Physical Applications of Geometric Algebra

Handout 6

Balanced Algebras and Projective Geometry

We have seen how to represent both rotation groups and unitary groups in terms of rotors. We will now see how all matrix groups can be represented by rotors, and hence that all possible Lie algebras can be realised as bivector algebras. This is the ultimate motivation for the treatment given in this course. Incorporating general linear functions as rotors is achieved by working in the balanced algebra, which is generated by equal numbers of vectors with positive and negative square. Some of the algebraic considerations for these types of spaces will be useful when we turn to Minkowski spacetime.

As a separate application of geometric algebra we look at the treatment of projective geometry. This will give us a different view on the role of bivectors, and will take us up to some areas of active research.

1 The Balanced Algebra $\mathcal{G}_{n,n}$

The key to representing general linear transformations (the group $\mathrm{Gl}(n)$) in terms of rotors is the introduction of a second space of *opposite* signature. Suppose that we start with an n-dimensional orthonormal basis $\{e_k\}$, $e_i{\cdot}e_j = \delta_{ij}$. We introduce a second frame $\{f_k\}$ with the property that

$$f_i{\cdot}f_j = -\delta_{ij}, \qquad e_i{\cdot}f_j = 0. \tag{1.1}$$

The introduction of vectors whose square is negative does not alter any of the axioms of geometric algebra. The full algebra is generated by equal numbers ('balanced') of vectors with positive and negative square. This algebra is labeled $\mathcal{G}_{n,n}$.

1.1 Null Spaces

We next introduce the balanced analog of the complex 'doubling' bivector by defining

$$K = e_i f_i = e_1{\wedge}f_1 + e_2{\wedge}f_2 + \cdots + e_n{\wedge}f_n. \tag{1.2}$$

This has the properties that

$$e_i \cdot K = f_i, \qquad f_i \cdot K = f_i \cdot (e_j \wedge f_j) = -f_i \cdot f_j \, e_j = e_i. \tag{1.3}$$

It follows that

$$(a \cdot K) \cdot K = K \cdot (K \cdot a) = a \quad \forall a. \tag{1.4}$$

There is a crucial sign difference compared with the doubling bivector J, which means that K does not generate a complex structure. Instead it generates a *null* structure. To see this, take any vector a in $\mathcal{G}_{n,n}$ and define

$$n = a \pm a \cdot K. \tag{1.5}$$

We see that

$$\begin{aligned} n^2 &= a^2 \pm 2a \cdot (a \cdot K) + (a \cdot K)^2 = a^2 - \langle a \cdot K \, K a \rangle \\ &= a^2 - [(a \cdot K) \cdot K] \cdot a = a^2 - a^2 = 0. \end{aligned} \tag{1.6}$$

Vectors whose square is zero are called *null* vectors. They are important in relativity for describing the paths of photons.

The bivector K splits the vectors in $\mathcal{G}^1_{n,n}$ into two separate null vectors

$$a = a_+ + a_- \tag{1.7}$$

where

$$a_+ = \tfrac{1}{2}(a + a \cdot K), \qquad a_- = \tfrac{1}{2}(a - a \cdot K). \tag{1.8}$$

In this manner the space of vectors $\mathcal{G}^1_{n,n}$ decomposes into a direct sum of two null spaces. We denote the space of vectors of form a_+ by $\mathcal{V}_n$. Vectors in $\mathcal{V}_n$ are characterised by

$$a_+ \cdot K = a_+ \quad \forall a_+ \in \mathcal{V}_n. \tag{1.9}$$

From this we can see that all vectors in $\mathcal{V}_n$ square to zero. Such a space defines a *Grassmann algebra*. These are important in fermionic quantum field theory and supersymmetry.

1.2 Statement of Theorem

Every linear function acting on an n-dimensional vector space, $a \mapsto \mathsf{f}(a)$, can be represented in $\mathcal{V}_n$ by a transformation of the form

$$a_+ \mapsto M a_+ M^{-1}, \tag{1.10}$$

where M is a geometric product of an even number of unit vectors.

The idea is that vectors a in $\mathcal{G}_n$ are mapped to null vectors a_+ in $\mathcal{G}_{n,n}$. These are acted on by the multivector M in such a way that

$$\mathsf{f}(a) + \mathsf{f}(a)\cdot K = M(a + a\cdot K)M^{-1}. \tag{1.11}$$

This defines a map between linear functions $\mathsf{f}(a)$ and multivectors $M \in \mathcal{G}_{n,n}$. The map is not quite an isomorphism because both M and $-M$ generate the same function — the multivectors M form a *'double-cover'* representation. In order for the map to work, the action of M must not take us outside the space $\mathcal{V}_n$. This implies that

$$(Ma_+M^{-1})\cdot K = Ma_+M^{-1}, \tag{1.12}$$

so we must have

$$\begin{aligned} a_+ &= M^{-1}\,(Ma_+M^{-1})\cdot K\,M \\ &= M^{-1}\tfrac{1}{2}(Ma_+M^{-1}K - KMa_+M^{-1})M = a_+\cdot(M^{-1}KM). \end{aligned} \tag{1.13}$$

It follows that we require $M^{-1}KM = K$, or

$$MK = KM. \tag{1.14}$$

Since M is a product of an even number of unit vectors we must have $M\tilde{M} = \pm 1$. The subgroup with $M\tilde{M} = 1$ are rotors in $\mathcal{G}_{n,n}$, and their generators (the Lie algebra elements) are bivectors. The condition $RK\tilde{R} = K$ is then the direct analog of the condition that defined the unitary group in terms of rotors.

1.3 The Lie Algebra

The bivector generators are the set of bivectors that commute with K. The Jacobi identity ensures that the commutator product of two bivectors which commute with K results in a third which also commutes with K. We proceed as with the unitary group and construct

$$[(a\cdot K)\wedge(b\cdot K)]\times K = a\wedge(b\cdot K) + (a\cdot K)\wedge b = (a\wedge b)\times K \tag{1.15}$$

so that

$$[a\wedge b - (a\cdot K)\wedge(b\cdot K)]\times K = 0. \tag{1.16}$$

We can again run through all combinations of $\{e_i, f_i\}$ to produce the following bivector basis for the Lie algebra of the general linear group,

$$\begin{aligned} E_{ij} &= e_ie_j - f_if_j \qquad (i < j = 1\ldots n) \\ F_{ij} &= e_if_j - f_ie_j \qquad (i < j = 1\ldots n) \\ K_i &= e_if_i. \end{aligned} \tag{1.17}$$

The difference in structure between the Lie algebras of the linear group and the unitary group is solely down to the different signatures of their underlying spaces.

1.4 Singular Value Decomposition

The next step towards proving our theorem is to find a suitable decomposition of a linear function $\mathsf{f}(a)$. The key to this is the *singular value decomposition* (SVD) of a function. From the non-singular function $\mathsf{f}(a)$ we form the symmetric function $\bar{\mathsf{f}}\mathsf{f}(a)$. This has a spectrum of orthonormal eigenvectors d_i and eigenvalues λ_i,

$$\bar{\mathsf{f}}\mathsf{f}(d_i) = \lambda_i d_i \tag{1.18}$$

with the summation convention dropped for this subsection. Each of the λ_i are positive, since

$$d_i \cdot \bar{\mathsf{f}}\mathsf{f}(d_i) = [\mathsf{f}(d_i)]^2 = \lambda_i (d_i)^2 \tag{1.19}$$

and in a Euclidean space all vectors have positive square. (This limits the application of the SVD to Euclidean spaces.) We can write

$$\bar{\mathsf{f}}\mathsf{f}(a) = \sum_k \lambda_k \, a \cdot d_k \, d_k, \tag{1.20}$$

which has the well-defined square root

$$\mathsf{d}(a) = \sum_k (\lambda_k)^{1/2} a \cdot d_k \, d_k. \tag{1.21}$$

We now define $\mathsf{S} = \mathsf{f}\mathsf{d}^{-1}$. This satisfies

$$\bar{\mathsf{S}}\mathsf{S} = \bar{\mathsf{d}}^{-1}\bar{\mathsf{f}}\mathsf{f}\mathsf{d}^{-1} = \bar{\mathsf{d}}^{-1}\mathsf{d}^2\mathsf{d}^{-1} = \mathsf{I}, \tag{1.22}$$

where I is the identity function. It follows that S is an orthonormal transformation. We can therefore write

$$\mathsf{f}(a) = \mathsf{S}\mathsf{d}(a), \tag{1.23}$$

which decomposes a general non-singular function into the product of a series of dilations (a symmetric function) followed by an orthonormal transformation.

A further rotation can be used to bring the $\{d_i\}$ frame onto the $\{e_i\}$ frame, which is usually required when working with matrices. The conclusion then is that every non-singular matrix can be written as a diagonal matrix of positive entries sandwiched between two distinct orthonormal matrices (check the degrees of freedom). This is the SVD of a matrix and is very useful in signal processing and data analysis. We do not need this second rotation for our proof as we can continue to work in a more frame-independent manner.

1.5 Proof of Theorem

To prove the theorem we need to demonstrate that both orthonormal transformations and positive dilations can be found as transformations of the type of Eq. (1.10) in $\mathcal{G}_{n,n}$. Rotations are present as they are generated by the E_{ij} bivectors in the Lie algebra. These bivectors jointly rotate the $\{e_i\}$ and $\{f_i\}$ vectors by the same amount. Next we need reflections. Suppose the reflection in $\mathcal{G}_n$ is generated by the unit vector n. We define

$$\hat{n} = n\cdot K, \qquad \hat{n}^2 = -1. \tag{1.24}$$

The multivector generator M is then $n\hat{n}$. This satisfies

$$n\hat{n}K = 2n\,\hat{n}\cdot K + nK\hat{n} = 2(n^2 + \hat{n}^2) + Kn\hat{n} = Kn\hat{n}, \tag{1.25}$$

so the multivector does commute with K. Since $(n\hat{n})^2 = +1$ this multivector is not a rotor. Its action on vectors $a_+ \in \mathcal{V}_n$ results in the vector

$$-n\hat{n}a_+\hat{n}n = -n\hat{n}a\hat{n}n - (n\hat{n}a\hat{n}n)\cdot K = -nan - (nan)\cdot K, \tag{1.26}$$

where we have used the result that $a\cdot\hat{n} = 0$. This holds because $\hat{n}$ is constructed entirely from the $\{f_i\}$ frame. Eq. (1.26) is the required result for a reflection. The need to incorporate reflections is what forces us to include multivectors with $M\tilde{M} = -1$.

The final step is to see how dilations are formulated with rotors. Suppose that we need a positive dilation in the n direction, where n is one of the eigenvectors of $\mathsf{d}(a)$. We again form the bivector $n\hat{n}$, which we can see is constructed from the F_{ij} and K_i Lie algebra generators. With $n_+ = n + \hat{n}$ the equivalent of the vector n in $\mathcal{V}_n$ we find that

$$\begin{aligned} \mathrm{e}^{-\lambda n\hat{n}/2}\, n_+\, \mathrm{e}^{\lambda n\hat{n}/2} &= \mathrm{e}^{-\lambda n\hat{n}}\, n_+ \\ &= [\cosh(\lambda) - n\hat{n}\sinh(\lambda)](n+\hat{n}) \\ &= [\cosh(\lambda) + \sinh(\lambda)](n+\hat{n}) \;=\; \mathrm{e}^{\lambda}\, n_+, \end{aligned} \tag{1.27}$$

which is a pure dilation. Furthermore, any vector perpendicular to n has an image in $\mathcal{V}_n$ which commutes with $n\hat{n}$ and so is unaffected by the action of the rotor. These are precisely the required properties of the positive dilation, which completes the proof.

We now have an alternative means of representing every matrix group within geometric algebra. Since *all* Lie algebras can be represented by matrices, we have proved that all Lie algebras can be realised as bivector algebras. The accompanying Lie group elements can then all be written as even products of unit vectors. This is potentially a very powerful idea, though it remains to be fully exploited.

2 Projective Geometry

It is a seldom stressed fact that the relationship between mathematics and physics is far from one-to-one. Most physical systems can be studied using different mathematics, and the same piece of mathematics will frequently find applications across a range of physical problems, with different interpretations placed on the same mathematical objects. We will now explore how some of the algebraic results we have derived can be pictured differently through *projective geometry*.

Points in 3-d space are projected onto a 2-d plane (see Fig. 1). This is obviously an important concept in *computer vision*. The key principle is that points in the plane (a_1, a_2) are represented by *vectors* in a space of one dimension higher. This is the essence of projective geometry. The magnitude of the vector is unimportant as both a and λa represent the same point. This is sometimes stated by saying that projective geometry does not require a metric (*i.e.* a measure of length). This is not the same as saying that we do not need an inner product. Now that we have placed a different interpretation on the role of vectors, the inner product also has a different interpretation attached. It no longer has a role in determining lengths, but it is crucial in forming projections onto planes and lines.

2.1 The Join

Now that we have a representation for points in a plane, the next thing we want to represent is the line joining them together. Fig. 1 shows that this line is the result of projecting the plane defined by a_1 and a_2 onto the projective plane. We therefore define the join of the points a and b by

$$\operatorname{join}(a, b) = a \wedge b. \tag{2.1}$$

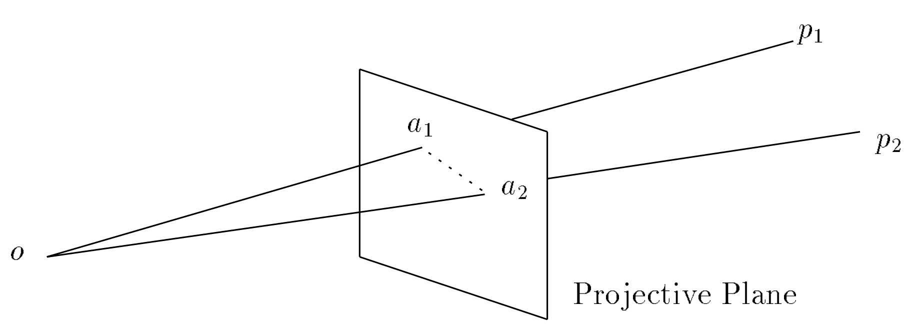

Figure 1: *Projective Geometry.* Points in the projective plane are represented by vectors in a space one dimension higher.

Bivectors are therefore used to define lines in projective geometry. Similarly, we can keep on taking exterior products to define (projectively) higher dimensional objects. For example, the join of a point a and a line $b\wedge c$ is the plane defined by the trivector $a\wedge b\wedge c$. This enables us to write down a condition for three points to be collinear. The points cannot define a projected area, so we set

$$a\wedge b\wedge c = 0, \tag{2.2}$$

which implies that the points a, b, c lie on a common line.

To handle complicated 3-d problems we need to work with a 4-d algebra. This algebra contains 6 bivectors, which represent lines in 3-d. The reason that 6 bivectors are required is because we are no longer restricting all vectors to have a common origin. To specify a line requires 5 components, three to specify a point on the line and two to determine the direction. The 6 possible bivector components are reduced to 5 by requiring that the bivector is a pure blade, formed by the join of two points on the line. The algebraic condition for this is

$$B\wedge B = 0, \tag{2.3}$$

which removes a degree of freedom. We also have a projective interpretation for commuting bivectors in 4-d. These represent lines in 3-d which do not share a common point.

2.2 Duality and the Meet

The next concept we require is that of the *meet*, which describes the intersection of geometric objects. This is encoded via the duality operation introduced in Handout 3. We denote the dual of an r-blade by

$$A_r^* = A_r I = A_r \cdot I = (-1)^{r(n-r)} I A \tag{2.4}$$

where I is the pseudoscalar. The result is a blade of grade $n-r$. In 3-d the dual of a line (a bivector) is a conjugate point (a vector). The pseudoscalar interchanges inner and outer products via

$$\begin{aligned} A_r\cdot(B_s I) &= A_r\wedge B_s\, I \qquad && r+s \leq n \\ A_r\wedge(B_s I) &= A_r\cdot B_s\, I \qquad && r \leq s. \end{aligned} \tag{2.5}$$

These are applicable not just when I is the overall pseudoscalar, but also when I is the pseudoscalar for any subspace, provided that I contains all of the vectors in A_r and B_s.

We define the meet $A \vee B$ by a 'de Morgan rule'

$$(A \vee B)^* = A^* \wedge B^*, \tag{2.6}$$

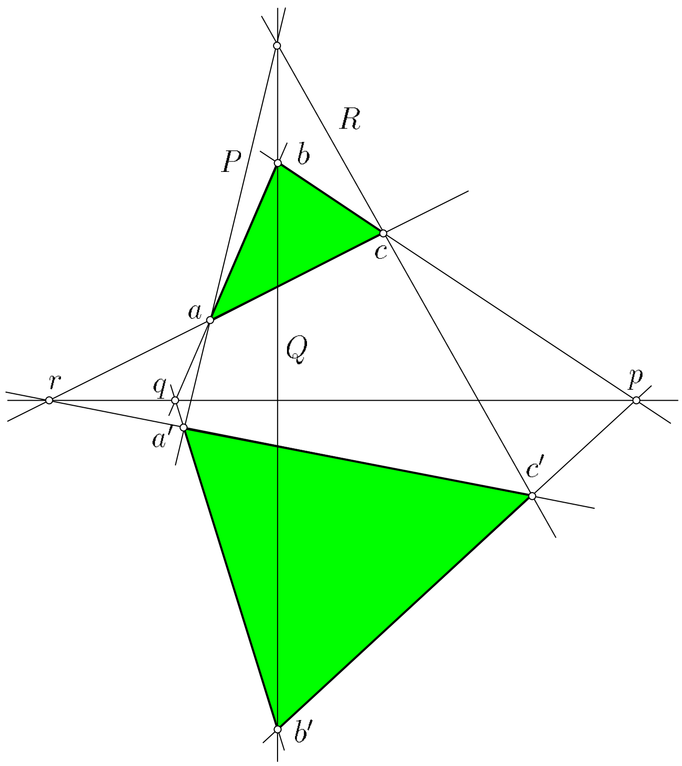

Figure 2: *Desargues' theorem.* The lines P, Q, R meet at a point if and only if the points p, q, r lie on a line. The two triangles are then projectively related.

where the dual is formed with respect to the highest grade blade that can be formed from the vectors in the blades A and B. For example, consider the meet of two lines in a plane. The appropriate pseudoscalar has grade 3, so we can work in $\mathcal{G}_3$. The dual of the meet is given by the join of two vectors. The meet of two lines is therefore described by the dual of a bivector, which is a vector, and so represents a point. That is, two lines meet at a point. In this case we have

$$A \vee B = (A^* \wedge B^*) I^{-1} = A \times B\, I, \tag{2.7}$$

where A and B are bivectors in $\mathcal{G}_3$.

2.3 Example — Desargues' Theorem

We can put the preceding definitions into practice with a simple proof of Desargues' theorem for two triangles (see Fig. 2). The two sets of points a, b, c and a', b', c' define two triangles, and we define the pseudoscalars

$$J = a \wedge b \wedge c, \qquad J' = a' \wedge b' \wedge c'. \tag{2.8}$$

We also define the lines

$$A = b\wedge c, \quad B = c\wedge a, \quad C = a\wedge b, \tag{2.9}$$

with the same definitions holding for A', B', C' in terms of a', b', c'. The two sets of points determine the lines

$$P = a\wedge a', \quad Q = b\wedge b', \quad R = c\wedge c', \tag{2.10}$$

and the two sets of lines determine the points

$$p = A\times A' I, \quad q = B\times B' I, \quad r = C\times C' I. \tag{2.11}$$

To find the condition that three lines meet at a point we use

$$(P \vee Q)\wedge R = \langle P\times Q\, RI\rangle_3 = \langle PQR\rangle I, \tag{2.12}$$

so that the condition becomes

$$\langle PQR\rangle = \langle a\wedge a'\, b\wedge b'\, c\wedge c'\rangle = 0. \tag{2.13}$$

Similarly, for p, q, r to fall on a line we form

$$\begin{aligned} p\wedge q\wedge r &= \langle A\times A' I\, B\times B' I\, C\times C' I\rangle_3 \\ &= -I\langle A\times A'\, B\times B'\, C\times C'\rangle. \end{aligned} \tag{2.14}$$

Desargues' theorem is then proved by the algebraic identity (exercise)

$$JJ'\langle a\wedge a'\, b\wedge b'\, c\wedge c'\rangle = \langle A\times A'\, B\times B'\, C\times C'\rangle \tag{2.15}$$

The left-hand side vanishes if and only if the right-hand side does, proving the theorem. Notice how a quite complicated and involved picture can be attached to a basic algebraic identity!

2.4 Homogeneous Coordinates

In many applications we are interested in the relationship between coordinates in the image plane (for example in terms of pixels relative to some origin) and the 3-d position vector. Suppose that the origin in the image plane is defined by the vector n, which is perpendicular to the plane. The line on the image plane from the origin to the image point is represented by the bivector $a\wedge n$ (see Fig. 3) . The vector OA belongs to a 2-d geometric algebra. We can relate this directly to the 3-d algebra by first writing

$$n + OA = \lambda a. \tag{2.16}$$

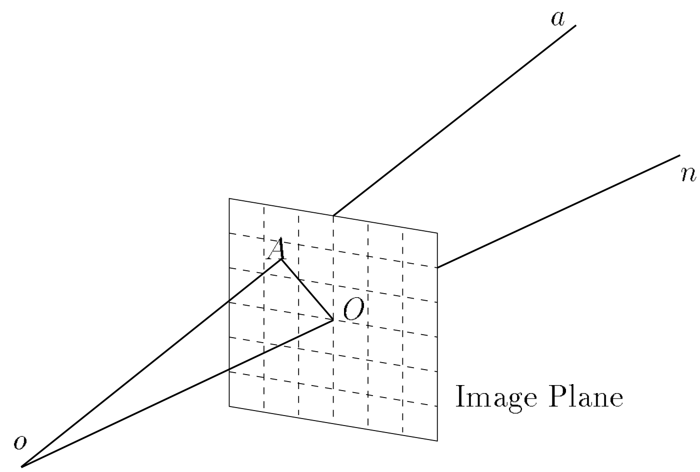

Figure 3: *The Image Plane.* Vectors in the image plane, OA, are described by bivectors in $\mathcal{G}_3$. The point A can be expressed in terms of *homogeneous* coordinates.

Dotting with n, and choosing units so that $n^2 = 1$, we find that $\lambda = (a\cdot n)^{-1}$ so

$$OA = \frac{a - a\cdot nn}{a\cdot n} = \frac{a\wedge n}{a\cdot n} n. \tag{2.17}$$

The final factor of n can be dropped so that we directly represent the line OA in 2-d with the bivector

$$A = \frac{a\wedge n}{a\cdot n}. \tag{2.18}$$

We can also write

$$an = a\cdot n + a\wedge n = a\cdot n(1 + A). \tag{2.19}$$

We will meet this type of projective transformation again in the context of relativity.

The map (2.18) relates bivectors in a higher dimensional space to vectors in a space of dimension one lower. If we introduce a coordinate frame with $n = e_3$ we see that the coordinates of the image of $a = a_i e_i$ are

$$A = \frac{a_1}{a_3} e_1 e_3 + \frac{a_2}{a_3} e_2 e_3 = A_1 E_1 + A_2 E_2. \tag{2.20}$$

The components $A_i = a_i/a_3$ are called *homogeneous coordinates*, as they are independent of scale. It is these that are usually measured. The map between a and A is a nonlinear map in 3-d. This can be turned into a linear map by representing points in 3-d with vectors in 4-d.

One small price to pay for these definitions is that our basis vectors $\{E_1, E_2\}$ have negative square, so span an anti-Euclidean space. This changes very little, though special relativity does offer an elegant alternative.

3 Invariants

An important problem in computer vision is to recover geometric information about the 3-d world from various 2-d image planes. Information from different views is then often used to construct a 3-d model of the world. Some of the most important objects to study are the *projective invariants*, which are quantities which are independent of the camera position. These are useful because they can be used to check that point identifications from different views are consistent.

Consider the situation described in Fig. 3. A set of 4 vectors a, b, c, d project out two sets of points on two distinct lines. We want to find an invariant formed from ratios of lengths. Vectors along these lines are handled projectively by bivectors. Suppose again that n is the unit normal vector to the line, so that

$$OA = A = \frac{a\wedge n}{a\cdot n}. \tag{3.1}$$

Choosing n to be a unit vector imposes a scale. Clearly it will only be ratios of lengths (scale invariants) that can be genuine invariants. We now form the bivector for AB,

$$AB = \frac{b\wedge n}{b\cdot n} - \frac{a\wedge n}{a\cdot n} = \frac{(a\cdot n\, b - b\cdot n\, a)\wedge n}{a\cdot n\, b\cdot n} = \frac{[(b\wedge a)\cdot n]\wedge n}{a\cdot n\, b\cdot n} = \frac{b\wedge a}{a\cdot n\, b\cdot n}. \tag{3.2}$$

So, as expected, AB is determined by the bivector $a\wedge b$. What we want to do is form an invariant which is independent of n, and so will be the same if measured on L or L'.

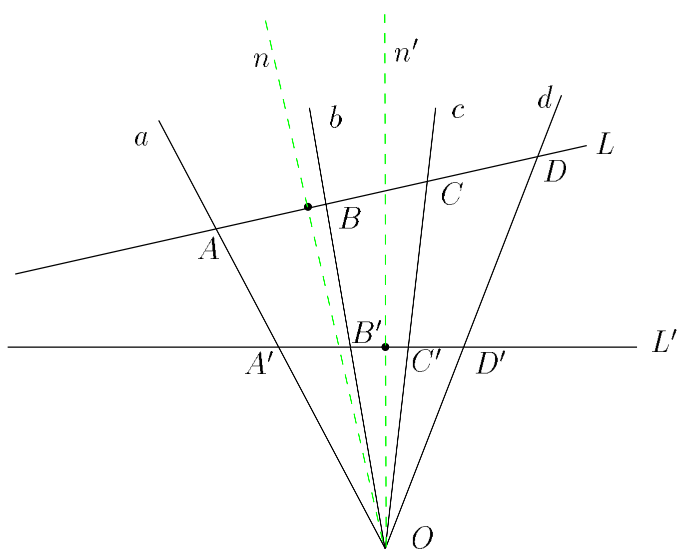

Figure 4: *A Line Invariant.* Points on the lines L and L' represent two different projective views of the same vectors in space.

The expression should only contain a, b, c, d, and it is clear that we need to assemble ratios of lengths to remove factors of $a \cdot n$, *etc.* In particular we form

$$\rho = \frac{AC}{BC}\frac{BD}{AD} = \frac{a \wedge c}{b \wedge c}\frac{b \wedge d}{a \wedge d} \tag{3.3}$$

which is manifestly independent of the chosen projection.

For the projection of a 3-d image onto a 2-d camera plane it is clear that the analogous objects must be ratios of trivectors, which represent areas in the camera plane. For example, suppose we have 6 points in space with position vectors $a_1 \cdots a_6$. These produce the 6 projected points $A_1 \cdots A_6$. An invariant is formed by

$$\frac{a_5 \wedge a_4 \wedge a_3}{a_5 \wedge a_1 \wedge a_3}\frac{a_5 \wedge a_2 \wedge a_1}{a_5 \wedge a_2 \wedge a_4} = \frac{A_{543}}{A_{513}}\frac{A_{521}}{A_{524}} \tag{3.4}$$

where A_{ijk} is the projected area of the triangle with vertices A_i, A_j, A_k. This example again demonstrates how geometric reasoning can quickly yield useful algebraic formulae when working with geometric algebra.