mu = \langle\bar{\psi}|\hat{\gamma}_\mu|\psi\rangle \;\leftrightarrow\; \langle\tilde{\psi}\gamma_\mu\psi\gamma_0\rangle - \langle\tilde{\psi}\gamma_\mu\psi I\gamma_3\rangle I\boldsymbol{\sigma}_3. \tag{3.7}$$

The final term contains $\langle\gamma_\mu\psi I\gamma_3\tilde{\psi}\rangle$, which vanishes because $\psi I\gamma_3\tilde{\psi}$ reverses to minus itself and so is a pure trivector. Similarly, $\psi\gamma_0\tilde{\psi}$ is a pure vector, and we are left with

$$J_\mu = \langle\bar{\psi}|\hat{\gamma}_\mu|\psi\rangle \;\leftrightarrow\; \gamma_\mu\cdot(\psi\gamma_0\tilde{\psi}). \tag{3.8}$$

As with the Pauli theory, the operation of taking the expectation value of a matrix operator is replaced by that of picking out a component of a vector. We can therefore reconstitute the full vector J and write

$$J = \psi\gamma_0\tilde{\psi} \tag{3.9}$$

for the first of our observables.

We can now follow a procedure first introduced in Handout 8. We write the scalar + pseudoscalar quantity $\psi\tilde{\psi}$ as

$$\psi\tilde{\psi} = \rho\,\mathrm{e}^{I\beta}\,, \tag{3.10}$$

and define a spacetime rotor R by

$$R = \psi\rho^{-1/2}\,\mathrm{e}^{-I\beta/2}\,, \qquad R\tilde{R} = 1. \tag{3.11}$$

We have now decomposed the spinor ψ into

$$\psi = \rho^{1/2}\,\mathrm{e}^{I\beta/2}\,R \tag{3.12}$$

which separates out a density ρ and the rotor R. The remaining factor of β is curious. It turns out that plane-wave particle states have $\beta = 0$, whereas antiparticle states have $\beta = \pi$. The picture for bound state wavefunctions is more complicated, however.

Bilinear Covariant	Standard Form	STA Equivalent	Frame-Free Form
Scalar	$\langle\bar{\psi}\vert\psi\rangle$	$\langle\psi\tilde{\psi}\rangle$	$\rho\cos\beta$
Vector	$\langle\bar{\psi}\vert\hat{\gamma}_\mu\vert\psi\rangle$	$\gamma_\mu\cdot(\psi\gamma_0\tilde{\psi})$	$\psi\gamma_0\tilde{\psi}=J$
Bivector	$\langle\bar{\psi}\vert i\hat{\gamma}_{\mu\nu}\vert\psi\rangle$	$(\gamma_\mu\wedge\gamma_\nu)\cdot(\psi \boldsymbol{I\sigma}_3\tilde{\psi})$	$\psi\boldsymbol{I\sigma}_3\tilde{\psi}=S$
Pseudovector	$\langle\bar{\psi}\vert\hat{\gamma}_\mu\hat{\gamma}_5\vert\psi\rangle$	$\gamma_\mu\cdot(\psi\gamma_3\tilde{\psi})$	$\psi\gamma_3\tilde{\psi}=s$
Pseudoscalar	$\langle\bar{\psi}\vert i\hat{\gamma}_5\vert\psi\rangle$	$\langle\psi\tilde{\psi}I\rangle$	$-\rho\sin\beta$

Table 1: *Observables in the Dirac theory.*

With this decomposition of ψ, the current becomes

$$J=\psi\gamma_0\tilde{\psi}=\rho\,\mathrm{e}^{I\beta/2}\,R\gamma_0\tilde{R}\,\mathrm{e}^{I\beta/2}\;=\rho R\gamma_0\tilde{R}. \tag{3.13}$$

So the rotor is now an instruction to rotate γ_0 onto the direction of the current. This is precisely the picture adopted for studying the dynamics of a relativistic point particle! A similar picture emerges for the spin. The spin becomes a rank 2 antisymmetric tensor (a bivector!) in the relativistic theory, with components given by the observables

$$\langle\bar{\psi}|i\tfrac{1}{2}(\hat{\gamma}_\mu\hat{\gamma}_\nu-\hat{\gamma}_\nu\hat{\gamma}_\mu)|\psi\rangle\;\leftrightarrow\;\langle\tilde{\psi}\gamma_\mu\wedge\gamma_\nu\psi\boldsymbol{I\sigma}_3\rangle_q=\langle\gamma_\mu\wedge\gamma_\nu\psi\boldsymbol{I\sigma}_3\tilde{\psi}\rangle, \tag{3.14}$$

where again there is no imaginary component. This time we are picking out the components of the spin bivector S, given by

$$S=\psi\boldsymbol{I\sigma}_3\tilde{\psi}. \tag{3.15}$$

This is the natural generalisation of the Pauli result.

There are 5 such observables in all, which are summarised in Table 1. Of particular interest is the spin vector $s=\rho R\gamma_3\tilde{R}$. This justifies the classical model of spin examined in Handout 9, Section 3.3. There it was shown that the rotor form of the Lorentz force law naturally gives rise to a reduced gyromagnetic ratio of $g=2$. This is usually viewed as a result of relativistic quantum mechanics as described by the Dirac equation. In fact, $g=2$ is the natural value once one sees how the spin vector is formed from a rotor.