Physical Applications of Geometric Algebra

Handout 14

The Gravitational Field Equations

The key to deriving the field equations in any gauge theory is the covariant *field strength tensor*. This encodes the content of the gauge fields which is not generated by gauge transformations alone. There are two of these in gravity, but it turns out that one is extremely weak and only enters at the quantum level. In the approximation that this term is zero, we recover a theory which is equivalent to GR. The second gauge field gives rise a field strength which has units of energy density. A version of this is equated with the matter stress-energy tensor to yield the full set of field equations.

Some of the more technical derivations in this handout have been relegated to an appendix. You are not expected to remember or be able to reproduce these derivations. They are included only for general interest and to give a feel for how geometric calculus is applied in gravitation and gauge theories in general.

1 The Field Strength

The field strength tensor is found in general by commuting covariant derivatives. Suppose, first, that we are dealing with electromagnetism again, so ψ transforms to ψR under rotor transformations. In this case, if a and b are constant vectors, we have

$$\begin{aligned}[D_a, D_b]\psi &= D_a\left(b\cdot\nabla\psi + \tfrac{1}{2}\psi\Omega(b)\right) - D_b\left(a\cdot\nabla\psi + \tfrac{1}{2}\psi\Omega(a)\right)\\ &= \tfrac{1}{2}\psi[a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a) - \Omega(a)\times\Omega(b)].\end{aligned} \tag{1.1}$$

Despite the fact that we formed commutators of derivatives on ψ, all of the derivatives of ψ have canceled.

Specialising to the case of electromagnetism, where $\Omega(a) = -2a\cdot A\, I\boldsymbol{\sigma}_3$, we find that the term multiplying ψ is

$$\begin{aligned}&b\cdot\nabla(a\cdot A\, I\boldsymbol{\sigma}_3) - a\cdot\nabla(b\cdot A\, I\boldsymbol{\sigma}_3) - 2a\cdot A\, b\cdot A\, I\boldsymbol{\sigma}_3\times I\boldsymbol{\sigma}_3 \\ &\quad = (a\wedge b)\cdot(\nabla\wedge A)I\boldsymbol{\sigma}_3 = (a\wedge b)\cdot F\, I\boldsymbol{\sigma}_3.\end{aligned} \tag{1.2}$$

This is a function which maps the bivector $a\wedge b$ linearly onto a pure phase term. In electromagnetism we lose the mapping nature of the field strength and instead work directly with the bivector $\nabla\wedge A$. For more complicated systems this is not appropriate. In forming our commutator we have extracted the correct field strength, $F = \nabla\wedge A$. This encodes the physically measurable content of the electromagnetic field, and vanishes if A is a pure gauge field, $A = \nabla\phi$.

1.1 The Rotation Gauge Field Strength

In analysing rotations the only difference to the above is that the rotors multiply ψ from the left. In this case we form

$$[D_a, D_b]\psi = \tfrac{1}{2}\mathsf{R}(a\wedge b)\psi \tag{1.3}$$

where

$$\mathsf{R}(a\wedge b) = a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a) + \Omega(a)\times\Omega(b). \tag{1.4}$$

Since the right-hand side is antisymmetric on a and b, the field strength can depend only on the bivector $a\wedge b$. This linear action on bivector blades is extended to general bivectors by defining

$$\mathsf{R}(a\wedge b + c\wedge d) = \mathsf{R}(a\wedge b) + \mathsf{R}(c\wedge d). \tag{1.5}$$

This means that we can write the field strength as,

$$\mathsf{R}(B) = \mathsf{R}(B; x) \tag{1.6}$$

which is a position dependent, linear function of the bivector B.

There are two main differences from the electromagnetic case. The first is that the commutator term $\Omega(a)\times\Omega(b)$ has not cancelled out. This has an important consequence for the field equations — they are no longer *linear*. If one adds together two configurations of $\Omega(a)$, the field strength of the resultant $\Omega(a)$ is not the same as that from the superposition of the original field strengths. This makes the equations much more difficult to solve than those of electromagnetism. The second main difference is that the field strength is now a general bivector, rather than being constrained to the $I\boldsymbol{\sigma}_3$ plane. That is, $\mathsf{R}(B)$ is a bivector-valued function of the bivector argument B. This means that $\mathsf{R}(a\wedge b)$ has 36 degrees of freedom, instead of the more manageable 6 of electromagnetism.

The definition of $\mathsf{R}(B)$ in terms of commutators makes it easy to establish its transformation properties under rotation gauge transformations. We see that

$$[D'_a, D'_b,]\psi' = \tfrac{1}{2}\mathsf{R}'(a\wedge b)R\psi = R[D_a, D_b]\psi = \tfrac{1}{2}R\mathsf{R}(a\wedge b)\psi, \tag{1.7}$$

from which we can read off that

$$\mathsf{R}'(a\wedge b) = R\mathsf{R}(a\wedge b)\tilde{R}. \tag{1.8}$$

Unlike electromagnetism, the field strength now transforms under gauge transformations, albeit in a straightforward way. This affords a quick proof that the field strength vanishes if $\Omega(a)$ is derived from a pure gauge transformation. In this case a gauge exists in which D_a is just $a\cdot\nabla$. Since partial derivatives commute, $\mathsf{R}(a\wedge b)$ vanishes in this gauge, and Eq. (1.8) ensures that $\mathsf{R}(a\wedge b)$ also vanishes in all other gauges.

1.2 The Displacement Gauge Field Strength

The displacement gauge field $\bar{\mathsf{h}}(a)$ couples to derivatives in a different manner, but it is still possible to define a sensible field strength. If we consider the commutator we form

$$\begin{aligned}
[a\cdot\bar{\mathsf{h}}(\nabla), b\cdot\bar{\mathsf{h}}(\nabla)]\psi &= b\cdot[a\cdot\bar{\mathsf{h}}(\nabla)\bar{\mathsf{h}}(\nabla)]\psi - a\cdot[b\cdot\bar{\mathsf{h}}(\nabla)\bar{\mathsf{h}}(\nabla)]\psi \\
&= (b\wedge a)\cdot[\bar{\mathsf{h}}(\nabla)\wedge\bar{\mathsf{h}}(\nabla)]\psi.
\end{aligned} \tag{1.9}$$

The result now is a differential operator, which is driven by the term $\bar{\mathsf{h}}(\nabla)\wedge\bar{\mathsf{h}}(\nabla)$. Whenever this acts on a scalar ϕ we pick up a term in

$$\bar{\mathsf{h}}(\nabla)\wedge\bar{\mathsf{h}}(\nabla\phi) = \bar{\mathsf{h}}(\dot{\nabla})\wedge\dot{\bar{\mathsf{h}}}(\nabla\phi) + \bar{\mathsf{h}}(\nabla\wedge\nabla\phi) = \bar{\mathsf{h}}(\dot{\nabla})\wedge\dot{\bar{\mathsf{h}}}(\nabla\phi) \tag{1.10}$$

where the over-dot on $\dot{\bar{\mathsf{h}}}$ denotes that only the position dependence in the $\bar{\mathsf{h}}$-field is differentiated, and not the position dependence of its argument. To generalise this we first write ∇ as $\bar{\mathsf{h}}^{-1}\bar{\mathsf{h}}(\nabla)$. Since it is $\bar{\mathsf{h}}(\nabla)$ that is covariant, we replace this by a constant vector and define

$$\mathsf{S}(a) = \bar{\mathsf{h}}(\dot{\nabla})\wedge\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(a) = -\bar{\mathsf{h}}[\nabla\wedge\bar{\mathsf{h}}^{-1}(a)], \tag{1.11}$$

where we have employed the result that

$$\nabla\wedge[\bar{\mathsf{h}}\bar{\mathsf{h}}^{-1}(a)] = \dot{\nabla}\wedge\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(a) + \dot{\nabla}\wedge\bar{\mathsf{h}}\dot{\bar{\mathsf{h}}}^{-1}(a) = \nabla\wedge a = 0. \tag{1.12}$$

Our function $\mathsf{S}(a)$ is a bivector-valued function of the vector argument a. It is covariant under displacements, which follows from its derivation, so is a candidate for the field strength.

The second property that $\mathsf{S}(a)$ must satisfy is that it vanishes if $\bar{\mathsf{h}}(a)$ is pure gauge. For $\bar{\mathsf{h}}(a)$ to be a pure gauge field we must have

$$\bar{\mathsf{h}}(a) = \bar{\mathsf{f}}^{-1}(a), \tag{1.13}$$

so that $\bar{\mathsf{h}}(\nabla) = \bar{\mathsf{f}}^{-1}(\nabla) = \nabla_{x'}$ is a pure vector derivative in some other gauge. In this case we have

$$\mathsf{S}(a) = -\bar{\mathsf{f}}^{-1}[\nabla\wedge\bar{\mathsf{f}}(a)]. \tag{1.14}$$

But if $\mathsf{f}(a)$ is the derivative of a displacement we have

$$\mathsf{f}(a) = a\cdot\nabla f(x), \tag{1.15}$$

so

$$\bar{\mathsf{f}}(a) = \partial_b\langle\mathsf{f}(b)a\rangle = \partial_b\langle b\cdot\nabla f(x)a\rangle = \nabla(a\cdot f(x)). \tag{1.16}$$

It follows that $\nabla\wedge\bar{\mathsf{f}}(a) = \nabla\wedge\nabla(a\cdot f(x)) = 0$, so $\mathsf{S}(a)$ does indeed vanish if the $\bar{\mathsf{h}}$-field is generated by a gauge transformation.

2 The Covariant Field Strengths

We have determined the field strengths of our two gravitational fields. The next step is to ensure that both of these are correctly expressed *covariantly.* We start with the rotation gauge field, $\Omega(a)$.

2.1 The Riemann Tensor

We have already determined the transformation law for $\mathsf{R}(B)$ under rotation gauge changes. Under displacements we first note that $\Omega(a)$ is the gauge field introduced to remove terms of the type $a\cdot\nabla R\tilde{R}$. It follows that under displacements we have

$$\Omega(a;x) \mapsto \Omega'(a;x) = \Omega[\mathsf{f}(a);x'] \tag{2.1}$$

As a result, the transformed field strength is

$$\begin{aligned}\mathsf{R}'(a\wedge b) =& a\cdot\nabla\Omega'(b) - b\cdot\nabla\Omega'(a) + \Omega'(a)\times\Omega'(b)\\ =& \mathsf{f}(a)\cdot\dot{\nabla}_{x'}\dot{\Omega}[\mathsf{f}(b);x'] - \mathsf{f}(b)\cdot\dot{\nabla}_{x'}\dot{\Omega}[\mathsf{f}(a);x'] + \Omega'(a)\times\Omega'(b)\\ &+ \Omega[a\cdot\nabla\mathsf{f}(b) - b\cdot\nabla\mathsf{f}(a);x']\\ =& \mathsf{R}[\mathsf{f}(a\wedge b);x'] + \Omega[a\cdot\nabla\mathsf{f}(b) - b\cdot\nabla\mathsf{f}(a);x'].\end{aligned} \tag{2.2}$$

But we know that

$$a\cdot\nabla\mathsf{f}(b) - b\cdot\nabla\mathsf{f}(a) = a\cdot\nabla(b\cdot\nabla x') - b\cdot\nabla(a\cdot\nabla x') = 0, \tag{2.3}$$

so the field strength has the simple displacement transformation law

$$\mathsf{R}(B) \mapsto \mathsf{R}'(B) = \mathsf{R}[\mathsf{f}(B); x']. \tag{2.4}$$

We see that $\mathsf{R}'(B)$ picks up an unwanted term in $\mathsf{f}(B)$, but this is easily removed. Since $\bar{\mathsf{h}}(a)$ has the transformation property

$$\bar{\mathsf{h}}(a) \mapsto \bar{\mathsf{h}}'(a) = \bar{\mathsf{h}}\bar{\mathsf{f}}^{-1}(a) \tag{2.5}$$

we see that the adjoint function transforms as

$$\mathsf{h}(a) \mapsto \mathsf{h}'(a) = \mathsf{f}^{-1}\mathsf{h}(a). \tag{2.6}$$

We therefore insert a term in $\mathsf{h}(a)$ into $\mathsf{R}(B)$ and define the covariant field strength

$$\mathcal{R}(B) = \mathsf{R}[\mathsf{h}(B)]. \tag{2.7}$$

The extra factor of $\mathsf{h}(B)$ alters the transformation properties under rotations now. We established in Handout 13 that $\bar{\mathsf{h}}(a)$ transforms under rotations as

$$\bar{\mathsf{h}}(a) \mapsto \bar{\mathsf{h}}'(a) = R\bar{\mathsf{h}}(a)\tilde{R}. \tag{2.8}$$

It follows that the adjoint transforms to

$$\mathsf{h}(a) \mapsto \mathsf{h}'(a) = \partial_b \langle aR\bar{\mathsf{h}}(b)\tilde{R}\rangle = \mathsf{h}(\tilde{R}aR). \tag{2.9}$$

The transformation properties of $\mathcal{R}(B)$ are therefore summarised by:

$$\begin{aligned} \text{Displacements:} \quad & \mathcal{R}'(B, x) = \mathcal{R}(B, x') \\ \text{Rotations:} \quad & \mathcal{R}'(B) = R\mathcal{R}(\tilde{R}BR)\tilde{R}. \end{aligned} \tag{2.10}$$

These are precisely the properties we want. The displacement law means that we can move the field strength around in the STA, in the same manner as our other covariant fields. The rotation law might look more complicated, but it is quite natural as well. Suppose, for example, that $\mathcal{R}(B)$ simply amounted to the instruction 'dilate all fields by the factor α'. This is a physical statement, so ought to be true in all gauges. This is what we find. The original statement corresponds to

$$\mathcal{R}(B) = \alpha B. \tag{2.11}$$

The transformed field is then

$$\mathcal{R}'(B) = R\mathcal{R}(\tilde{R}BR)\tilde{R} = R(\alpha\tilde{R}BR)\tilde{R} = \alpha B \tag{2.12}$$

so does correspond to the same physical information. Any linear function with the transformation properties of Eq. (2.10) is called a *covariant tensor*. As $\mathcal{R}(B)$ plays the same role in the gauge theory approach as the curvature tensor in GR, we refer to $\mathcal{R}(B)$ as the *Riemann tensor*. We employ this notational device of writing covariant tensors in calligraphic symbols to help keep track of which objects are gauge invariant.

2.2 Examples

A few examples should give a flavour of how the STA simplifies the study of the Riemann tensor. All of the expressions below are considerably simpler than their tensor calculus counterparts.

I. The Schwarzschild Solution

The fields outside a non-rotating spherically symmetric source of mass M are described by the Schwarzschild solution. For this case the Riemann tensor is given by

$$\mathcal{R}(B) = -\frac{M}{2r^3}(B + 3\boldsymbol{\sigma}_r B \boldsymbol{\sigma}_r) \tag{2.13}$$

where $r = |x \wedge \gamma_0|$, $\boldsymbol{\sigma}_r = x \wedge \gamma_0 / r$, and the source is at rest in the γ_0 frame. The scalar term $M/2r^3$ controls the magnitude of the tidal forces due to the source. In empty space it is the residual effects of tidal forces that are measurable.

II. The Kerr Solution

The fields outside a rotating black hole are described by the Kerr solution. This is beyond the scope of this course, but it is worth seeing the form of the Riemann tensor for this solution. It is

$$\mathcal{R}(B) = -\frac{M}{2(r + IL\cos\theta)^3}(B + 3\boldsymbol{\sigma}_r B \boldsymbol{\sigma}_r). \tag{2.14}$$

This only differs from the Schwarzschild case through the pre-factor, which generalises r to the scalar+pseudoscalar combination $r + IL\cos\theta$. Here L controls the angular momentum of the source, and θ has its usual meaning for a spherical polar coordinate system. It has been known for many years that a natural complex structure underlies the Kerr solution. The form of the Riemann tensor explains why this is so.

III. Cosmic Strings

The Riemann tensor inside an infinite, pressure-free string with density ρ is

$$\mathcal{R}(B) = 8\pi\rho \langle B I \boldsymbol{\sigma}_3 \rangle I \boldsymbol{\sigma}_3 \tag{2.15}$$

where the string lies along the γ_3 axis and is at rest in the γ_0 frame. Tidal forces are only exerted in the $I\boldsymbol{\sigma}_3$ plane and are controlled by the density, which is what one would expect physically.

IV. Cosmology

The Riemann tensor for an isotropic, homogeneous cosmology is

$$\mathcal{R}(B) = 4\pi(\rho + P)B\cdot e_t\, e_t - \tfrac{1}{3}(8\pi\rho + \Lambda)B. \tag{2.16}$$

Here P and ρ are the pressure and density and are functions of cosmic time, Λ is the cosmological constant, and e_t is the 'rest-frame' of the universe (defined by the cosmic microwave background radiation). Note that no other direction is contained in $\mathcal{R}(B)$, as one expects for an isotropic solution.

2.3 The Displacement Gauge Field Strength

Next we return to the displacement gauge field strength. As we found in Section 1.2, the key quantity is

$$\mathsf{S}(a) = -\bar{\mathsf{h}}[\nabla\wedge\bar{\mathsf{h}}^{-1}(a)] = \bar{\mathsf{h}}(\dot{\nabla})\wedge\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(a). \tag{2.17}$$

This is already covariant under displacements, but under rotations the $\bar{\mathsf{h}}(a)$ field picks up some additional rotors. We must therefore replace the directional derivatives of this with rotationally covariant derivatives, so we define

$$\mathcal{S}(a) = \bar{\mathsf{h}}(\partial_b)\wedge\left(b\cdot\dot{\nabla}\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(a) + \Omega(b)\cdot a\right). \tag{2.18}$$

This guarantees that we have the required transformation law under displacements and rotations:

$$\begin{aligned} \text{Displacements:} \quad & \mathcal{S}'(a,x) = \mathcal{S}(a,x') \\ \text{Rotations:} \quad & \mathcal{S}'(a) = R\mathcal{S}(\tilde{R}aR)\tilde{R}. \end{aligned} \tag{2.19}$$

3 The Field Equations

The full, formal development of gauge theory gravity generates the field equations from a Lagrangian. Here we adopt a more heuristic approach, based on some simple physical arguments. We first consider the dimensions of the various constants at our disposal. We know, from experiment, that the scale of gravitational interactions is fixed by Newton's constant G. Combined with Planck's constant $\hbar$ and the speed of light c, this fixes a natural scale for interactions as follows:

$$\begin{aligned} \text{Planck length} \quad & l_P = \sqrt{\hbar G/c^3} = 1.6\times10^{-35}\text{m} \\ \text{Planck mass} \quad & M_P = \hbar/L_P c = 2.2\times10^{-8}\text{kg} \\ \text{Planck time} \quad & t_P = L_P/c = 5.3\times10^{-44}\text{s}. \end{aligned} \tag{3.1}$$

The natural length scale is therefore tiny — far smaller than any scale currently accessible to experiment. The same goes for the Planck time. The mass, on the other hand, is large and far in excess of the masses of any fundamental particles observed to date.

If we now consider the two field strengths we have defined, $\mathcal{S}(a)$ and $\mathcal{R}(B)$, we see that these differ in dimensions by a factor of length. This is because $\Omega(a)$ has dimensions of $(\text{length})^{-1}$, whereas $\bar{\mathsf{h}}(a)$ is dimensionless. It follows that we expect the $\mathcal{S}(a)$ term to be of comparable magnitude to $l_P\mathcal{R}(B)$, and so is extremely small. If we ignore quantum effects, we therefore expect that $\mathcal{S}(a)$ will vanish, and this yields the first of our field equations,

$$\mathcal{S}(a) = 0. \tag{3.2}$$

This is borne out by the Lagrangian analysis, which shows that $\mathcal{S}(a)$ is driven by *quantum spin*, and so is negligible for most interactions. In setting $\mathcal{S}(a)$ to zero, we are saying that the field strength of the $\bar{\mathsf{h}}$ field vanishes. We might expect this to mean that $\bar{\mathsf{h}}(a)$ is a pure gauge field. But in constructing the covariant field strength $\mathcal{S}(a)$ we had to couple in the $\Omega(a)$ field. Remarkably, it is this coupling which generates some dynamics. It is also Eq. (3.2) that ensures that the equations we derive are (locally) equivalent to those of GR! The revealing feature of this approach is that GR is only recovered in the limit where quantum interactions are ignored. This has a number of implications for attempts to unite quantum theory and gravity.

3.1 Consequences of the First Field Equation

To establish some consequences of our first field equation $\mathcal{S}(a) = 0$ it is first convenient to express the equation in the form

$$\mathcal{D}\wedge\bar{\mathsf{h}}(a) = \bar{\mathsf{h}}(\partial_b)\wedge[b\cdot\nabla\bar{\mathsf{h}}(a) + \Omega(b)\cdot\bar{\mathsf{h}}(a)] = 0, \tag{3.3}$$

where, as usual, a is a constant vector. Perhaps more usefully, suppose that $A(x)$ is a vector-valued field. In this case we have

$$\mathcal{D}\wedge\bar{\mathsf{h}}(A) = \bar{\mathsf{h}}(\nabla\wedge A). \tag{3.4}$$

This result enables us to move easily between covariant expressions and 'flat space' expressions involving the vector derivative. The result also removes a potential ambiguity for the electromagnetic field strength. The vector potential is A, and its covariant form is $\mathcal{A} = \bar{\mathsf{h}}(A)$ (since A picks up a term in $\nabla\phi$ under local phase changes). Given this, we might be unsure whether the covariant form of the electromagnetic field strength should be $\bar{\mathsf{h}}(\nabla\wedge A)$ or $\mathcal{D}\wedge\mathcal{A}$. Equation (3.3) ensures that both give the same result, which is

$$\mathcal{F} = \bar{\mathsf{h}}(\nabla\wedge A) = \mathcal{D}\wedge\mathcal{A} = \bar{\mathsf{h}}(F). \tag{3.5}$$

(When quantum effects are included this ambiguity resurfaces, and only the Lagrangian approach can tell us which is the correct field strength.)

Equation (3.3) extends to the case of a general multivector. For example, we see that

$$\begin{aligned}\mathcal{D}\wedge\bar{\mathsf{h}}(a\wedge b) &= \bar{\mathsf{h}}(\partial_c)\wedge[\mathcal{D}_c\bar{\mathsf{h}}(a)\wedge\bar{\mathsf{h}}(b) + \bar{\mathsf{h}}(a)\wedge\mathcal{D}_c\bar{\mathsf{h}}(b)] \\ &= [\mathcal{D}\wedge\bar{\mathsf{h}}(a)]\wedge\bar{\mathsf{h}}(b) - \bar{\mathsf{h}}(a)\wedge[\mathcal{D}\wedge\bar{\mathsf{h}}(b)] \\ &= 0. \end{aligned} \tag{3.6}$$

Again the result for position dependent fields is also useful,

$$\mathcal{D}\wedge\bar{\mathsf{h}}(M) = \bar{\mathsf{h}}(\nabla\wedge M). \tag{3.7}$$

Now suppose that we apply Eq. (3.3) a second time. We find that

$$\mathcal{D}\wedge[\mathcal{D}\wedge\bar{\mathsf{h}}(A)] = \mathcal{D}\wedge\bar{\mathsf{h}}(\nabla\wedge A) = \bar{\mathsf{h}}(\nabla\wedge\nabla\wedge A) = 0. \tag{3.8}$$

This can be used to derive an algebraic identity for the Riemann tensor. The details are not important and are contained in Appendix A, but the conclusion is simply that

$$\partial_a\wedge\mathcal{R}(a\wedge b) = 0. \tag{3.9}$$

This expresses all of the symmetries of the Riemann tensor in one simple equation. This equation says that the trivector $\partial_a\wedge\mathcal{R}(a\wedge b)$ vanishes for all values of the vector b, so amounts to a set of 16 constraints. Since the Riemann tensor originally had 36 degrees of freedom, we are left with 20 degrees of freedom. We will see later what these correspond to.

3.2 The Second Field Equation

The Riemann tensor has dimensions of (length)$^{-2}$. Given that we are ignoring quantum effects, the only constant we can scale this with is G. If we form the quantity $\mathcal{R}(B)/G$, we see that this has dimensions of *energy density*. In this respect gravity is different from other gauge theories. The energy density is linear in $\mathcal{R}(B)$, whereas for electromagnetism, for example, the energy density is quadratic in the field strength (going as $\mathsf{T}(a) = -\frac{1}{2}FaF$).

On dimensional grounds, we expect to equate the Riemann tensor with some version of the energy density of the physical fields. But the latter is described by the stress-energy tensor, which is a linear function mapping vectors to vectors, not bivectors to bivectors. Since we have already established that $\partial_a\wedge\mathcal{R}(a\wedge b) = 0$, we next consider the contraction of $\mathcal{R}(a\wedge b)$ and define

$$\mathcal{R}(b) = \partial_a\cdot\mathcal{R}(a\wedge b). \tag{3.10}$$

This is called the Ricci tensor. It is still a covariant tensor, and divided by G it still has dimensions of energy density. Notice that we use the same symbol $\mathcal{R}(a)$ and $\mathcal{R}(a\wedge b)$ for the Ricci and Riemann tensors. Which is intended is defined by the grade of the linear argument. (This is the STA equivalent of using the number of indices on a tensor to determine which is intended.) We can perform one further contraction to define the Ricci scalar

$$\mathcal{R} = \partial_a \cdot \mathcal{R}(a). \tag{3.11}$$

This is a covariant scalar field, so is invariant under rotations, and transforms covariantly under displacements. The Ricci scalar is the first scalar observable we have constructed from the gravitational fields. It also turns out that $\mathcal{R}$ is the appropriate Lagrangian density for a formal development of gauge theory gravitation.

The key to deciding which of the gravitational fields to equate with the matter stress-energy tensor is provided by the *Jacobi Identity*. There is a version of this for all gauge theories and it arises straightforwardly from the identity

$$[D_a,[D_b,D_c]]\psi + [D_c,[D_a,D_b]]\psi + [D_b,[D_c,D_a]]\psi = 0. \tag{3.12}$$

Evaluating the commutators we find that

$$\mathcal{D}_a \mathsf{R}(b\wedge c) + \mathcal{D}_c \mathsf{R}(a\wedge b) + \mathcal{D}_b \mathsf{R}(c\wedge a) = 0 \tag{3.13}$$

which is the Jacobi identity. In electromagnetism this identity amounts to the statement $\nabla\wedge F = 0$, which records the fact that F is derived from a gauge field. The version for the rotation gauge field is slightly more complicated, and is derived in Appendix B. The conclusion is that

$$\bar{\mathsf{h}}(\partial_a)\wedge[\mathcal{D}_a\mathcal{R}(B) - \mathcal{R}(\mathcal{D}_a B)] = \dot{\mathcal{D}}\wedge\dot{\mathcal{R}}(B) = 0. \tag{3.14}$$

As is common in the STA, yet another of the useful, practical results has a very simple simple, memorable expression. Despite their formal simplicity, equations like Eq. (3.14) and Eq. (3.9) contain a wealth of information. They are also simple to implement on a computer in a symbolic algebra package such as Maple or Mathematica. These days, computer packages are an essential part of analysing the gravitational field equations, and it certainly appears that the gauge theory approach is better suited to this than the traditional, metric approach.

On contracting Eq. (3.14) we find that the Ricci tensor does not satisfy a covariant conservation equation. Instead, it is the *Einstein tensor* which has this property. This is defined by

$$\mathcal{G}(a) = \mathcal{R}(a) - \tfrac{1}{2}a\mathcal{R}. \tag{3.15}$$

We obtain our second field equation by equating the Einstein tensor with the covariant matter stress-energy tensor $\mathcal{T}(a)$,

$$\mathcal{G}(a) = \kappa\mathcal{T}(a), \tag{3.16}$$

where κ is a constant. This constant is determined by looking at spherically symmetric solutions and comparing with the Newtonian equations. From this analysis we conclude that $\kappa = 8\pi G$.

A Symmetries of $\mathcal{R}(B)$

Expanding out Eq. (3.8) we find that the left-hand side becomes

$$\begin{aligned}\mathcal{D}\wedge[\mathcal{D}\wedge\bar{\mathsf{h}}(A)] &= \bar{\mathsf{h}}(\partial_a)\wedge\mathcal{D}_a[\bar{\mathsf{h}}(\partial_b)\wedge\mathcal{D}_b\bar{\mathsf{h}}(A)] \\ &= [\mathcal{D}\wedge\bar{\mathsf{h}}(\partial_b)]\wedge\mathcal{D}_b\bar{\mathsf{h}}(A) + \bar{\mathsf{h}}(\partial_a)\wedge\bar{\mathsf{h}}(\partial_b)\wedge[\mathcal{D}_a\mathcal{D}_b A].\end{aligned} \tag{A.17}$$

The first term here vanishes, and the second contains a factor of $\bar{\mathsf{h}}(\partial_a\wedge\partial_b)$. We can therefore antisymmetrise the covariant derivatives to get

$$\begin{aligned}\tfrac{1}{2}\bar{\mathsf{h}}(\partial_a)\wedge\bar{\mathsf{h}}(\partial_b)\wedge[(\mathcal{D}_a\mathcal{D}_b - \mathcal{D}_b\mathcal{D}_a)A] &= \tfrac{1}{2}\bar{\mathsf{h}}(\partial_a)\wedge\bar{\mathsf{h}}(\partial_b)\wedge(\mathsf{R}(a\wedge b)\cdot A) \\ &= \tfrac{1}{2}\partial_a\wedge\partial_b\wedge[\mathcal{R}(a\wedge b)\cdot A] = 0.\end{aligned} \tag{A.18}$$

Here we have used the general result that, when forming a contraction between $\bar{\mathsf{h}}(\partial_a)$ and a, we can replace this with a contraction between ∂_a and $\mathsf{h}(a)$. The result is an algebraic identity satisfied by the Riemann tensor. This must be true for all A, so we can write

$$\partial_a\wedge\partial_b\wedge[\mathcal{R}(a\wedge b)\cdot c] = 0 \quad \forall\, a, b, c. \tag{A.19}$$

We next take the exterior product with ∂_c to get

$$\begin{aligned}\partial_c\wedge\partial_a\wedge\partial_b\wedge[\mathcal{R}(a\wedge b)\cdot c] &= -\partial_a\wedge\partial_b\wedge[\partial_c\wedge(c\cdot\mathcal{R}(a\wedge b))] \\ &= -2\partial_a\wedge\partial_b\wedge\mathcal{R}(a\wedge b) = 0.\end{aligned} \tag{A.20}$$

Now if we dot this result back with c again, we get

$$c\cdot[\partial_a\wedge\partial_b\wedge\mathcal{R}(a\wedge b)] = \partial_b\wedge\mathcal{R}(c\wedge b) - \partial_a\wedge\mathcal{R}(a\wedge c) + \partial_a\wedge\partial_b\wedge[c\cdot\mathcal{R}(a\wedge b)] = 0 \tag{A.21}$$

and using Eq. (A.19) again we arrive at the simple result

$$\partial_a\wedge\mathcal{R}(a\wedge b) = 0, \tag{A.22}$$

as used in the main text. A corollary of this result is that the Riemann tensor is symmetric, so satisfies

$$B_1\cdot\mathcal{R}(B_2) = \mathcal{R}(B_1)\cdot\mathcal{R}(B_2). \tag{A.23}$$

This alone reduces the degrees of freedom in the Riemann tensor from 36 to $\frac{1}{2}\cdot 6\cdot 7 = 21$, so is not quite as restrictive as Eq. (A.22).

B The Bianchi Identity

We seek a covariant expression of the Jacobi identity (3.13). The identity currently says that a bivector vanishes for all totally antisymmetrised combinations of a, b, c. The bivector must therefore be a function of the trivector $a\wedge b\wedge c$. We construct the adjoint version of the identity by forming

$$\partial_a\wedge\partial_b\wedge\partial_c\langle[a\cdot\nabla\mathsf{R}(b\wedge c)+\Omega(a)\times\mathsf{R}(b\wedge c)]B\rangle = 0, \tag{B.24}$$

which says that the the trivector on the left vanishes for all values of the bivector B. We next use the symmetry of the Riemann tensor to write

$$\mathsf{R}(B_1)\cdot B_2 = \mathcal{R}[\mathsf{h}^{-1}(B_1)]\cdot B_2 = B_1\cdot\bar{\mathsf{h}}^{-1}[\mathcal{R}(B_2)]. \tag{B.25}$$

The two terms in the Jacobi identity now become

$$\begin{aligned}\partial_a\wedge\partial_b\wedge\partial_c\langle a\cdot\nabla\mathsf{R}(b\wedge c)B\rangle &= \partial_a\wedge\partial_b\wedge\partial_c\langle b\wedge c\, a\cdot\nabla\big(\bar{\mathsf{h}}^{-1}[\mathcal{R}(B)]\big)\rangle\\ &= -2\nabla\wedge\big(\bar{\mathsf{h}}^{-1}[\mathcal{R}(B)]\big)\end{aligned} \tag{B.26}$$

and

$$\begin{aligned}\partial_a\wedge\partial_b\wedge\partial_c\langle\Omega(a)\times\mathsf{R}(b\wedge c)\,B\rangle &= \partial_a\wedge\partial_b\wedge\partial_c\langle\mathsf{R}(b\wedge c)\,B\times\Omega(a)\rangle\\ &= 2\partial_a\wedge\big(\bar{\mathsf{h}}^{-1}[\mathcal{R}(\Omega(a)\times B)]\big)\,.\end{aligned} \tag{B.27}$$

We next act of the full bivector with the $\bar{\mathsf{h}}$-field to get

$$\bar{\mathsf{h}}(\nabla)\wedge\mathcal{R}(B) - \bar{\mathsf{h}}(\dot{\nabla})\wedge\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}[\mathcal{R}(B)] - \bar{\mathsf{h}}(\partial_a)\wedge\mathcal{R}[\Omega(a)\times B] = 0. \tag{B.28}$$

Finally, we use our first field equation in the form of Eq. (3.7) to write the Jacobi identity in the form

$$\bar{\mathsf{h}}(\partial_a)\wedge[\mathcal{D}_a\mathcal{R}(B) - \mathcal{R}(\mathcal{D}_a B)] = 0, \tag{B.29}$$

which is now valid if B is position dependent. This is often referred to as the Bianchi identity. We adapt the over-dot notation for this form of covariant derivative, so that the Bianchi identity finally reduces to the simple expression

$$\dot{\mathcal{D}}\wedge\dot{\mathcal{R}}(B) = 0. \tag{B.30}$$

One advantage of the over-dot notation of Eq (B.30) is that it commutes with contractions. We can therefore easily form the contracted Bianchi identity

$$\partial_a\cdot[\dot{\mathcal{D}}\wedge\dot{\mathcal{R}}(a\wedge b)] = \dot{\mathcal{R}}(\dot{\mathcal{D}}\wedge b) - \mathcal{D}\wedge\mathcal{R}(b) \tag{B.31}$$

and contracting once more gives

$$-2\dot{\mathcal{R}}(\dot{\mathcal{D}}) + \mathcal{D}\mathcal{R} = -2\dot{\mathcal{G}}(\dot{\mathcal{D}}) = 0, \tag{B.32}$$

where $\mathcal{G}(a)$ is the *Einstein tensor*. It is this tensor which is covariantly conserved, so satisfies the same relation as the matter stress-energy tensor.