Physical Applications of Geometric Algebra

Handout 16

Spherically Symmetric Fields

In the previous handout we used some physical arguments to motivate a solution of the field equations, which turned out to describe the fields outside any spherically symmetric source. In this final handout we explore some of the properties of this solution by looking at the motion of particles and photons. We find that a *horizon* is present and that once inside the horizon all matter, including photons, must terminate on a central singularity. The horizon is located at precisely the place predicted by Newtonian arguments. We also look at some of the properties of stationary observers, and end with a comparison with the Schwarzschild line element employed in GR.

1 Freely-Falling Observers

We motivated our solution with considerations of Newtonian trajectories for infall. However, we know that the correct covariant equation for free-fall is

$$v\cdot\mathcal{D}v = \dot{v} + \Omega(\dot{x})\cdot v = 0. \tag{1.1}$$

The first thing to establish, then, is whether $v = e_t$ is a solution of this equation. The trajectory still has

$$\dot{x} = \mathsf{h}(v) = \mathsf{h}(e_t) = e_t + ue_r \tag{1.2}$$

with

$$u = -\sqrt{(2GM/r)}. \tag{1.3}$$

(Note that we have defined the velocity u to be *negative*.) We therefore have

$$\Omega(\dot{x}) = \Omega[\mathsf{h}(e_t)] = \Omega(e_t) + u\Omega(e_r) = 0. \tag{1.4}$$

It follows that

$$e_t\cdot\mathcal{D}e_t = \partial_\tau e_t + \Omega[\mathsf{h}(e_t)]\cdot e_t = 0, \tag{1.5}$$

and we do indeed have a solution. Observers freely-falling from infinity do follow the *Newtonian* trajectory. This ultimately justifies the approach used to arrive at a solution.

1.1 General, Radial Free-Fall

Now consider general, radial free-fall where the particle has not necessarily fallen from rest at infinity. We must sill have $v^2 = 1$ and v must be constructed from e_t and e_r only for radial motion. We can therefore write

$$v = \mathrm{e}^{\alpha \boldsymbol{\sigma}_r} e_t = \cosh(\alpha) e_t + \sinh(\alpha) e_r. \tag{1.6}$$

It follows that

$$\dot{v} = \dot{\alpha} \boldsymbol{\sigma}_r \cdot v \tag{1.7}$$

and

$$\Omega(\dot{x}) = \cosh(\alpha)\Omega[\mathsf{h}(e_t)] + \sinh(\alpha)\Omega[\mathsf{h}(e_r)] = -\sinh(\alpha)\frac{GM}{r^2 u}\boldsymbol{\sigma}_r. \tag{1.8}$$

The free-fall equation therefore reduces to

$$\dot{\alpha} = \sinh(\alpha)\frac{GM}{r^2 u} \tag{1.9}$$

and $\dot{x} = \mathsf{h}(v)$ gives

$$\dot{t} = \cosh(\alpha) \tag{1.10}$$

$$\dot{r} = \sinh(\alpha) + u\cosh(\alpha). \tag{1.11}$$

We arrive at a set of three first-order equations, which are sufficient to specify a unique trajectory, given initial values of position (r and t) and velocity ($\tanh \alpha$).

Taking the second derivative of the $\dot{r}$ equation, we find that

$$\ddot{r} = -\frac{GM}{r^2} \tag{1.12}$$

so the Newtonian force law is still present. The differences with Newtonian physics now lie in the meaning of the variables. The variable r is now a *local* observable, fixed by the magnitude of the tidal force. It is no longer just a coordinate, and it is also no longer the proper distance from the source. Similarly, the derivatives in $\ddot{r}$ are taken with respect to the local particle proper time, rather than a global Newtonian time. This transition from global to local variables is in keeping with the gauging process. Often the trick is to find a gauge and a set of global coordinates such that the values of the coordinates coincide with local, physical observables.

The equation for $\dot{r}$ contains a further surprise. On writing

$$\dot{r}/\cosh(\alpha) = \tanh(\alpha) - \surd(2GM/r) \tag{1.13}$$

we see that if $2GM/r > 1$ then $\dot{r}$ is necessarily negative. There is no way for the particle to escape. The place where this happens, $r = 2GM = 2GM/c^2$, is the radius where the escape velocity $\sqrt{2GM/r}$ is greater than the speed of light. This is called the Schwarzschild radius, though the possibility of bodies becoming so dense that light could not escape was first suggested by John Michell ($\sim$ 1782).

1.2 Incoming Photons

The simplest way to study the properties of electromagnetic waves in a gravitational background is to use the geometric optics approximation and work with photons as point particles. These particles follow *null* trajectories with

$$k = \mathsf{h}^{-1}(\dot{x}), \qquad k^2 = 0. \tag{1.14}$$

The trajectories are still specified by the equation $k\cdot\mathcal{D}k = 0$. For radial *infall* we must have

$$k = \omega(e_t - e_r), \tag{1.15}$$

where $\omega = k \cdot e_t$ is the frequency measured by free-falling observers (at rest at infinity). The path for this k has

$$\dot{x} = \mathsf{h}(v) = \omega[e_t - (1 + \sqrt{}(2GM/r))e_r] \tag{1.16}$$

and so we find that

$$\frac{dr}{dt} = -(1 + \sqrt{}(2GM/r)). \tag{1.17}$$

This integrates straightforwardly to give the photon path. We have therefore found the path without employing the equation of motion. What the latter tells us, in this case, is how the frequency changes along the path. To find this we need

$$\Omega(\dot{x}) = \omega\Omega[\mathsf{h}(e_t)] - \omega\Omega[\mathsf{h}(e_r)] = \omega\frac{GM}{r^2 u}\boldsymbol{\sigma}_r, \tag{1.18}$$

from which we see that

$$\dot{\omega} = \omega^2\frac{GM}{r^2 u}. \tag{1.19}$$

This equations is more usefully expressed in terms of r. We use

$$\dot{r} = -\omega[1 + \sqrt{}(2GM/r)] \tag{1.20}$$

to arrive at

$$\frac{1}{\omega}\frac{d\omega}{dr} = \frac{GM}{r}\frac{1}{2GM + \sqrt{}(2GMr)} = \frac{1}{2r}\frac{1}{\sqrt{r/r_S} + 1}, \tag{1.21}$$

where $r_S = 2GM$ is the Schwarzschild radius. This equation can again be integrated straightforwardly to tell us how frequency ω changes with radius. We see that nothing untoward happens until $r = 0$ is reached.

1.3 Outgoing Photons

We now repeat the previous analysis for outgoing photons. For this case we have

$$k = \omega(e_t + e_r) \tag{1.22}$$

and the path is

$$\dot{x} = \mathsf{h}(v) = \omega[e_t + (1 - \sqrt{(2GM/r)})e_r]. \tag{1.23}$$

It follows that

$$\frac{dr}{dt} = 1 - \sqrt{(2GM/r)}. \tag{1.24}$$

But now, when $r < 2GM$ the path is still *inwards*. Inside $r = 2GM$, not even light can escape. The surface $r = 2GM$ is called the *event horizon*. It marks the boundary between two regions, one of which (the interior in this case) cannot signal to the other.

If any object collapses to within its event horizon, it must carry on collapsing to form a central *singularity*. There is no possible force capable of preventing the collapse. This is because matter is always constrained to follow timelike paths, and if the entire future light-cone points inwards towards the singularity, no matter can escape. The object remaining at the end of this process is called a *black hole*. All paths for infalling matter terminate on the singularity. There has been much discussion of the properties of singularities in the GR literature. They are viewed as being highly problematic because the entire structure of spacetime breaks down at a singularity. The gauge theory perspective is rather different. In gauge theory gravity, gravitational singularities are no more difficult to deal with than singularities in the electromagnetic field due to point sources. They are also analysed in much the same way — using integral equations. This appears to be one of the main areas of difference between GR and gauge theory gravity. Sadly, these differences are hidden behind horizons, so are likely to prove difficult to investigate!

We also now find that

$$\frac{1}{\omega}\frac{d\omega}{dr} = \frac{GM}{r}\frac{1}{2GM - \sqrt{(2GMr)}} = -\frac{1}{2r}\frac{1}{\sqrt{r/r_S} - 1}, \tag{1.25}$$

which is negative outside the horizon. So, as photons climb out of a gravitational field, they are *red-shifted*. This is one of the best tested predictions of GR (and gauge theory gravity). The redshift becomes increasingly large as the horizon is approached, so photons emitted from near the horizon are strongly redshifted as they climb out to infinity. The various features of radial motion in a black hole background are shown in Figure 1. One conclusion from this plot is that, as seen by *external observers*, any object falling through the horizon appears to hover outside the horizon and just fade out of existence as the redshift increases.

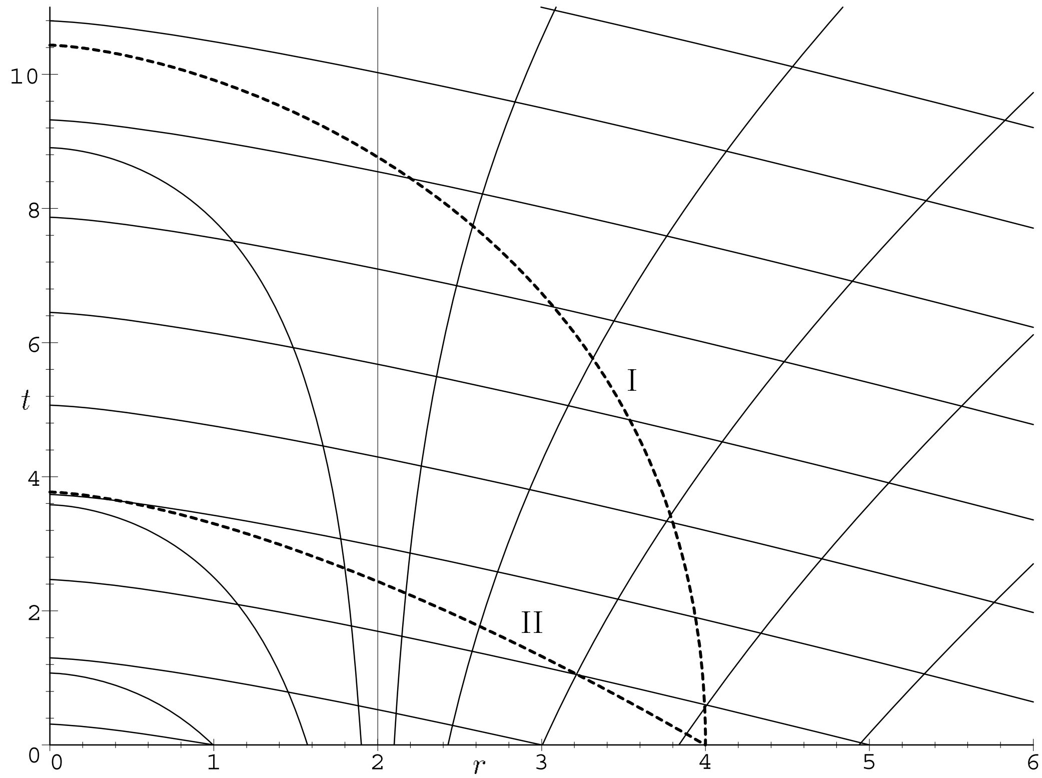

Figure 1: *Matter and photon trajectories in a black hole background.* The solid lines are photon trajectories, and the horizon lies at r=2. Outside the horizon it is possible to send photons out to infinity, and hence communicate with the rest of the universe. As one approaches the horizon, these photons are strongly redshifted and take a long time to escape. Once inside the horizon, all photon paths end on the singularity. The broken lines represent two possible trajectories for infalling matter. Trajectory I is for a particle released from rest at $r = 4$. Trajectory II is for a particle released from rest at $r = \infty$.

2 Stationary Observers

As well as observers in free-fall, it is useful to see how physics looks from the point of view of *stationary observers*. These have constant r, θ, ϕ, so

$$\dot{x} = \dot{t} e_t. \tag{2.1}$$

It follows that

$$v = \dot{t}(e_t + \sqrt{(2GM/r)} e_r). \tag{2.2}$$

But we require that $v^2 = 1$ for the path to be parameterised by the observer's proper time, so

$$\dot{t}^2(1 - 2GM/r) = 1, \qquad \dot{t} = (1 - 2GM/r)^{-1/2}. \tag{2.3}$$

This is a constant, since r is fixed for these observers. We can see immediately that it is only possible to remain at rest *outside* the horizon. This is reasonable given the preceding considerations, though the picture is not quite so clear if the black hole is rotating. For this case there is a region outside the horizon within which it is impossible to remain at rest (though it is still possible to escape).

We can define the acceleration bivector *covariantly* as

$$v\cdot\mathcal{D}v\, v = \dot{v}v + \Omega(\dot{x})\cdot v\, v. \tag{2.4}$$

This gives the acceleration required to follow a given path. For stationary observers we have

$$v\cdot\mathcal{D}v\, v = \Omega(\dot{x}) = \frac{GM}{r^2(1 - 2GM/r)^{1/2}}\boldsymbol{\sigma}_r. \tag{2.5}$$

So an observer with mass m needs to apply force of $GMm/r^2 \times (1 - 2GM/r)^{-1/2}$ to remain at rest. This is the Newtonian value multiplied by a relativistic correction term. This correction gets increasingly large as the horizon is approached, as one would expect.

We can now look at physics from point of view of these observers, which can be viewed as both being stationary and having constant acceleration. For example, if a second observer has velocity γ_0, (so is in free-fall) the relative velocity the two observers measure when their positions coincide is

$$\frac{v\wedge\gamma_0}{v\cdot\gamma_0} = \sqrt{(2GM/r)}\boldsymbol{\sigma}_r \tag{2.6}$$

which is precisely the Newtonian result! This prediction is gauge invariant. The magnitude of this velocity is a gauge scalar expressed in terms of local observables, and the direction is also physically fixed as an eigendirection of the Riemann tensor.

3 The Schwarzschild Metric

We finish with a brief comparison with the GR treatment of this problem. Our h-field produces the line element

$$ds^2 = (1 - 2GM/r)dt^2 - 2\sqrt{(2GM/r)}dt\, dr - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{3.1}$$

The off-diagonal term here makes the line element appear unnecessarily complicated from a GR perspective, though of course we know that the underlying gauge field is remarkably simple! In GR one often removes the off-diagonal term by introducing a new time coordinate t', with

$$dt = dt' + \alpha(r) dr \tag{3.2}$$

and $\alpha(r)$ is chosen to make the line element diagonal. In this case we arrive at the Schwarzschild form of the metric,

$$ds^2 = (1 - 2GM/r) dt'^2 - (1 - 2GM/r)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2(\theta) d\phi^2). \tag{3.3}$$

But if we set $r = 2GM$ in Eq. (3.1), we see that the coefficient of the $dr\, dt$ cross term is -2. This cannot be changed by a coordinate transformation of the form of Eq. (3.2) unless the term $\alpha(1 - 2GM/r)$ is finite at the horizon. In this case the transformation is *singular* there.

In gauge theory terms we say that the displacement necessary to arrive at a diagonal form is not defined globally. So, if a horizon is present, the diagonal gauge is *not valid* and one has to work with a global solution which generates an off-diagonal term in the line element. GR expresses this rather differently, by saying that the coordinates are only valid locally, whereas the full solution requires patching together different coordinate systems for different parts of spacetime. This is a further area with the potential for disagreement between the gauge theory approach and GR. Again, outside horizons, the theories are in complete agreement.