Physical Applications of Geometric Algebra

Examples 1 — Answers

1. Expand out the products and collect the terms.

2. The equation for U is

$$\frac{d^2U}{ds^2} = \frac{E}{2m}U = -\omega^2 U,$$

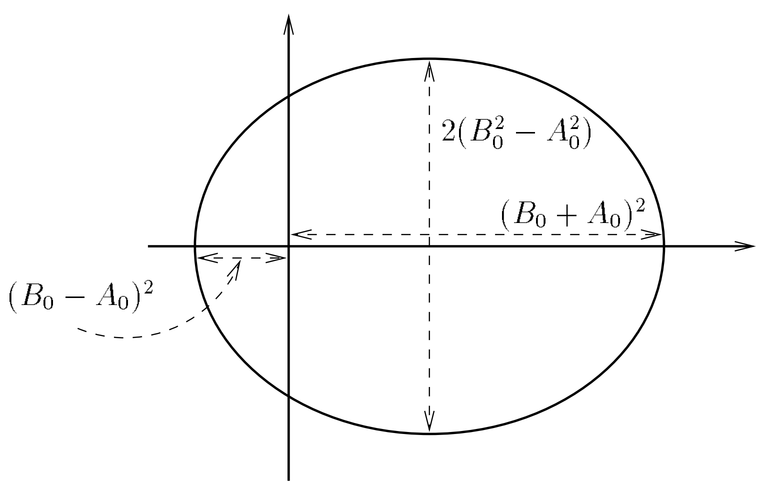
since $E < 0$ for a bound orbit. General solution is

$$U = A_0 \, \mathrm{e}^{I\omega s} + B_0 \, \mathrm{e}^{-I\omega s} .$$

Four real constants, as expected. At $s = 0$ want U^2 real and positive, so must have $A_0 + B_0$ real. Also have

$$\begin{aligned} r\dot{x} = 2\frac{dU}{ds}Ue_1 &= 2\omega I \left(A_0 \, \mathrm{e}^{I\omega s} - B_0 \, \mathrm{e}^{-I\omega s}\right)\left(A_0 \, \mathrm{e}^{I\omega s} + B_0 \, \mathrm{e}^{-I\omega s}\right) e_1 \\ &= -2\omega \left(A_0^2 \, \mathrm{e}^{2I\omega s} - B_0^2 \, \mathrm{e}^{-2I\omega s}\right) e_2 \end{aligned}$$

so at $t = 0$, $r\dot{x} = -2\omega(A_0^2 - B_0^2)e_2$. So must also have $A_0 - B_0$ real. Hence both A_0 and B_0 are real, with $B_0 > A_0$. $\dot{x}$ in $-e_1$ direction when $s = \pi/(4\omega)$. At this point height from e_1 axis is $B_0^2 - A_0^2$. Orbit is as follows:



Has semi-major axis $a = \frac{1}{2}[(B_0 + A_0)^2 + (B_0 - A_0)^2] = B_0^2 + A_0^2$ and eccentricity

$$\epsilon = \frac{\sqrt{[(B_0^2 + A_0^2)^2 - (B_0^2 - A_0^2)^2]}}{B_0^2 + A_0^2} = \frac{2A_0B_0}{a}$$

Invert to get

$$B_0 + A_0 = \sqrt{a(1+\epsilon)}, \qquad B_0 - A_0 = \sqrt{a(1-\epsilon)}.$$

Now get answer by re-expressing exponentials in terms of sin and cos.

3. Expand out to get

$$a\, a\wedge b = \tfrac{1}{2}(aab - aba) = \tfrac{1}{2}(ba - ab)a = -a\wedge b\, a$$

Any vector perpendicular to plane anticommutes with a and b, so commutes past $a\wedge b$.

4. $(a\wedge b)\cdot(a\wedge b) = a\cdot(a\cdot b\, b - b^2\, a) = (a\cdot b)^2 - a^2 b^2 = -a^2\, b^2 \sin^2(\theta)$.

5. First establish that $e_1 e_2 = I e_3$, *etc.* so that

$$e_i\wedge e_j = I\, e_k \epsilon_{ijk}, \qquad \epsilon_{ijk} = -I\, e_i\wedge e_j\wedge e_k$$

Now kth component of $a\boldsymbol{\times} b$ is

$$a_i b_j \epsilon_{ijk} = -I(a_i e_i)\wedge(b_j e_j)\wedge e_k = -I\, a\wedge b\wedge e_k = -(I\, a\wedge b)\cdot e_k.$$

Hence result. Move I around using duality relations. Have

$$a\boldsymbol{\times}(b\boldsymbol{\times} c) = -a\cdot(I\, b\boldsymbol{\times} c) = -a\cdot(b\wedge c)$$

and

$$a\cdot(b\boldsymbol{\times} c) = a\cdot(-I\, b\wedge c) = a\wedge b\wedge c\, I^{-1}.$$

6. Expanding gives

$$a\wedge(b\wedge c) = \tfrac{1}{4}(abc - acb + bca - cba)$$

Antisymmetric on a and b because

$$abc - cba = (a\cdot b + a\wedge b)c - c(b\cdot a + b\wedge a) = a\wedge b\, c - c\, b\wedge a$$

For $a\wedge b\wedge c$ want trivector part of abc. This is part equal to minus its reverse, so

$$a\wedge b\wedge c = \tfrac{1}{2}[abc - (abc)^{\sim}] = \tfrac{1}{2}(abc - cba).$$

Now have

$$8a\wedge b\wedge c = (ab - ba)c - 2acb + 2bca - c(ba - ab).$$

So, subtracting these,

$$6a\wedge b\wedge c = abc - bac - acb + bca - cba + cab + (-acb - abc + bca + cba)$$

and final term in brackets is zero.

7. Particle has $\dot{x} = v$, with v^2 constant. Differentiate to get

$$\frac{d}{dt}v^2 = 2\dot{v}\cdot v = 0$$

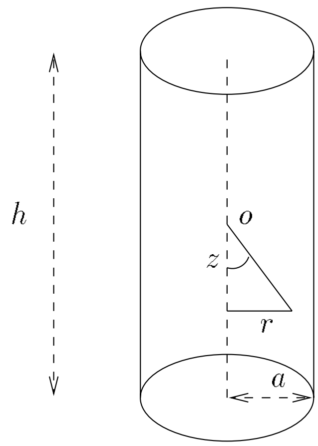


Figure 1: A cylinder, height h, radius a. Origin o at centre of mass. Radial distance from centre axis r, $0 \leq r \leq a$. Height from origin z, $-h/2 \leq z \leq h/2$

So

$$\dot{v} = \frac{1}{v^2}\dot{v}vv = \frac{1}{v^2}\dot{v}\wedge v\, v = \left(\frac{1}{v^2}\dot{v}\wedge v\right)\cdot v$$

which gives Ω as term in brackets. Can add any multiple of Iv. In 4-d, can have

$$\Omega = \frac{1}{v^2}\dot{v}\wedge v + B$$

where B is any bivector with $B\cdot v = 0$ (a 3-space of possible bivectors).

8. Cylinder, uniform density ρ, dimensions and coordinates as in Fig 1. If $\{e_k\}$ are principal axis, have

$$\mathcal{I}(Ie_k) = i_k I e_k$$

so

$$\begin{aligned} i_k = (-Ie_k)\cdot\mathcal{I}(Ie_k) &= \int d^3x\, \rho(-Ie_k)\cdot[x\wedge(x\cdot(Ie_k))] \\ &= \int d^3x\, \rho|(x\wedge e_k)^2| \end{aligned}$$

With e_3 the symmetry axis, have

$$i_3 = 2\pi h\rho \int_0^a r^3\, dr = \frac{a^2}{2}M$$

Similarly, in perpendicular direction get, with ϕ angle in e_1e_2 plane,

$$i_1 = i_2 = \int d^3x\, \rho(r^2\sin^2\phi + z^2) = \frac{Ma^2}{4} + \frac{Mh^2}{12}$$

Now have

$$\mathcal{I}(B) = i_1 B + (i_3 - i_1) B\wedge e_3 e_3 = \frac{Mh^2}{12}(B - B\wedge e_3\, e_3) + \frac{Ma^2}{4}(B + B\wedge e_3\, e_3)$$

9. Have velocity $v = R\, x\cdot\Omega_B\, \tilde{R} + v_0$. Kinetic energy is

$$T = \int d^3x\, \tfrac{1}{2}\rho v^2 = \tfrac{1}{2}\int d^3x\, \rho \left(v_0^2 + 2v_0\cdot(Rx\cdot\Omega_B\tilde{R}) + (x\cdot\Omega_B)^2 \right).$$

First term is $mv_0^2/2$ and second term vanishes because origin at centre of mass. For final term use

$$(x\cdot\Omega_B)^2 = \langle x\cdot\Omega_B\, x\cdot\Omega_B\rangle = -\langle\Omega_B x\, x\cdot\Omega_B\rangle = -\Omega_B\cdot[x\wedge(x\cdot\Omega_B)]$$

Now have

$$T = \tfrac{1}{2}mv_0^2 - \tfrac{1}{2}\int d^3x\, \rho\Omega_B\cdot[x\wedge(x\cdot\Omega_B)] = \tfrac{1}{2}mv_0^2 - \tfrac{1}{2}\Omega_B\cdot\mathcal{I}(\Omega_B).$$

The minus sign arises because bivectors have negative square.

10. The equations of motion for a torque-free rigid body are

$$\mathcal{I}(\dot{\Omega}_B) = \Omega_B\times\mathcal{I}(\Omega_B)$$

Introduce body angular momentum

$$\Pi = \mathcal{I}(\Omega_B), \qquad \Omega_B = \mathcal{I}^{-1}(\Pi)$$

and write equations as

$$\dot{\Pi} = \mathcal{I}^{-1}(\Pi)\times\Pi.$$

Get

$$\frac{d}{dt}\Pi^2 = 2\dot{\Pi}\cdot\Pi = 2[\mathcal{I}^{-1}(\Pi)\times\Pi]\cdot\Pi$$

and

$$\frac{d}{dt}[\Pi\cdot\mathcal{I}^{-1}(\Pi)] = 2\dot{\Pi}\cdot\mathcal{I}^{-1}(\Pi) = 2[\mathcal{I}^{-1}(\Pi)\times\Pi]\cdot\mathcal{I}^{-1}(\Pi).$$

In second have used fact that $\mathcal{I}(B)$ is time-independent and symmetric. But for bivectors A and B have

$$A\cdot(A\times B) = \tfrac{1}{2}\langle A^2B - ABA\rangle = 0$$

since no grade-0 term present. Follows that both terms are conserved. First is magnitude of the angular momentum bivector $L = R\Pi\tilde{R}$. Second, $\Pi\cdot\mathcal{I}^{-1}(\Pi)$, is twice the rotational energy (cf. $p^2/(2m)$). Writing $\Pi = \Pi_i I e_i$ get that

$$\Pi_1^2 + \Pi_2^2 + \Pi_3^2 = \text{constant}, \qquad \frac{\Pi_1^2}{i_1} + \frac{\Pi_2^2}{i_2} + \frac{\Pi_3^2}{i_3} = \text{constant}.$$

First defines a sphere in Π space. Second an ellipsoid. Paths in Π-space are orbits formed from intersection of sphere and ellipsoid.

11. If $a\wedge b + c\wedge d$ is a blade $e\wedge f$, then

$$(a\wedge b + c\wedge d)\wedge(a\wedge b + c\wedge d) = 2a\wedge b\wedge c\wedge d = (e\wedge f)\wedge(e\wedge f) = 0.$$

Conversely, if $a\wedge b\wedge c\wedge d = 0$, then d must be linearly dependent on a, b and c. Write

$$d = \alpha a + \mu b + \nu c$$

Get

$$a\wedge b + c\wedge d = a\wedge b + c\wedge(\alpha a + \mu b) = (a/\mu + c)\wedge(\alpha a + \mu b)$$

which is a blade. If $\mu = 0$, result is $a\wedge(b - \alpha c)$.

12. From basic axioms

$$\begin{aligned} A_r\wedge(B_s\wedge C_t) &= A_r\wedge\langle B_s C_t\rangle_{s+t} \\ &= \langle A_r\langle B_s C_t\rangle_{s+t}\rangle_{r+s+t} \\ &= \langle A_r B_s C_t\rangle_{r+s+t} \end{aligned}$$

Now get associativity of exterior product from properties of geometric product.

13. One proof given in Handout 3. Second is to build up using

$$\begin{aligned} &a\cdot(a_1\wedge M_r) \\ &\quad = \tfrac{1}{4}[a(a_1 M_r + (-1)^r M_r a_1) - (-1)^{r+1}(a_1 M_r + (-1)^r M_r a_1)a] \\ &\quad = a\cdot a_1\, M_r - \tfrac{1}{4}[a_1(a M_r + (-1)^{r+1} M_r a) - (-1)^r(a M_r + (-1)^{r+1} M_r a)a_1] \\ &\quad = a\cdot a_1\, M_r - a_1\wedge(a\cdot M_r). \end{aligned}$$

Result then builds up inductively.

14. Can prove result geometrically. Another way is to look at image of m, which is $nmmmn = nmn$. Angle with m is

$$m\cdot(nmn) = \langle mnmn\rangle = 2(m\cdot n)^2 - \langle nmmn\rangle = 2\cos^2\theta - 1 = \cos(2\theta)$$

so m is rotated by 2θ. To rotate a onto b, rotor is

$$R = \mathrm{e}^{-\hat{B}\theta/2}, \quad a\cdot b = \cos\theta, \quad \hat{B} = a\wedge b/\sin\theta.$$

Expanding out, get

$$R = \cos(\theta/2) - \sin(\theta/2)\frac{a\wedge b}{\sin\theta} = \frac{2\cos^2(\theta/2) - a\wedge b}{2\cos(\theta/2)}$$

But

$$2\cos^2(\theta/2) = 1 + \cos\theta = 1 + a\cdot b$$

so

$$R = \frac{1 + a\cdot b + b\wedge a}{[2(1 + a\cdot b)]^{1/2}} = \frac{1 + ba}{[2(1 + a\cdot b)]^{1/2}}.$$

15. Write $\hat{B} = \hat{c}\hat{a}$ and $\hat{A} = \hat{b}\hat{c}$. Then

$$\hat{B}\hat{A} = \hat{c}\hat{a}\hat{b}\hat{c} = (-\hat{a}\hat{c})(-\hat{c}\hat{b}) = \hat{a}\hat{b}.$$

But $\hat{a}\cdot\hat{b} = \cos(\gamma)$, and $\hat{a}\wedge\hat{b}$ has orientation of $I\hat{c}$. Hence

$$\hat{B}\hat{A} = \cos(\gamma) + \sin(\gamma)I\hat{c} = \mathrm{e}^{I\gamma\hat{c}} = \mathrm{e}^{Ic}.$$

Similarly, if $|C|$ is length of arc between $\hat{a}$ and $\hat{b}$, then

$$\hat{a}\hat{b} = \cos|C| + \sin|C|\hat{C} = \mathrm{e}^{C}$$

(Check orientation!) Now have

$$\begin{aligned} \mathrm{e}^{C}\,\mathrm{e}^{A}\,\mathrm{e}^{B} &= \hat{a}\hat{b}\hat{b}\hat{c}\hat{c}\hat{a} = 1 \\ \mathrm{e}^{Ic}\,\mathrm{e}^{Ib}\,\mathrm{e}^{Ia} &= \hat{B}\hat{A}\hat{A}\hat{C}\hat{C}\hat{B} = (-1)^3 = -1. \end{aligned}$$

Final gives

$$\begin{aligned} \cos(\gamma) = \langle\,\mathrm{e}^{-Ic}\,\rangle &= -\langle\,\mathrm{e}^{Ib}\,\mathrm{e}^{Ia}\,\rangle \\ &= -\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\langle\hat{b}\hat{a}\rangle \\ &= -\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\cos|C|. \end{aligned}$$