# Physical Applications of Geometric Algebra

## Handout 2

## Geometric Algebra of 3-d Space

The geometric algebra (GA) of 3-d space is a remarkably powerful tool for solving problems in geometry and classical mechanics. It describes vectors, planes and volumes in a single algebra, which contains all of the familiar vector operations for 3-d space. These include the vector cross product, which is revealed as a disguised form of bivector. The algebra provides a very clear and compact method for encoding rotations, which is considerably more powerful than working with matrices. An example of the power of the GA approach is provided by rigid body mechanics, where it provides a simplified treatment of a spinning top.

The webpage for this course is `www.mrao.cam.ac.uk/~clifford/ptIIIcourse/`.

# 1 Geometric Algebra in 3-d

In Lecture 1 we constructed the geometric algebra of a 2-d plane. We now add a third vector $e_3$ to our 2-d set $\{e_1, e_2\}$. All three vectors are assumed to be orthonormal, so they all *anticommute*. From these 3 basis vectors, we can generate 3 independent bivectors: $e_1e_2$, $e_2e_3$ and $e_3e_1$. This is the expected number of independent planes in 3-d space.

Our expanded algebra gives us a number of new products to consider. The first is the product of a bivector and an orthogonal vector,

$$(e_1 \wedge e_2)e_3 = e_1e_2e_3. \tag{1.1}$$

This corresponds to sweeping the bivector $e_1 \wedge e_2$ along the vector $e_3$. the result is a 3-dimensional volume element and is called a *trivector*. This has *grade*-3, where the word 'grade' refers to the number of independent vectors forming the object. The term 'grade' is preferred to 'dimension' as the latter is reserved for the size of a linear space.

We continue to use the wedge symbol for the operation of sweeping one element along another. Given three vectors, $a$, $b$ and $c$, the trivector $a \wedge b \wedge c$ is formed by sweeping $a \wedge b$ along $c$ (see Fig. 1). The same result is obtained by sweeping $b \wedge c$ along $a$. The

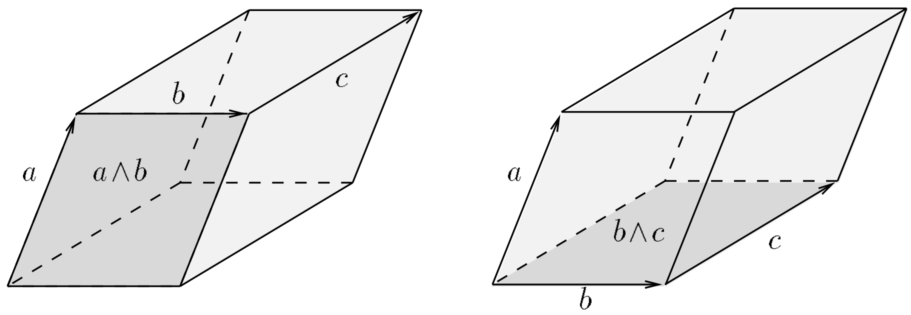



Figure 1: *The Trivector.* The result of sweeping $a\wedge b$ along $c$ is a directed volume, or trivector. The same trivector is obtained by sweeping $b\wedge c$ along $a$.

mathematical expression of this is that the outer product is associative:

$$(a\wedge b)\wedge c = a\wedge(b\wedge c) = a\wedge b\wedge c. \tag{1.2}$$

The other main property of the outer product is that it is antisymmetric on every pair of vectors,

$$a\wedge b\wedge c = -b\wedge a\wedge c = c\wedge a\wedge b, \quad etc. \tag{1.3}$$

This is because swapping any two vectors reverses the orientation of the product.

One can keep forming exterior products of independent vectors to form a wealth of higher-grade objects. In 3-d, however, there are no further directions to use and trivectors are unique up to scale (or volume) and handedness. Our 3-d algebra is therefore spanned by

$$\begin{array}{cccc} 1 & \{e_i\} & \{e_i\wedge e_j\} & e_1\wedge e_2\wedge e_3 \\ 1 \text{ scalar} & 3 \text{ vectors} & 3 \text{ bivectors} & 1 \text{ trivector} \end{array} \tag{1.4}$$

These define a linear space of dimension $8 = 2^3$. We call this algebra $\mathcal{G}_3$. Notice that the dimesnions of each subspace are given by the binomial coefficients.

## 1.1 Vectors and Bivectors

Each of the basis bivectors shares the properties of the 2-d bivector studied in Lecture 1. In particular,

$$(e_1e_2)^2 = (e_2e_3)^2 = (e_3e_1)^2 = -1 \tag{1.5}$$

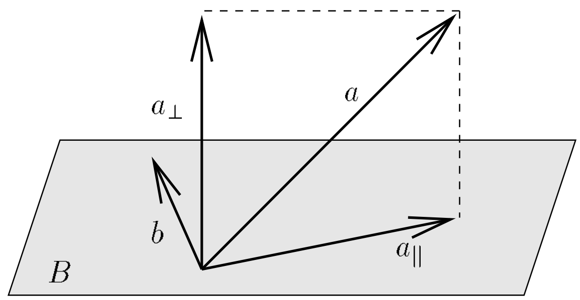
and each bivector generates $90^{\rm O}$ rotations in its own plane. So, for example, we recall that

$$e_1(e_1\wedge e_2) = e_1(e_1e_2) = e_2, \tag{1.6}$$

which returns a vector. The geometric product for vectors extends to all objects in the algebra, so we can form expressions such as $aB$, where $B$ is a bivector. But we have also just seen that $e_1(e_2 \wedge e_3)$ is a trivector, so the result of the product $aB$ can clearly contain both vector and trivector terms.

To understand the properties of the product $aB$ we first decompose $a$ into terms in and out of the plane:

$$a = a_{||} + a_{\perp}$$




We can now write $aB = (a_{||} + a_{\perp})B$. Suppose that we also write

$$B = a_{||} \wedge b \tag{1.7}$$

where $b$ is orthogonal to $a_{||}$ in the $B$ plane. We see that

$$a_{||}B = a_{||}(a_{||} \wedge b) = a_{||}(a_{||}b) = {a_{||}}^2 b \tag{1.8}$$

which is a vector, whereas

$$a_{\perp}B = a_{\perp}(a_{||} \wedge b) = a_{\perp} \wedge a_{||} \wedge b \tag{1.9}$$

which is a trivector. We therefore write

$$aB = a \cdot B + a \wedge B \tag{1.10}$$

where the dot is generalised to mean the *lowest* grade part of the result, while the wedge means the *highest* grade part of the result.

From Eq. (1.8) we see that the $a \cdot B = a_{||} \cdot B$ term projects onto the component of $a$ in the plane, and then rotates this through $90^{\text{o}}$ and dilates by the magnitude of $B$. We also see that

$$a \cdot B = {a_{||}}^2 b = -(a_{||}b)a_{||} = -B \cdot a, \tag{1.11}$$

so the dot product between a vector and a bivector is antisymmetric. Similarly, from Eq. (1.9), the $a \wedge B$ term projects onto the component perpendicular to the plane, and returns a trivector. This term is symmetric

$$a \wedge B = a_{\perp} \wedge a_{||} \wedge b = a_{||} \wedge b \wedge a_{\perp} = B \wedge a. \tag{1.12}$$

The separate vector and trivector terms are wrapped up in the single geometric product $aB$. Again, the advantage of this is that the product is invertible. As with vectors, we can now write the separate dot and wedge products in terms of the geometric product

$$\begin{aligned} a\cdot B &= \tfrac{1}{2}(aB - Ba) \\ a\wedge B &= \tfrac{1}{2}(aB + Ba). \end{aligned} \tag{1.13}$$

## 1.2 The Bivector Algebra

Our three independent bivectors also give us a further new product to consider. When multiplying two bivectors we find, for example, that

$$(e_1\wedge e_2)(e_2\wedge e_3) = e_1e_2e_2e_3 = e_1e_3, \tag{1.14}$$

resulting in a third bivector. We also find that

$$(e_2\wedge e_3)(e_1\wedge e_2) = e_3e_2e_2e_1 = e_3e_1 = -e_1e_3, \tag{1.15}$$

so the product is antisymmetric. The symmetric contribution vanishes because the two planes are perpendicular. If we introduce the following labelling for the basis bivectors:

$$B_1 = e_2e_3, \quad B_2 = e_3e_1, \quad B_3 = e_1e_2 \tag{1.16}$$

we find that the commutator satisfies

$$B_iB_j - B_jB_i = -2\epsilon_{ijk}B_k. \tag{1.17}$$

Not surprisingly, this algebra is closely linked to 3-d rotations, and will be familiar from the quantum theory of angular momentum. The commutator of 2 bivectors always results in a third bivector (or zero). We will learn more about this in later lectures. We also now have ${B_1}^2 = {B_2}^2 = {B_3}^2 = -1$, and $B_1B_2 = -B_2B_1$ *etc.* This recovers the *quaternion* algebra $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$, $\boldsymbol{ij} = -\boldsymbol{ji}$. Quaterions were *bivectors* all along!

## 1.3 Properties of The Trivector

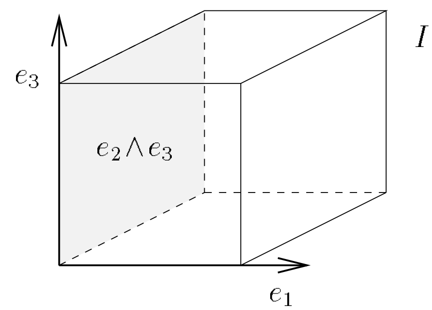
The trivector represents the unique volume element in 3-d. It is the highest grade element in the algebra and again is called the *pseudoscalar*, or *directed volume element.* The latter name is more accurate, but the former is seen more often. (Though be careful of this usage — pseudoscalar can mean different things in different contexts). To simplify, we introduce the symbol $I$,

$$I = e_1e_2e_3. \tag{1.18}$$

As in 2-d, the pseudoscalar is defined by convention to be *right-handed*. This means that in the expression $I = e_1 \wedge e_2 \wedge e_3$, the $\{e_1, e_2, e_3\}$ frame is right-handed. If a left-handed set of orthonormal vectors is multiplied together the result is $-I$.

Now consider the product of a vector and the pseudoscalar,

$$e_1 I = e_1(e_1 e_2 e_3) = e_2 e_3$$




This returns a bivector — the plane perpendicular to the original vector. The product of a grade-1 vector with the grade-3 pseudoscalar is therefore a grade-2 bivector. Multiplying from the left we find that

$$I e_1 = e_1 e_2 e_3 e_1 = -e_1 e_2 e_1 e_3 = e_2 e_3. \tag{1.19}$$

The result is therefore independent of order — the pseudoscalar commutes with all vectors in 3-d, $Ia = aI$. It follows that $I$ commutes with all elements in the algebra. This is always the case for the pseudoscalar in spaces of odd dimension. In even dimensions, the pseudoscalar anticommutes with all vectors, as we have already seen in 2-d.

We can now express each of our basis bivectors as the product of the pseudoscalar and a *dual* vector,

$$e_1 e_2 = I e_3, \qquad e_2 e_3 = I e_1, \qquad e_3 e_1 = I e_2. \tag{1.20}$$

This operation of multiplying by the pseudoscalar is called a *duality* transformation. Again, we can write

$$aI = a \cdot I \tag{1.21}$$

with the dot used to denote the lowest grade term in the product. The result of this can be understood as a projection — projecting onto the component of $I$ perpendicular to $a$.

We next form the square of the pseudoscalar

$$I^2 = e_1 e_2 e_3 e_1 e_2 e_3 = e_1 e_2 e_1 e_2 = -1. \tag{1.22}$$

So the pseudoscalar commutes with all elements and squares to $-1$. It is therefore a further candidate for a unit imaginary. In some physical applications this is the correct one to use, whereas for others it is one of the bivectors. These different possibilities give us a much richer language and provide a number of geometric insights.

Finally, we consider the product of a bivector and the pseudoscalar:

$$I(e_1 \wedge e_2) = Ie_1e_2e_3e_3 = IIe_3 = -e_3 \tag{1.23}$$

So the result of the product of $I$ with the bivector formed from $e_1$ and $e_2$ is $-e_3$, that is, minus the vector perpendicular to the $e_1 \wedge e_2$ plane. This affords a definition of the vector cross product in 3-d as

$$a \boldsymbol{\times} b = -I(a \wedge b) = -I\, a \wedge b. \tag{1.24}$$

The bold $\boldsymbol{\times}$ symbol is used here as we will soon encounter a better use for the $\times$ symbol. We have also started to employ the useful *operator ordering convention* that, in the absence of brackets, *dot and wedge products are performed before geometric products.* This cleans up expressions by enabling us to remove unnecessary brackets.

Equation (1.24) shows how the cross product is a bivector in disguise, the bivector being mapped to a vector by a duality operation. It is also now clear why the product only exists in 3-d — this is the only space for which the dual of a bivector is a vector. We will have little further use for the cross product and will rarely employ it from now on. This means we can also do away with the awkward distinction between axial and polar vectors. Instead we just talk of vectors and bivectors.

## 1.4 Reversion

An important operation in GA is that of reversing the order of vectors in any product. This is denoted with a tilde, $\tilde{A}$. Scalars and vectors are invariant under reversion, but bivectors change sign,

$$(e_1e_2)^\sim = e_2e_1 = -e_1e_2. \tag{1.25}$$

Similarly, we see that

$$\tilde{I} = e_3e_2e_1 = e_1e_3e_2 = -e_1e_2e_3 = -I. \tag{1.26}$$

A general multivector in 3-d can be written

$$M = \alpha + a + B + \beta I. \tag{1.27}$$

From the above we see that

$$\tilde{M} = \alpha + a - B - \beta I. \tag{1.28}$$

### 1.5 Aside — Quantum Spin

The full geometric product for vectors can now be written

$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j = \delta_{ij} + I\epsilon_{ijk} e_k. \tag{1.29}$$

This should be familiar — it is the Pauli algebra of quantum mechanics! This suggests that the matrix structure of spin in quantum theory might have a more geometric origin, and also calls into question one or two of the interpretations currently attached to the Pauli matrix operators (though this is controversial). The relation also shows us that the Pauli matrices form a matrix representation of $\mathcal{G}_3$, providing an alternative way of performing multivector manipulations. The matrix method is usually slower, however.

## 2 Rotations

Recall from Lecture 1 that in 2-d a vector can be rotated through $\theta$ in the $e_1e_2$ plane by any one of the expressions

$$a \mapsto a' = \mathrm{e}^{-e_1e_2\theta}\, a = a\, \mathrm{e}^{e_1e_2\theta} = \mathrm{e}^{-e_1e_2\theta/2}\, a\, \mathrm{e}^{e_1e_2\theta/2}. \tag{2.1}$$

We now want to find a version of this formula appropriate for 3-d. This is a problem with which Hamilton struggled for many years. Any of the above formulae will do for rotating a vector lying in the $e_1e_2$ plane, but we also require that any component outside the plane be unaffected. The key to finding the correct formula is to note that $e_3$ commutes with $e_1e_2$, so

$$\begin{aligned} \mathrm{e}^{-e_1e_2\theta}\, e_3 &= [\cos(\theta) - \sin(\theta)e_1e_2]e_3 \\ &= e_3(\cos(\theta) - \sin(\theta)e_1e_2) \\ &= e_3\, \mathrm{e}^{-e_1e_2\theta}. \end{aligned} \tag{2.2}$$

This makes it clear that only the intermediate, double-sided formula has all of the required properties. It rotates vectors in the $e_1e_2$ plane and leaves vectors perpendicular to the plane untouched:

$$\mathrm{e}^{-e_1e_2\theta/2}\, e_3\, \mathrm{e}^{e_1e_2\theta/2} = e_3\, \mathrm{e}^{-e_1e_2\theta/2}\, \mathrm{e}^{e_1e_2\theta/2} = e_3. \tag{2.3}$$

We therefore arrive at the result that in 3-d a vector is rotated through an angle $\theta$ in the $\hat{B}$ plane ($\hat{B}^2 = -1$) by

$$a \mapsto a' = Ra\tilde{R}, \qquad R = \mathrm{e}^{-\hat{B}\theta/2} \tag{2.4}$$

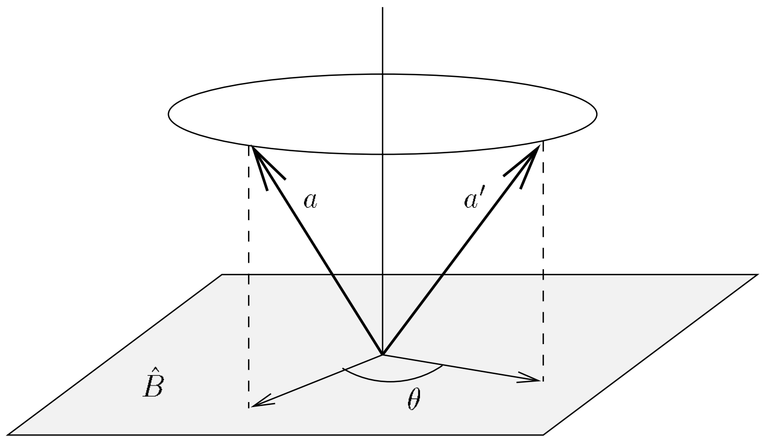



Figure 2: *The GA description of a rotation in 3D.* The vector $a$ is rotated to $a' = Ra\tilde{R}$. The rotor $R$ is defined by $R = \exp(-\hat{B}\theta/2)$ which describes the rotation directly in terms of the plane and angle.

(see Fig. 2). We will derive this result from an alternative route in Lecture 3. The object $R$ is called a *rotor*. It satisfies the normalisation condition

$$R\tilde{R} = \tilde{R}R = 1. \tag{2.5}$$

So $R$ is formed from the scalar + bivector algebra with one constraint, leaving 3 degrees of freedom, as expected for a general rotation.

Now suppose that the two vectors forming the bivector $B = a\wedge b$ are both rotated. What is the expression for the resulting bivector? To find this we form

$$\begin{aligned} B' = a'\wedge b' &= \tfrac{1}{2}(a'b' - b'a') = \tfrac{1}{2}(Ra\tilde{R}Rb\tilde{R} - Rb\tilde{R}Ra\tilde{R}) \\ &= \tfrac{1}{2}(Rab\tilde{R} - Rba\tilde{R}) = \tfrac{1}{2}R(ab - ba)\tilde{R} = Ra\wedge b\tilde{R} = RB\tilde{R}. \end{aligned} \tag{2.6}$$

Bivectors are rotated using precisely the same formula as vectors! The same turns out to be true for all multivectors. This is one of the most attractive features of geometric algebra.

# 3 Angular Momentum

Replacing axial vectors with bivectors forces us to reassess one of the fundamental concepts of mechanics — angular momentum. The angular momentum of a particle with momentum $p$ and position vector $x$ from some origin is usually defined in 3-d in terms of the cross product

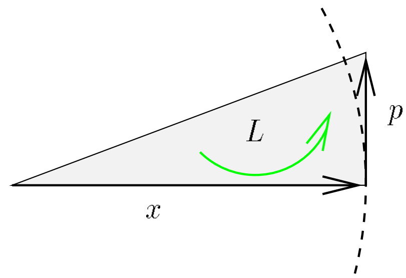
$$L = x \times p. \tag{3.1}$$

But do we want to keep this definition? Suppose we were formulating dynamics entirely in 2-d. Angular momentum is still a sensible notion, but it is certainly not a vector any more, as there is nowhere for it to point. It is clear then that the correct concept for angular momentum is a bivector, so we replace the above definition by

$$L = x \wedge p. \tag{3.2}$$

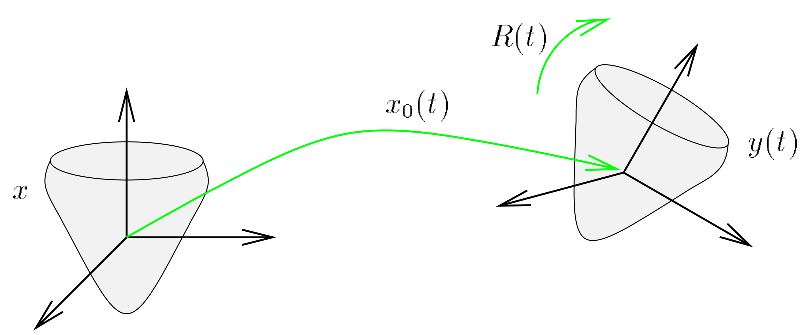
This replaces the clumsy notion of angular momentum as an 'axial vector' with an expression that directly encodes our understanding of angular momentum in terms of a particle sweeping out a plane.

The particle sweeps out the plane $L = x \wedge p$




# 4 Rigid Body Dynamics

We are now in a position to give our first major application of GA. Suppose that a rigid body is moving through space. The vector position of points in the moving body $y(t)$ can be related back to the equivalent positions in a 'reference' body, fixed for all time.




Here $x_0$ is the position in space of the centre of mass. The vectors $y(t)$ and $x$ are related by

$$y(t) = R(t) x \tilde{R}(t) + x_0(t) \tag{4.1}$$

This places all of the rotational motion in the time-dependent rotor $R(t)$.

## 4.1 Angular Velocity

With angular momentum described as a bivector, angular velocity must be as well. This arises naturally in the rotor description. Suppose that the frame of vectors $\{f_k\}$ is rotating in space. These can be related to a fixed orthonormal frame $\{e_k\}$ by

$$f_k(t) = R(t) e_k \tilde{R}(t). \tag{4.2}$$

(These frames can be chosen to be the body and space principal axes of a rigid body, though the formulae below are general.) The angular momentum vector $w$ is traditionally defined by the formula

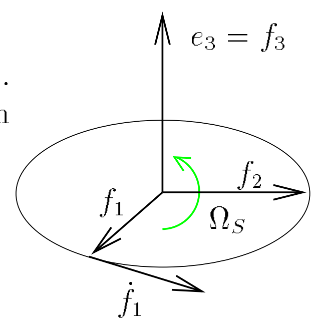
$$\dot{f}_k = \omega \times f_k = -I\,\omega \wedge f_k = (-I\omega)\cdot f_k. \tag{4.3}$$

We therefore introduce the (space) angular-velocity bivector

$$\Omega_S = I\omega. \tag{4.4}$$

The sign ensures that the angular-velocity bivector has the orientation implied by the rotation, as is easily seen for the case of motion about the $\omega = e_3$ axis.

$\Omega_S$ has the orientation of $f_1 \wedge \dot{f}_1$. It must therefore have orientation $+e_1 \wedge e_2$ when $\omega = e_3$.




We next look at the time dependence,

$$\dot{f}_k = \dot{R} e_k \tilde{R} + R e_k \dot{\tilde{R}} = \dot{R}\tilde{R} f_k + f_k R \dot{\tilde{R}}. \tag{4.5}$$

From the normalisation equation $R\tilde{R} = 1$ we see that

$$0 = \partial_t (R\tilde{R}) = \dot{R}\tilde{R} + R\dot{\tilde{R}}. \tag{4.6}$$

It follows that

$$\dot{R}\tilde{R} = -R\dot{\tilde{R}} = -(\dot{R}\tilde{R})^{\sim} \tag{4.7}$$

since the order of differentiation and reversion is interchangeable. The quantity $\dot{R}\tilde{R}$ is equal to minus its own reverse and has even grade, so must be a pure bivector. The equation for $f_k$ now becomes

$$\dot{f}_k = \dot{R}\tilde{R} f_k - f_k \dot{R}\tilde{R} = (2\dot{R}\tilde{R})\cdot f_k. \tag{4.8}$$

Comparing with Eq. (4.3) and Eq. (4.4) we see that $2\dot{R}\tilde{R}$ must equal minus the angular velocity bivector $\Omega_S$,

$$2\dot{R}\tilde{R} = -\Omega_S. \tag{4.9}$$

The dynamics therefore reduces to the single *rotor equation*

$$\dot{R} = -\tfrac{1}{2}\Omega_S R. \tag{4.10}$$

The reversed form is also useful,

$$\dot{\tilde{R}} = \tfrac{1}{2}\tilde{R}\Omega_S. \tag{4.11}$$

Equations like these are very common in physics, and are much easier to solve than their matrix counterparts. We will see many further examples in later lectures.

The equations can be expressed in terms of either the 'space' $\Omega_S$ or 'body' $\Omega_B$ angular velocities. The body angular velocity is the bivector $\Omega_S$ expressed back in the fixed reference copy. The two are related by

$$\Omega_S = R\Omega_B\tilde{R}. \tag{4.12}$$

In terms of these we have

$$\dot{R} = -\tfrac{1}{2}\Omega_S R = -\tfrac{1}{2}R\Omega_B, \quad \text{and} \quad \dot{\tilde{R}} = \tfrac{1}{2}\Omega_B\tilde{R}. \tag{4.13}$$

As an elementary example, suppose that the body is rotating on a fixed axle, so that $\Omega_S$ is a fixed constant. The rotor equation then integrates immediately to give

$$R(t) = \mathrm{e}^{-\Omega_S t/2}\, R(0) \tag{4.14}$$

which is the rotor for a constant frequency rotation in the positive sense in the $\Omega_S$ plane.

## 4.2 The Inertia Tensor

Suppose now that the rigid body has density $\rho = \rho(x)$. The position vector $x$ is taken relative to the centre of mass, so we have

$$\int d^3x\, \rho = m, \quad \text{and} \quad \int d^3x\, \rho x = 0. \tag{4.15}$$

The velocity of the point $y = Rx\tilde{R} + x_0$ is

$$\begin{aligned} v(t) &= \dot{R}x\tilde{R} + Rx\dot{\tilde{R}} + \dot{x}_0 \\ &= -\tfrac{1}{2}R\Omega_B x\tilde{R} + \tfrac{1}{2}Rx\Omega_B\tilde{R} + v_0 \\ &= R\, x\cdot\Omega_B\, \tilde{R} + v_0 \end{aligned} \tag{4.16}$$

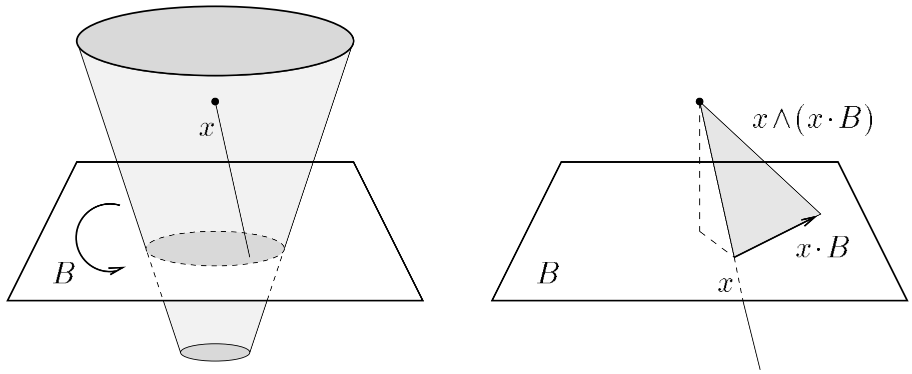



Figure 3: *The Inertia Tensor.* The inertia tensor $\mathcal{I}(B)$ is a linear function mapping its bivector argument $B$ onto a bivector. It returns the total angular momentum about the centre of mass for rotation in the $B$ plane.

where $v_0$ is the velocity of the centre of mass. Again we have suppressed unnecessary brackets to write $R\, x\cdot\Omega_B\, \tilde{R}$ in place of $R(x\cdot\Omega_B)\tilde{R}$.

The quantity we require is the angular momentum bivector $L$ for the body about its centre of mass. We therefore form

$$\begin{aligned} L &= \int d^3x\, \rho(y-x_0)\wedge v \\ &= \int d^3x\, \rho(Rx\tilde{R})\wedge(R\, x\cdot\Omega_B\, \tilde{R}+v_0) \\ &= R\left(\int d^3x\, \rho x\wedge(x\cdot\Omega_B)\right)\tilde{R} \end{aligned} \tag{4.17}$$

We now introduce the *inertia tensor* $\mathcal{I}(B)$, defined by

$$\mathcal{I}(B) = \int d^3x\, \rho x\wedge(x\cdot B). \tag{4.18}$$

This is a linear function mapping bivectors to bivectors. (Linear functions will be dealt with in more detail in lecture 5). This replaces the idea of the inertia tensor mapping vectors to vectors. Again this accords well with our geometrical intuition of this tensor (see Fig. (3)). If the body rotates in the $B$ plane, with rotation rate fixed by $|B|$, then the momentum denisty is $\rho x\cdot B$. The angular momentum density bivector is therefore $x\wedge(\rho x\cdot B)$. Integrating this over the entire body returns the total angular momentum bivector for rotation in the $B$ plane. In general the result will not be the same plane as $B$, but it will be if $B$ is perpendicular to one of the principal axes.

The inertia tensor is constructed from the point of view of the fixed body. The space angular momentum requires a further rotation,

$$L = R\mathcal{I}(\Omega_B)\,\tilde{R}. \tag{4.19}$$

The equations of motion are $\dot{L} = T$, where $T$ is the external torque (also a bivector). The inertia tensor is *time-independent*, as it only refers to the fixed 'reference' copy of the the rigid body, so we find that

$$\begin{aligned} \dot{L} &= \dot{R}\mathcal{I}(\Omega_B)\tilde{R} + R\mathcal{I}(\Omega_B)\dot{\tilde{R}} + R\mathcal{I}(\dot{\Omega}_B)\tilde{R} \\ &= R[\mathcal{I}(\dot{\Omega}_B) - \tfrac{1}{2}\Omega_B\mathcal{I}(\Omega_B) + \tfrac{1}{2}\mathcal{I}(\Omega_B)\Omega_B]\tilde{R} \\ &= R[\mathcal{I}(\dot{\Omega}_B) - \Omega_B \times \mathcal{I}(\Omega_B)]\tilde{R}. \end{aligned} \tag{4.20}$$

Here we have introduced the extremely useful commutator product

$$A \times B = \tfrac{1}{2}(AB - BA). \tag{4.21}$$

There should be little potential for confusion with the cross product, $a \boldsymbol{\times} b$, as we have eliminated any need for the latter. The torque-free equation $\dot{L} = 0$ reduces to

$$\mathcal{I}(\dot{\Omega}_B) - \Omega_B \times \mathcal{I}(\Omega_B) = 0. \tag{4.22}$$

We usually align our body frame $\{e_k\}$ with the principal axes of the rigid body. If the principal moments of inertia are $i_k, k = 1 \ldots 3$, we then have

$$\Omega_B = \sum_k \omega_k I e_k, \qquad \Omega_S = \sum_k \omega_k I f_k \tag{4.23}$$

and

$$L = \sum_k i_k \omega_k I f_k. \tag{4.24}$$

Expanding out Eq. (4.22) recovers the Euler equations in component form.

## 4.3 Example — The Symmetric Top

We will now work through an example to show how the GA approach can simplify the steps to finding a solution. Suppose the body has two equal moments of inertia, $i_1 = i_2 \neq i_3$. We can write

$$\mathcal{I}(B) = i_1 B + (i_3 - i_1)(B \wedge e_3)e_3, \tag{4.25}$$

where we have used the fact $B \wedge e_3$ is a trivector to write the final term as a geometric product $(B \wedge e_3)e_3$. It follows that

$$\mathcal{I}(\dot{\Omega}_B) = \Omega_B \times \mathcal{I}(\Omega_B) = (i_3 - i_1)\Omega_B \times [(\Omega_B \wedge e_3)e_3]. \tag{4.26}$$

But since $\Omega_B \wedge e_3$ is a trivector, we can use the result that

$$B \times (Ia) = \tfrac{1}{2}(BIa - IaB) = -a \wedge (IB) \tag{4.27}$$

to arrive at

$$\mathcal{I}(\dot{\Omega}_B) = -(i_3 - i_1)e_3 \wedge [(\Omega_B \wedge e_3)\Omega_B]. \tag{4.28}$$

It follows that

$$e_3 \wedge \mathcal{I}(\dot{\Omega}_B) = i_3 \dot{\omega}_3 I = -(i_3 - i_1)e_3 \wedge e_3 \wedge [(\Omega_B \wedge e_3)\Omega_B] = 0, \tag{4.29}$$

which shows that $\omega_3$ is a constant. This ability to deduce useful consequences without dropping down to the individual component equations becomes ever more valuable as the complexity of the system increases.

Next we use the result that

$$i_1 \Omega_B = \mathcal{I}(\Omega_B) - (i_3 - i_1)(\Omega_B \wedge e_3)e_3 = \mathcal{I}(\Omega_B) + (i_1 - i_3)\omega_3 I e_3 \tag{4.30}$$

to write

$$\Omega_S = R\Omega_B \tilde{R} = \frac{1}{i_1} L + \frac{i_1 - i_3}{i_1} \omega_3 R I e_3 \tilde{R}. \tag{4.31}$$

Our rotor equation now becomes

$$\dot{R} = -\tfrac{1}{2}\Omega_S R = -\frac{1}{2i_1}(LR + R(i_1 - i_3)\omega_3 I e_3). \tag{4.32}$$

We therefore define two *constant* precession rates, $\Omega_l$ and $\Omega_r$, acting to the left and right of $R$:

$$\Omega_l = \frac{1}{i_1} L, \qquad \Omega_r = \omega_3 \frac{i_1 - i_3}{i_1} I e_3. \tag{4.33}$$

In terms of these the rotor equation becomes

$$\dot{R} = -\tfrac{1}{2}\Omega_l R - \tfrac{1}{2} R \Omega_r, \tag{4.34}$$

which integrates immediately to give

$$R(t) = \exp(-\tfrac{1}{2}\Omega_l t) R(0) \exp(-\tfrac{1}{2}\Omega_r t). \tag{4.35}$$

This fully describes the motion of a symmetric top. It shows that there is an 'internal' rotation in the $e_1 e_2$ plane (a symmetry of the body). This is responsible for the precession of a symmetric top. The resultant body is then rotated in the plane of its angular momentum.