Physical Applications of Geometric Algebra

Handout 7

Geometric Calculus

Geometric algebra has provided us with a new, invertible product for vectors. We now investigate what insights this brings to the subject of vector calculus. We are used to the grad, div and curl operators, which are obtainable from a single, *vector derivative*. This operator also turns out to lie at the heart of complex analysis, providing a means of extending the results of complex analysis to higher dimensions. This synthesis of vector differentiation and geometric algebra is called '*geometric calculus*'.

1 The Vector Derivative

For studying fields in geometric algebra we represent position in an n-dimensional space by the vector x. The *vector derivative*, ∇ is the derivative with respect to position x. In terms of a fixed frame $\{e^k\}$, with coordinates $x^k = e^k \cdot x$, we can write

$$\nabla = \sum_k e^k \frac{\partial}{\partial x^k}. \tag{1.1}$$

The essential new feature is that the $\{e^k\}$ vectors belong to a geometric algebra, so inherit a full geometric product.

If we dot ∇ with the vector a we obtain the *directional derivative* in the a direction

$$a \cdot \nabla F(x) = \lim_{\epsilon \mapsto 0} \frac{F(x + \epsilon a) - F(x)}{\epsilon}. \tag{1.2}$$

This can be used as an alternative starting point to determine the properties of ∇. The argument $F(x)$ here can be any multivector-valued function of position, or more generally a position-dependent linear function.

1.1 Grad, Div and Curl

Acting on a scalar field $\phi(x)$, the vector derivative returns the *gradient* $\nabla\phi$. This is a vector whose components in the $\{e^k\}$ frame are the partial derivatives with respect to the x^k coordinates. In Euclidean spaces $\nabla\phi$ points in the direction of steepest increase of ϕ. In mixed signature spaces, such as Minkowski spacetime, the picture is not always so obvious.

The next thing we can do with the vector derivative is to dot it with a vector field $J(x)$. This gives

$$\nabla\cdot J = \frac{\partial}{\partial x^k} e^k \cdot J = \frac{\partial J^k}{\partial x^k} = \partial_k J^k, \tag{1.3}$$

which is the *divergence* of $J(x)$. Here we have employed the useful abbreviation

$$\partial_i = \frac{\partial}{\partial x^i}. \tag{1.4}$$

This is only the scalar part of the full geometric product of two vectors. What is the other term? We form

$$\nabla\wedge J = e^i \wedge (\partial_i J) = e^i \wedge e^j \, \partial_i J_j. \tag{1.5}$$

The components are the antisymmetrised terms in $\partial_i J_j$. In 3-d these are the components of the *curl*, so

$$\nabla\wedge J = I\, \nabla \boldsymbol{\times} J \tag{1.6}$$

Of course, $\nabla\wedge J$ is a bivector, rather than an (axial) vector. We now have a successful generalisation of the curl to arbitrary dimensions.

1.2 Multivector Fields

The preceding definitions extend simply to the case of the vector derivative acting on a multivector field. We have

$$\nabla A = e^k \partial_k A, \tag{1.7}$$

and for an r-grade multivector field A_r we write

$$\nabla\cdot A_r = \langle \nabla A_r \rangle_{r-1}, \quad \nabla\wedge A_r = \langle \nabla A_r \rangle_{r+1}. \tag{1.8}$$

These are usually referred to as the divergence and curl respectively. A useful result for the curl is that the curl of a curl vanishes,

$$\nabla\wedge(\nabla\wedge A) = e^i \wedge \partial_i (e^j \wedge \partial_j A) = e^i \wedge e^j \wedge (\partial_i \partial_j A) = 0, \tag{1.9}$$

which holds because $e^i \wedge e^j$ is antisymmetric and partial derivatives commute. Similarly, the divergence of a divergence vanishes,

$$\nabla\cdot(\nabla\cdot A) = 0, \tag{1.10}$$

which is proved in the same way, or by using duality. (By convention, the inner product of a vector and a scalar is zero.)

Because ∇ is a vector, it does not necessarily commute with other multivectors. We therefore need to be careful in describing the scope of the operator. We use the following conventions:

1. In the absence of brackets, ∇ acts on the object to its immediate right.
2. When the ∇ is followed by brackets, the derivative acts on all of the terms in the brackets.
3. When the ∇ acts on a multivector to which it is not adjacent, we use overdots to describe the scope.

The overdots work as follows:

$$\dot{\nabla} A \dot{B} = e^k A \, \partial_k B \tag{1.11}$$

which encodes the fact that the A is not differentiated. With this notation we can write

$$\nabla(AB) = \nabla AB + \dot{\nabla} A \dot{B} \tag{1.12}$$

This is a form of *Leibniz' rule*. We also employ this notation for linear functions

$$\dot{\nabla}\dot{\mathsf{f}}(a) = \nabla \mathsf{f}(a) - e^k \mathsf{f}(\partial_k a) \tag{1.13}$$

so that $\dot{\nabla}\dot{\mathsf{f}}(a)$ only differentiates the position-dependence in the linear function, and not in its argument.

1.3 Linear Algebra

A number of derivations in linear algebra can be cleaned up by replacing frame contractions by vector derivatives. For example, we have the basic relation

$$\nabla(x \cdot a) = e^i \partial_i (x^j e_j) \cdot a = e^i \, e_j \cdot a \, \delta_i^j = e^i \, e_i \cdot a = a \tag{1.14}$$

So differentiating a function which depends linearly on x is equivalent to forming contractions over frame indices. To exploit this, we often introduce a new vector variable, usually a, and denote the derivative with respect to a by ∂_a. We can then write the formulae from Handout 5, Section 2.9 in the form

$$\begin{aligned} \partial_a \, a \cdot A_r &= r A_r \\ \partial_a \, a \wedge A_r &= (n-r) A_r \\ \dot{\partial}_a A_r \dot{a} &= (-1)^r (n-2r) A_r . \end{aligned} \tag{1.15}$$

We can also write the trace of a linear function as

$$\mathrm{Tr}(\mathsf{f}) = \partial_a \cdot \mathsf{f}(a). \tag{1.16}$$

This is a useful notational device. It enables us to write terms which do not depend on a frame in a way that reflects this independence. This helps to bring out the intrinsic, geometric content of an equation.

1.4 Curvilinear Coordinates

For many applications we work in non-Cartesian coordinate systems. A coordinate system is defined by a set of scalar functions $\{x^i(x)\}$ defined over some region. A function $F(x)$ can be re-expressed in terms of the coordinates as $F(x^i)$. A simple application of the chain rule gives

$$\nabla F = \nabla x^i\, \partial_i F = e^i\, \partial_i F. \tag{1.17}$$

This defines the (contravariant) frame vectors $\{e^i\}$ as

$$e^i = \nabla x^i. \tag{1.18}$$

In Euclidean spaces these vectors are perpendicular to the surfaces of constant x^i. The vectors have vanishing curl, since

$$\nabla \wedge e^i = \nabla \wedge (\nabla x^i) = 0. \tag{1.19}$$

The reciprocal frame vectors are the (covariant) coordinate vectors

$$e_i = \partial_i x. \tag{1.20}$$

These are formed by increasing the x^i coordinate, keeping all others fixed. The two frames are reciprocal because

$$e_i \cdot e^j = (\partial_i x) \cdot \nabla x^j = \partial_i x^j = \delta_i^j. \tag{1.21}$$

In practice it is useful to be able to work with both frames where necessary, with the two related by Eq. 2.8 of Handout 5. In elementary approaches this distinction is often obscured by restricting to orthogonal frames and introducing 'weighting factors' to write

$$e_i = h_i \hat{e}_i, \quad e^i = h_i^{-1} \hat{e}_i. \tag{1.22}$$

This is not always a good approach, however, particularly if the signature is not Euclidean.

2 Geometric Calculus in 2-d

The vector derivative combines the algebraic properties of geometric algebra with vector calculus in a simple and natural way. Many of the preceding formulae can be constructed using other systems, such as tensors (and differential forms). What makes geometric algebra unique is the geometric product. Does this offer any advantages when it comes to studying the properties of the vector derivative? To answer this, we return to 2 dimensions.

2.1 The 2-d Vector Derivative

We start by looking at the vector derivative in 2-d. We write the vector x in terms of a right-handed orthonormal frame as

$$x = x^1 e_1 + x^2 e_2 = \mathrm{x} e_1 + \mathrm{y} e_2. \tag{2.1}$$

Note the different fonts to distinguish the scalar coordinate x from the vector x. The vector derivative is now

$$\nabla = e_1 \partial_{\mathrm{x}} + e_2 \partial_{\mathrm{y}} = e_1(\partial_{\mathrm{x}} + I \partial_{\mathrm{y}}) \tag{2.2}$$

with $I = e_1 e_2$. Now suppose we let this act on the vector $a = u e_1 - v e_2$. We find that

$$\nabla a = (e_1 \partial_{\mathrm{x}} + e_2 \partial_{\mathrm{y}})(u e_1 - v e_2) = \frac{\partial u}{\partial \mathrm{x}} - \frac{\partial v}{\partial \mathrm{y}} - I \left(\frac{\partial v}{\partial \mathrm{x}} + \frac{\partial u}{\partial \mathrm{y}} \right). \tag{2.3}$$

The two terms here are precisely the ones that vanish in the Cauchy-Riemann equations! There is clearly a close link between complex analysis and the 2-d vector derivative. To bring this out, we introduce the 'complex' field ψ,

$$\psi = a e_1 = u + I v. \tag{2.4}$$

The statement that ψ is analytic (satisfies the Cauchy-Riemann equations) now reduces to the equation

$$\nabla \psi = 0. \tag{2.5}$$

This is the fundamental equation which can be generalised immediately to higher dimensions. Applications include:

- In 3-d, with ψ an arbitrary even-grade multivector, Eq. (2.5) defines the spin harmonics, which are fundamental to the Pauli and Dirac theories of electron orbitals.

- In spacetime we have

$$\nabla = e^0 \partial_t + e^i \partial_{x^i}, \qquad i = 1 \dots 3. \tag{2.6}$$

  We will see shortly that special relativity requires the e^0 vector to have opposite sign magnitude to the $\{e^i\}$. With ψ an even-grade multivector, the equation $\nabla\psi = 0$ is then the wavefunction for a massless fermion (*e.g.* a neutrino).

- A mass term on the right-hand side of the neutrino equation converts it to the equation for a massive fermion (*e.g.* an electron).

- Restricting ψ to be a pure bivector in spacetime gives the equation $\nabla F = 0$, which encodes *all* of the Maxwell equations for free-field electromagnetism.

It is no exaggeration to say that that Eq. (2.5) is one of the most studied equations in physics, yet few people are aware of the fact that all of the preceding examples are special cases of the same underlying mathematics.

2.2 Analytic Functions

To complete the link with complex analysis we first recall that

$$\begin{aligned} z &= e_1 x = \mathrm{x} + I\mathrm{y} \\ z^* &= \mathrm{x} - I\mathrm{y} = x e_1 = e_1(-e_2 x e_2), \end{aligned} \tag{2.7}$$

where e_1 is the real axis. The final form of z^* is included to illustrate how the operation of complex conjugation is a reflection. The complex partial derivatives are defined to have the properties

$$\begin{aligned} \partial_z z &= 1 \qquad & \partial_z z^* &= 0 \\ \partial_{z^*} z &= 0 \qquad & \partial_{z^*} z^* &= 1 \end{aligned} \tag{2.8}$$

From these we see that

$$\partial_z = \tfrac{1}{2}(\partial_\mathrm{x} - I\partial_\mathrm{y}), \quad \partial_{z^*} = \tfrac{1}{2}(\partial_\mathrm{x} + I\partial_\mathrm{y}). \tag{2.9}$$

An analytic function is one that depends on z alone. That is, we can write $\psi(\mathrm{x}+I\mathrm{y}) = \psi(z)$. The function is therefore independent of z^*, and we have

$$\partial_{z^*}\psi(z) = 0. \tag{2.10}$$

This is what the 'limit' argument often presented is actually all about! Comparing the preceding forms, we see that this equation is equivalent to

$$\tfrac{1}{2}\left(\frac{\partial}{\partial \mathrm{x}} + I\frac{\partial}{\partial \mathrm{y}}\right)\psi = \tfrac{1}{2}e_1\nabla\psi = 0 \tag{2.11}$$

recovering our vector equation.

It is instructive to see why solutions to $\nabla\psi = 0$ can be constructed as power series in z. We first see that

$$\nabla z = \nabla(e_1 x) = 2e_1 \cdot \nabla x - e_1 \nabla x = 2e_1 - 2e_1 = 0. \tag{2.12}$$

This little manipulation drives most of analytic function theory! It follows immediately, for example, that

$$\nabla(e_1 x - z_0)^n = n\nabla(e_1 x - z_0)(e_1 x - z_0)^{n-1} = 0, \tag{2.13}$$

so a Taylor series expansion in z about z_0 automatically returns an analytic function. We will delay looking at Laurent series until we have established the link with integration.

2.3 Generalisation Higher Dimensions

There are two problems with the standard presentation of complex analytic function theory that prevent a natural generalisation to higher dimensions:

1. Both the vector operator ∇ and the functions it operates on are mapped into the same algebra by picking out a preferred direction for the real axis. This only works in 2-d.

2. The 'complex limit' argument does not generalise to higher dimensions. Indeed, from our point of view it is not very satisfactory in 2-d as it confuses the concepts of directional derivatives and being independent of z^*.

These are solved by keeping the derivative operator ∇ as a vector while letting it act on general multivectors, and replacing the 'analytic' requirement with the equation $\nabla\psi = 0$. To see where this gets us, we must first look at directed integrals.

3 Directed Integration Theory

To date you have met four important integral theorems. The first is the *divergence theorem*

$$\int d^n x \, \nabla \cdot J = \oint ds \, n \cdot J \tag{3.1}$$

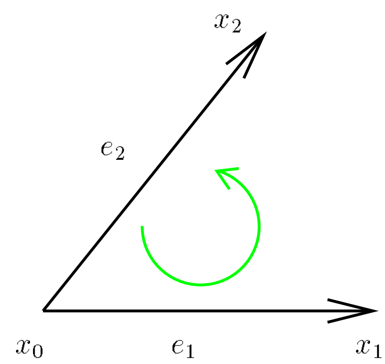

Figure 1: *A 2-d simplex.* The points x_0, x_1, x_2 define a triangle. The boundary of the triangle is defined in the usual, right-handed sense.

where n is the outward normal to the surface forming the (closed) boundary to the region of integration. The second is *Green's theorem*

$$\iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \oint P dx + Q dy \tag{3.2}$$

with the boundary integral the line integral taken around the perimeter of the area in the positive sense. This is a special case of *Stokes' theorem*

$$\int dA\, n \cdot (\nabla \times a) = \oint dl \cdot a \tag{3.3}$$

with the line integral taken around the perimeter of the surface. Finally, we recall the important *Cauchy integral formula*

$$f(a) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z-a} dz. \tag{3.4}$$

We will now demonstrate that these are all special cases of a single integral formula!

3.1 The Vector Derivative Again

We first need to establish an alternative formula for the vector derivative of a multivector. We will simplify matters by restricting to 2-d, though the argument generalises easily. Consider a multivector-valued function $M(x)$ at points x_0, x_1, x_2. The triangle formed by these points is called a 2-d *simplex* (See Fig. 1). We introduce the vectors

$$e_1 = x_1 - x_0, \qquad e_2 = x_2 - x_0, \tag{3.5}$$

so we can write the vector derivative at x_0 as

$$\nabla M = \lim_{x_i \mapsto x_0} e^1(M_1 - M_0) + e^2(M_2 - M_0) \tag{3.6}$$

where $M_i = M(x_i)$. We want to relate this to a surface integral. We extrapolate the function M linearly between its base points with the function

$$m(x) = M_0 + \sum_{i=1}^{2} (x - x_0)\cdot e^i \,(M_i - M_0) \tag{3.7}$$

We can now calculate the integral around the perimeter of $m(x)$ to obtain (exercise)

$$\begin{aligned} \oint dS\, m(x) &= \tfrac{1}{2} e_1(M_0 - M_2) + \tfrac{1}{2} e_2(M_1 - M_0) \\ &= \tfrac{1}{2} e_2 \wedge e_1 [e^1(M_1 - M_0) + e^2(M_2 - M_0)]. \end{aligned} \tag{3.8}$$

This relates the surface integral to the vector derivative. We therefore have

$$\nabla M = \lim_{x_i \mapsto x_0} \oint dS\, \tfrac{1}{2} e^2 \wedge e^1\, m. \tag{3.9}$$

Now the bivector $e^2 \wedge e^1/2$ equals $(IV)^{-1}$, where V is the scalar area of the triangle. Also, in the limit, we can replace m by M. From this we arrive at the formula

$$\nabla M = \lim_{V \mapsto 0} \frac{1}{V} \oint dS\, I^{-1} M. \tag{3.10}$$

This holds in any dimension and can be taken as an alternative definition of the vector derivative ∇. The derivative is then defined in terms of the limit a surface integral. In general, dS is the right-handed oriented surface element and I the right-handed pseudoscalar. The quantity dSI^{-1} is therefore vector-valued.

3.2 The Fundamental Theorem of Calculus

Now imagine building up the preceding result over a *triangulated* surface (Fig. 2). The contributions from each interior line integral cancel, and we arrive at

$$\int_V \dot{\nabla}\, dX\, \dot{M} = \oint_{\partial V} dS\, M \tag{3.11}$$

where $dX = IdV$ is the infinitesimal volume element and dS is the directed surface element, both defined in the right-handed sense. This is the *fundamental theorem of calculus*. It relates the integral over a volume of the derivative of a multivector to the integral over the boundary surface of the multivector. Note that dX is now a

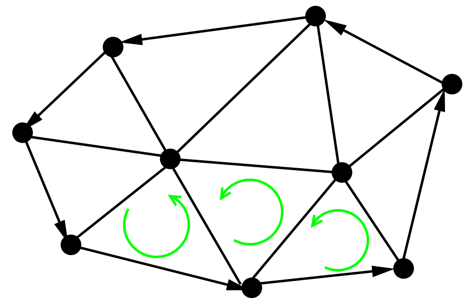

Figure 2: *A Triangulated Surface.* Each of the internal line integral cancels in the sum, as adjacent simplices have opposite orientations.

multivector, so we have to pay attention to the order of the terms. By 'reversing' the preceding derivation we also have

$$\int_V \dot{M}\dot{\nabla}\, dX = \oint_{\partial V} M\, dS \tag{3.12}$$

Alternatively, we can introduce a multilinear function $\mathsf{L}(A)$, which takes a grade $n-1$ multivector as its argument. The fundamental theorem then takes its simplest form as

$$\int_V \dot{\mathsf{L}}(\dot{\nabla}\, dX) = \oint_{\partial V} \mathsf{L}(dS) \tag{3.13}$$

This result even holds for curved surfaces, where dX becomes a function of position!

3.3 The Divergence Theorem

As a first example, let

$$\mathsf{L}(A) = \langle I^{-1} J A\rangle \tag{3.14}$$

where J is a vector. We find that

$$\int_V \langle \dot{J}\dot{\nabla} dX I^{-1}\rangle = \int_V \nabla\cdot J\, dV = \oint_{\partial V} \langle dS I^{-1} J\rangle. \tag{3.15}$$

We therefore recover the divergence theorem provided the normal to the surface is defined by

$$n\, dA = dS\, I^{-1} \tag{3.16}$$

where dA is a scalar measure. This does indeed point outwards in Euclidean spaces, though its behaviour is more complicated in mixed signature spaces. With this definition we arrive at

$$\int_V \nabla\cdot J\, dV = \oint_{\partial V} n\cdot J\, dA \tag{3.17}$$

as expected. We can similarly go on to recover Green's theorem.

3.4 Cauchy's Theorem Recovered

We now return to 2-d space and let ψ be a 'complex' valued multivector (a scalar + pseudoscalar). The fundamental theorem says

$$\int \nabla\psi \, dX = \oint dS \, \psi. \tag{3.18}$$

Recalling that $z = e_1 x$ we can write this as

$$\oint \psi \, dz = \int e_1 \nabla\psi \, dX. \tag{3.19}$$

Now suppose that $\psi = f(z)/(z - z_0)$ with $f(z)$ an analytic function ($\nabla f = 0$). We need to establish the properties of the $(z - z_0)^{-1}$ function (the Cauchy Kernel). We have

$$\frac{1}{z - z_0} = \frac{(z - z_0)^*}{|(z - z_0)|^2} = \frac{x - x_0}{(x - x_0)^2} e_1, \tag{3.20}$$

where $x_0 = e_1 z_0$. But now consider the 2-d Green's function $\ln|x - x_0|$. This has

$$\nabla \ln|x - x_0| = \frac{x - x_0}{(x - x_0)^2}. \tag{3.21}$$

Hence the Cauchy kernel satisfies

$$\nabla \frac{1}{z - z_0} = \nabla \frac{x - x_0}{(x - x_0)^2} e_1 = \nabla^2 \ln|x - x_0| \, e_1 = 2\pi\delta(x - x_0) e_1. \tag{3.22}$$

The Cauchy kernel is simply the Green's function for the 2-d vector derivative! Putting this together we arrive at

$$\begin{aligned} \oint \frac{f(z)}{z - z_0} dz &= e_1 \int \nabla \left(\frac{x - x_0}{(x - x_0)^2} e_1 f(x) \right) dX \\ &= e_1 \int 2\pi\delta(x - x_0) e_1 f(x) I \, |dx| = 2\pi I f(z_0), \end{aligned} \tag{3.23}$$

which recovers the Cauchy integral formula. We can now understand what each of the terms is doing:

- The dz encodes the tangent vector and forms a *geometric* product in the integrand.
- The $(z - z_0)^{-1}$ is the Green's function for the vector derivative ∇ and ensures that the area integral only picks up a the value at z_0.

- The I (or i) comes from the directed volume element $dX = I\,dV$.

We also see now that the residue term in the Laurent expansion of a function

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} \cdots \frac{a_{-1}}{z-z_0} + \sum_{i=0} a_i (z-z_0)^i \tag{3.24}$$

is simply a weighted Green's function. The residue theorem just recovers the weight! We have unified the theory of poles and residues, supposedly unique to complex analysis, with that of Green's functions and δ-functions.

3.5 Extension to Arbitrary Dimensions

Non-Examinable

The extension to arbitrary (Euclidean) dimensions is now straightforward. The analog of an analytic function is an even-grade multivector satisfying $\nabla M = 0$. The Green's function for the vector derivative in n-d is

$$G(x, x_0) = \frac{1}{S_n} \frac{x - x_0}{|x - x_0|^n} \tag{3.25}$$

where S_n is the surface area of the $(m-1)$-dimensional unit ball. The Green's function satisfies

$$\nabla G(x, x_0) = \nabla \cdot G(x, x_0) = \delta(x - x_0). \tag{3.26}$$

A version of Cauchy's theorem in n-d is therefore constructed from

$$\oint_{\partial V} \frac{x - x_0}{|x - x_0|^n} dS\, M = \int_V \left(\frac{x - x_0}{|x - x_0|^n} \right) \overleftarrow{\nabla}\; dX\, M + \int_V \frac{x - x_0}{|x - x_0|^n} \dot{\nabla}\, dX\, \dot{M} \tag{3.27}$$

where the $\overleftarrow{\nabla}$ shows that the ∇ acts on the object to its immediate left. Since M commutes with dX the final term vanishes, leaving

$$M(x_0) = \frac{1}{IS_n} \oint_{\partial V} \frac{x - x_0}{|x - x_0|^n} dS\, M. \tag{3.28}$$

This relates the value of an analytic function at a point to the value of a surface integral over a region surrounding the point.

This derivation is for Euclidean spaces. In non-Euclidean spaces we cannot guarantee that $|x - x_0|$ is non-zero for $x \neq x_0$. This possibility is particularly significant in relativity, as we shall shortly see.