Physical Applications of Geometric Algebra

Handout 8

Spacetime Algebra

The geometric algebra of spacetime is called the *spacetime algebra* or STA. This forms the basis for most of the remainder of this course, where we will deal mainly with applications of geometric algebra to relativistic physics and gravitation. The algebra is constructed from four basis vectors, three spatial and one timelike. The spacelike and timelike vectors have opposite signs for their squares. Rotors in this algebra provide the simplest means of performing Lorentz transformations and help us to understand the structure of the Lorentz group in more detail.

1 An Algebra for Spacetime

Special relativity is often introduced with the postulate that the speed of light is constant for all observers. From this one deduces the Lorentz transformation law before, finally, the concept of unifying space and time into a single spacetime is introduced. This partly mirrors the historical development of relativity. We will not follow this order. Instead, we jump straight to spacetime as the appropriate arena for relativistic physics. Our aim then is to construct the geometric algebra of spacetime. We start by recalling that the invariant interval of special relativity is

$$s^2 = c^2t^2 - x^2 - y^2 - z^2. \tag{1.1}$$

This is the 'particle physics' choice of signature. General relativists often flip all the signs. We work throughout in units where *c=1*. It is clear that we must build our algebra from four vectors $\{e_0, e_i\}, i = 1 \ldots 3$ with the following properties:

$$e_0{}^2 = 1, \qquad e_0 \cdot e_i = 0, \qquad e_i \cdot e_j = -\delta_{ij}. \tag{1.2}$$

These are summarised as

$$e_\mu \cdot e_\nu = \operatorname{diag}(+ - - -)\ , \quad \mu, \nu = 0 \ldots 3. \tag{1.3}$$

1.1 The Bivector Algebra

There are $4 \cdot 3/2 = 6$ bivectors in our algebra. These fall into two classes; those that contain a timelike component (*e.g.* $e_i \wedge e_0$), and those that do not (*e.g.* $e_i \wedge e_j$). For any pair of vectors a and b with $a \cdot b = 0$ we have

$$(a \wedge b)^2 = abab = -abba = -a^2 b^2. \tag{1.4}$$

The two types of bivectors therefore have different signs of their squares. First, we have

$$(e_i \wedge e_j)^2 = -{e_i}^2 {e_j}^2 = -1, \tag{1.5}$$

which is the familiar result for Euclidean bivectors. Each of these generate rotations in a plane. For bivectors containing a timelike component, however, we have

$$(e_i \wedge e_0)^2 = -{e_i}^2 {e_0}^2 = +1. \tag{1.6}$$

Bivectors with positive square have a number of new properties. One immediate result we notice, for example, is that

$$\begin{aligned} \mathrm{e}^{\alpha e_1 e_0} &= 1 + \alpha e_1 e_0 + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} e_1 e_0 + \cdots \\ &= \mathrm{ch}(\alpha) + \mathrm{sh}(\alpha) e_1 e_0. \end{aligned} \tag{1.7}$$

This shows us that we are dealing with *hyperbolic geometry*. This will prove crucial to our treatment of Lorentz transformations. We have started to employ the useful abbreviations

$$\mathrm{ch}(\alpha) = \cosh \alpha, \qquad \mathrm{sh}(\alpha) = \sinh \alpha, \qquad \mathrm{th}(\alpha) = \tanh \alpha. \tag{1.8}$$

1.2 The Pseudoscalar

We define the pseudoscalar I by

$$I = e_0 e_1 e_2 e_3. \tag{1.9}$$

This is still taken to be right-handed. Projecting down the e_3 axis, for example, the set $\{e_0, e_1, e_2\}$ form a right-handed triple (see Fig. 1). We have to be careful in applying these definitions, because we traditionally draw our spacetime diagrams with the time axis vertical. For these planes the 'right-handed' volume element is, for example, $e_1 e_0$. The reasons for our convention for I will emerge soon.

Since I is grade 4, it has

$$\tilde{I} = e_3 e_2 e_1 e_0 = I. \tag{1.10}$$

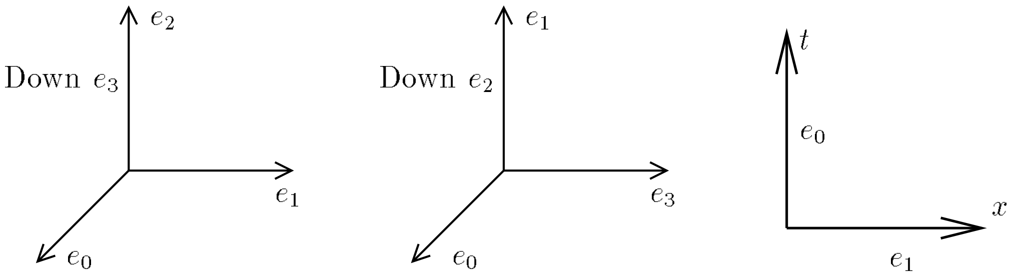

Figure 1: *Orientation of I.* The combination $e_0e_1e_2e_3$ is defined to be right-handed. Projecting down different axes gives the frames pictured. Care is required because spacetime diagrams traditionally have the t-axis vertical.

This makes it easy to compute the square of I :

$$I^2 = I\tilde{I} = (e_0e_1e_2e_3)(e_3e_2e_1e_0) = -1. \tag{1.11}$$

Multiplication of a bivector by I results in a multivector of grade $4 - 2 = 2$, so returns another bivector. This provides a map between the positive and negative square bivectors, *e.g.*

$$Ie_1e_0 = e_1e_0I = e_1e_0e_0e_1e_2e_3 = -e_2e_3. \tag{1.12}$$

If we define $B_i = e_ie_0$ then the bivector algebra can be written

$$\begin{aligned} B_i \times B_j &= \epsilon_{ijk}\, IB_k \\ (IB_i)\times(IB_j) &= -\epsilon_{ijk}\, IB_k \\ (IB_i)\times B_j &= -\epsilon_{ijk} B_k. \end{aligned} \tag{1.13}$$

As well as the four vectors, we also have four trivectors in our algebra. These are interchanged by a duality transformation,

$$e_1e_2e_3 = e_0e_0e_1e_2e_3 = e_0I = -Ie_0. \tag{1.14}$$

Note that I *anticommutes* with vectors and trivectors, as we are in a space of even dimensions. As always, I commutes with all even-grade multivectors.

1.3 The Spacetime algebra

In many applications we are interested in physics in a single preferred orthonormal frame. We denote this frame by $\{\gamma_\mu\}$. Putting the preceding together, we arrive at an algebra with 16 terms:

$$\begin{array}{ccccc} 1 & \{\gamma_\mu\} & \{\gamma_\mu \wedge \gamma_\nu\} & \{I\gamma_\mu\} & I \\ \text{1 scalar} & \text{4 vectors} & \text{6 bivectors} & \text{4 trivectors} & \text{1 pseudoscalar} \end{array} \tag{1.15}$$

This is the *spacetime algebra* or *STA*. We also introduce the following notation for the bivectors:

$$\boldsymbol{\sigma}_i = \gamma_i \gamma_0. \tag{1.16}$$

In the literature the symbol i is often used for the pseudoscalar. We have departed from this practice to avoid confusion with the i of quantum theory. Using the latter symbol presents a potential problem because of the fact that the pseudoscalar anticommutes with vectors.

1.4 The Dirac Matrix Algebra

The vector generators of the STA satisfy

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} \tag{1.17}$$

These are the defining relations of the Dirac matrix algebra, though without an identity matrix on the right-hand side. It follows that the Dirac matrices define a representation of the STA. This also explains our notation of writing $\{\gamma_\mu\}$ for an orthonormal frame. But it must be remembered that the $\{\gamma_\mu\}$ are basis *vectors*, not a set of matrices in 'isospace'.

2 Frames and Trajectories

Suppose that $x(\lambda)$ describes a curve in spacetime. The tangent vector to the curve is

$$x' = \frac{\partial x(\lambda)}{\partial \lambda} \tag{2.1}$$

There are two important cases to consider:

Timelike, $x'^2 > 0$

For these we introduce the preferred parameter along the curve, τ, defined so that

$$v = \partial_\tau x = \dot{x}, \qquad v^2 = 1. \tag{2.2}$$

The parameter τ is the proper time for the curve, and an observer moving along the curve measures this time. The unit timelike vector v then defines the instantaneous rest frame.

Null, $x'^2 = 0$

This defines a null trajectory, which are the paths taken by photons and other massless particles. There is no preferred parameter along these curves, and the proper distance (or time) measured along them is 0. Photons do still carry an intrinsic clock (their frequency), but this can tick at an arbitrary rate.

2.1 Relative Vectors

Now suppose that we are an observer on a timelike path with instantaneous velocity v. What sort of things do we measure? First we construct a frame of rest vectors $\{e_i\}$ perpendicular to v. We also take the point on the worldline as our origin. In this frame a general event x will have time coordinate

$$t = x \cdot v \tag{2.3}$$

and spatial coordinates

$$x^i = x \cdot e^i. \tag{2.4}$$

Suppose now that the event is a point on the worldline of an object at rest in our frame. The 3-d vector to this object is

$$x^i e_i = x \cdot e^\mu \, e_\mu - x \cdot e^0 \, e_0 = x - x \cdot v \, v = x \wedge v \, v. \tag{2.5}$$

Wedging with v projects onto the components of the vector x in the rest frame of v. The key quantity is the spacetime bivector $x \wedge v$. We call this the *relative* vector and write

$$\boldsymbol{x} = x \wedge v. \tag{2.6}$$

With these definitions we have

$$xv = x \cdot v + x \wedge v = t + \boldsymbol{x}. \tag{2.7}$$

The invariant distance now decomposes as

$$\begin{aligned} x^2 = xvvx &= (x \cdot v + x \wedge v)(x \cdot v + v \wedge x) \\ &= (t + \boldsymbol{x})(t - \boldsymbol{x}) = t^2 - \boldsymbol{x}^2, \end{aligned} \tag{2.8}$$

recovering the usual result. This is built into the definition of the STA.

2.2 The Even Subalgebra

Each observer sees a set of relative vectors, which we model as spacetime bivectors. What algebraic properties do these have? To simplify things, we take the timelike velocity vector to be γ_0 so that the relative vectors are given by $\boldsymbol{\sigma}_i = \gamma_i\gamma_0$. These satisfy

$$\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = \tfrac{1}{2}(\gamma_i\gamma_0\gamma_j\gamma_0 + \gamma_j\gamma_0\gamma_i\gamma_0) = \tfrac{1}{2}(-\gamma_i\gamma_j - \gamma_j\gamma_i) = \delta_{ij}. \tag{2.9}$$

These act as vector generators for a 3-d algebra. This is the geometric algebra of the 3-d relative space in the rest frame defined by γ_0. Furthermore, the volume element of this algebra is

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = (\gamma_1\gamma_0)(\gamma_2\gamma_0)(\gamma_3\gamma_0) = -\gamma_1\gamma_0\gamma_2\gamma_3 = I, \tag{2.10}$$

so this subalgebra shares the same pseudoscalar as spacetime. Of course, we still have

$$\tfrac{1}{2}(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j\boldsymbol{\sigma}_i) = \epsilon_{ijk} I \boldsymbol{\sigma}_k, \tag{2.11}$$

so that both relative vectors and relative bivectors are spacetime bivectors. We have projected everything onto the even subalgebra of the STA.

$$\begin{array}{ccccccccccc}
1 & \cdots & \{\gamma_\mu\} & \cdots & \{\boldsymbol{\sigma}_i, I\boldsymbol{\sigma}_i\} & \cdots & \{I\gamma_\mu\} & \cdots & I & & 4-d \\
1 & & \{\boldsymbol{\sigma}_i\} & & & & \{I\boldsymbol{\sigma}_i\} & & I & & 3-d
\end{array}$$

The 6 spacetime bivectors get split into relative vectors and relative bivectors. This split is *observer dependent.*

2.3 Conventions

Spacetime bivectors which are also used as relative vectors are written in bold. This is the only place we use bolds for vectors. (In written work we use a curly underline). It is not strictly necessary to put the $\{\boldsymbol{\sigma}_i\}$ in bold, though for consistency we have.

There is a potential ambiguity here - how are we to interpret the expression $\boldsymbol{a} \wedge \boldsymbol{b}$? Our convention is that if all of the terms in an expression are bold, the dot and wedge symbols drop down to their 3-d meaning, otherwise they take their spacetime definition. This works pretty well in practice.

2.4 Examples

i. Velocity

Suppose that an observer with constant velocity v measures the relative velocity of a particle with proper velocity $u(\tau)$, $u^2 = 1$. We have

$$uv = \partial_\tau(x(\tau)v) = \partial_\tau(t + \boldsymbol{x}), \tag{2.12}$$

so that

$$\partial_\tau t = u\cdot v. \tag{2.13}$$

The relative velocity is therefore

$$\boldsymbol{u} = \frac{\partial \boldsymbol{x}}{\partial t} = \frac{\partial \boldsymbol{x}}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{u\wedge v}{u\cdot v}. \tag{2.14}$$

This construction is familiar — it is precisely the one discovered in the context of projective geometry! We have also ensured that the projective vectors have positive square. This turns out to be very convenient for applications in computer vision, where we now routinely perform calculations using the 'relativistic' STA!

ii. Momentum and Wave Vectors

Now suppose we observe a particle with energy-momentum p. The energy measured is $p\cdot v$, and the relative momentum is $p\wedge v$, so

$$pv = p\cdot v + p\wedge v = E + \boldsymbol{p}. \tag{2.15}$$

From this we recover the invariant

$$m^2 = p^2 = pvvp = (E + \boldsymbol{p})(E - \boldsymbol{p}) = E^2 - \boldsymbol{p}^2. \tag{2.16}$$

Similarly, for a photon with wave-vector k we have

$$kv = k\cdot v + k\wedge v = \omega + \boldsymbol{k}, \tag{2.17}$$

and for photons in empty space $k^2 = 0$ so

$$0 = kvvk = (\omega + \boldsymbol{k})(\omega - \boldsymbol{k}) = \omega^2 - \boldsymbol{k}^2. \tag{2.18}$$

This recovers the relation $|\boldsymbol{k}| = \omega$, which holds in all frames.

This idea of projecting onto the even subalgebra to study physics in a rest frame is a very powerful technique. Our next task is to study Lorentz transformations to see how different observers see the same physics.

3 Lorentz Transformations

Lorentz Transformations are usually expressed in the form of a coordinate transformation, *e.g.*

$$\begin{array}{ll} \mathrm{x}' = \gamma(\mathrm{x} - \beta t) & t' = \gamma(t - \beta \mathrm{x}) \\ \mathrm{x} = \gamma(\mathrm{x}' + \beta t') & t = \gamma(t' + \beta \mathrm{x}') \end{array} \tag{3.1}$$

where $\gamma = (1-\beta^2)^{-1/2}$ and β is the scalar velocity in units of c. Our first task is to manipulate these relations into a transformation law for vectors. The vector x has been decomposed in two frames, $\{e_\mu\}$ and $\{e'_\mu\}$, so that

$$x = x^\mu e_\mu = x^{\mu\prime} e'_\mu. \tag{3.2}$$

We then have, for example

$$t = e^0 \cdot x, \qquad t' = e^{0\prime} \cdot x. \tag{3.3}$$

Concentrating on the 0, 1, components we have

$$te_0 + \mathrm{x}e_1 = t'e'_0 + \mathrm{x}'e'_1, \tag{3.4}$$

and from this we derive the vector relations

$$e'_0 = \gamma(e_0 + \beta e_1), \quad e'_1 = \gamma(e_1 + \beta e_0). \tag{3.5}$$

These define the new frame in terms of the old.

3.1 Rotor Form of a Lorentz Transformation

We saw earlier that bivectors with positive square lead to hyperbolic geometry. This suggests that we introduce an 'angle' α with

$$\mathrm{tanh}\alpha = \beta, \qquad (\beta < 1), \tag{3.6}$$

so that

$$\gamma = (1 - \tanh^2\alpha)^{-1/2} = \cosh\alpha. \tag{3.7}$$

The vector e'_0 is now

$$\begin{aligned} e'_0 &= \mathrm{ch}(\alpha)e_0 + \mathrm{sh}(\alpha)e_1 \\ &= (\mathrm{ch}(\alpha) + \mathrm{sh}(\alpha)e_1e_0)e_0 = \mathrm{e}^{\alpha\, e_1 e_0}\, e_0, \end{aligned} \tag{3.8}$$

where we have expressed the scalar + bivector as an exponential. Similarly, we have

$$e_1' = \mathrm{ch}(\alpha)e_1 + \mathrm{sh}(\alpha)e_0 = \mathrm{e}^{\alpha\, e_1 e_0}\, e_1. \tag{3.9}$$

Now recall that these are just two of four frame vectors, with the other pair untouched. The relationship between the two frames is simply expressed by

$$e_\mu' = Re_\mu\tilde{R}, \quad e^{\mu\prime} = Re^\mu\tilde{R}, \quad R = \mathrm{e}^{\alpha\, e_1 e_0/2}. \tag{3.10}$$

The same rotor prescription works for boosts as well as rotations! Now we really are treating spacetime as a unified entity.

3.2 Examples

i. Addition of Velocities

As a simple example, suppose that we are in a frame with basis vectors $\{e_0, e_1\}$. We observe two objects flying apart with 4-velocities

$$v_1 = \mathrm{e}^{\alpha_1 e_1 e_0}\, e_0, \qquad v_2 = \mathrm{e}^{-\alpha_2 e_1 e_0}\, e_0. \tag{3.11}$$

What is the relative velocity they see for each other? We form

$$\frac{v_1\wedge v_2}{v_1\cdot v_2} = \frac{\langle \mathrm{e}^{(\alpha_1+\alpha_2)e_1e_0}\rangle_2}{\langle \mathrm{e}^{(\alpha_1+\alpha_2)e_1e_0}\rangle_0} = \frac{\sinh(\alpha_1+\alpha_2)e_1e_0}{\cosh(\alpha_1+\alpha_2)}. \tag{3.12}$$

Both observers therefore measure a relative velocity of

$$\tanh(\alpha_1+\alpha_2) = \frac{\tanh\alpha_1 + \tanh\alpha_2}{1-\tanh\alpha_1\tanh\alpha_2} \tag{3.13}$$

Addition of velocities is achieved by adding hyperbolic angles, which recovers the familiar formula.

ii. Photons and Redshifts

Often in studying the properties of electromagnetic waves we use the geometric optics approximation and work directly with null vectors k. This provides for simple formulae for Doppler shifts and aberration. Suppose that two particles follow different worldlines and that particle 1 emits a photon which is received by particle 2 (see Fig. 2). The frequency seen by particle 1 is $\omega_1 = v_1\cdot k$, and by particle 2 is $\omega_2 = v_2\cdot k$. The ratio of these describes the Doppler effect, often expressed as a redshift, z:

$$1 + z = \omega_1/\omega_2. \tag{3.14}$$

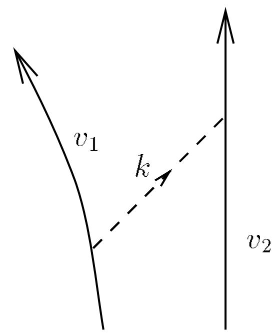

Figure 2: *Photon Emission and Absorption.* A photon is emitted by particle 1 and received by particle 2.

This can be applied in many ways. For example, suppose that the emitter is receding in the e_1 direction, and $v_2 = e_0$. We have

$$k = \omega_2(e_0 + e_1), \qquad v_1 = \cosh\alpha\, e_0 - \sinh\alpha\, e_1, \tag{3.15}$$

so that

$$1 + z = \frac{\omega_2(\cosh\alpha + \sinh\alpha)}{\omega_2} = \mathrm{e}^{\alpha}. \tag{3.16}$$

This transformation of a null frame producing a dilation is an example of the type of rotor transformation of the balanced algebra $\mathcal{G}_{n,n}$ considered in Handout 6. The velocity of the emitter in the e_0 frame is $\tanh\alpha$, and it is easy to check that

$$\mathrm{e}^{\alpha} = \left(\frac{1 + \tanh\alpha}{1 - \tanh\alpha}\right)^{1/2}, \tag{3.17}$$

recovering the standard expression for the relativistic Doppler effect. Aberration formulae can be obtained in the same way.

4 The Lorentz Group

The full Lorentz group consists of the transformation group for vectors which preserves lengths and angles. These include reflections and rotations. A reflection in the hyperplane perpendicular to n is achieved by

$$a \mapsto -nan^{-1}. \tag{4.1}$$

The n^{-1} is necessary to accommodate both timelike $n^2 > 0$ and spacelike $n^2 < 0$ cases. (We cannot have null n). A timelike n generates time-reversal transformations, whereas spacelike reflections preserve time ordering. Pairs of either of these result in a transformation which preserves time ordering. However, a combination of one spacelike and one timelike reflection does not preserve the time ordering. The full Lorentz group therefore contains 4 sectors.

		Space Reflection
	I Proper Orthochronous	*II* *I* with space reflection
Time reversal	*III* *I* with time reversal	*IV* *I* with $a \mapsto -a$

4.1 STA Description

The structure of the Lorentz group is easily understood in the STA. First we combine even numbers of reflections, producing a transformation of the form

$$a \mapsto \psi a \psi^{-1}, \tag{4.2}$$

where ψ is an even multivector. This expression is currently too general, as we have not ensured that the right-hand side is a vector. To see how to do this we decompose ψ into invariant terms. We first note that

$$\psi\tilde{\psi} = (\psi\tilde{\psi})^{\tilde{}} \tag{4.3}$$

so $\psi\tilde{\psi}$ is even-grade and equal to its own reverse. It can therefore only contain a scalar and a pseudoscalar,

$$\psi\tilde{\psi} = \alpha_1 + I\alpha_2 = \rho\,\mathrm{e}^{I\beta}\,, \tag{4.4}$$

where $\rho \neq 0$ in order for ψ^{-1} to exist. We can now define a rotor R by

$$R = \psi(\rho\,\mathrm{e}^{I\beta}\,)^{-1/2}, \tag{4.5}$$

so that

$$R\tilde{R} = \psi\tilde{\psi}(\rho\,\mathrm{e}^{I\beta}\,)^{-1} = 1, \tag{4.6}$$

as required. We now have

$$\psi = \rho^{1/2}\,\mathrm{e}^{I\beta/2}\,R, \qquad \psi^{-1} = \rho^{-1/2}\,\mathrm{e}^{-I\beta/2}\,\tilde{R} \tag{4.7}$$

and our general transformation becomes

$$a \mapsto \mathrm{e}^{I\beta/2}\,Ra\,\mathrm{e}^{-I\beta/2}\,\tilde{R} = \mathrm{e}^{I\beta}\,Ra\tilde{R}. \tag{4.8}$$

The term $Ra\tilde{R}$ is necessarily a vector (equal to its own reverse), so we must restrict β to either 0 or π, leaving the transformation

$$a \mapsto \pm Ra\tilde{R}. \tag{4.9}$$

4.2 The Restricted Lorentz Group

The transformation $a \mapsto Ra\tilde{R}$ preserves causal ordering as well as parity. Transformations of this type are called 'proper orthochronous' transformations. We can prove that rotor driven transformations are proper orthochronous by starting with the velocity γ_0 and transforming it to $v = R\gamma_0\tilde{R}$. We need the γ_0 component of v to be positive, that is

$$\gamma_0 \cdot v = \langle \gamma_0 R \gamma_0 \tilde{R} \rangle > 0. \tag{4.10}$$

Decomposing in the γ_0 frame we can write

$$R = \alpha + \boldsymbol{a} + I\boldsymbol{b} + I\beta \tag{4.11}$$

and we find that

$$\langle \gamma_0 R \gamma_0 \tilde{R} \rangle = \alpha^2 + \boldsymbol{a}^2 + \boldsymbol{b}^2 + \beta^2 > 0 \tag{4.12}$$

as required. Our rotor transformation law describes the group of proper orthochronous transformations, often called the *restricted Lorentz group*. These are the transformations of most physical relevance. The other sign, corresponding to $\beta = \pi$ in Eq. (4.8), gives class IV transformations.