Physical Applications of Geometric Algebra

Handout 12

The Dirac Equation

The relativistic wave equation for a spin-1/2 particle is the *Dirac equation*. This is a first order wave equation, which is necessary to achieve an equation which is Lorentz invariant and which has a future-pointing conserved current. The theory of the Dirac equation is a large subject and we will only touch briefly on a few of its properties, paying particular attention to the areas where the STA version offers significant advantages. We start by looking again at a relativistic model for a particle with $g = 2$.

1 The Gyromagnetic Ratio

In Handout 11 we saw that the spin vector observable formed from a relativistic spinor is $s = \rho R \gamma_3 \tilde{R}$. This is the justification behind the model for a point particle with spin described in Handout 9. The dynamics of the particle is described by the rotor equation

$$\dot{R} = \frac{e}{2m} F R, \tag{1.1}$$

and the particle has velocity $v = R\gamma_0\tilde{R}$. It was shown in Handout 9 that this model reproduces the spin-precession equations for a particle with $g = 2$. We can see this a different way by making contact directly with a Pauli rotor equation. We first decompose R as

$$R = LU \tag{1.2}$$

where L is a pure boost and U a pure rotation in the γ_0 frame (see Handout 9, Section 1.4). With this decomposition the rotor equation becomes

$$\dot{R}\tilde{R} = \dot{L}\tilde{L} + L\dot{U}\tilde{U}\tilde{L} = \frac{e}{2m} F, \tag{1.3}$$

hence

$$\tilde{L}\dot{L} + \dot{U}\tilde{U} = \frac{e}{2m} \tilde{L} F L. \tag{1.4}$$

This shows that U responds to the de-boosted field $\tilde{L}FL$.

The non-relativistic limit is taken by setting

$$L \approx 1 + \boldsymbol{v}/2, \qquad |\boldsymbol{v}| < 1, \tag{1.5}$$

and working to $\mathrm{O}(\boldsymbol{v}^2)$. Suppose then that $F = I\boldsymbol{B}$ is a pure magnetic field in the laboratory (γ_0) frame. In this case

$$\tilde{L}FL = I\boldsymbol{B} - \boldsymbol{v}\cdot(I\boldsymbol{B}) + \mathrm{O}(\boldsymbol{v}^2) \tag{1.6}$$

so the only relative bivector present in the non-relativistic limit is just $I\boldsymbol{B}$. The relative bivector component of $\tilde{L}\dot{L}$ is also $\mathrm{O}(\boldsymbol{v}^2)$ so in this approximation we are left with

$$\dot{U}\tilde{U} = \frac{e}{2m}I\boldsymbol{B}, \quad \Longrightarrow \quad \dot{U} = \frac{e}{2m}I\boldsymbol{B}U. \tag{1.7}$$

In the non-relativistic limit we can also replace the proper-time derivative by the derivative with respect to coordinate time t. In this case U satisfies precisely the non-relativistic rotor equation identified in Handout 11. Comparing directly with Eq. 2.11 of Handout 11, we see that

$$\gamma = g\frac{e}{2m} = \frac{e}{m} \tag{1.8}$$

hence $g = 2$ again drops out naturally in this model.

2 The Dirac Equation

While much of the preceding is both suggestive about the role of spinors in quantum theory, and algebraically very useful, one has to remember that quantum mechanics deals with *wave equations*. We therefore need to construct a relativistic wave equation for our Dirac spinor ψ, where ψ is an element of the 8-dimensional even subalgebra of the STA. Like Pauli spinors, ψ also has a single-sided rotor transformation law, $\psi \mapsto R\psi$, where R is a Lorentz rotor. To write down a covariant equation, we can therefore only place other covariant objects on the left of ψ. The available objects are any scalar or pseudoscalar, the vector derivative ∇, and any gauge fields describing interactions. On the right of ψ we can place combinations of γ_0, γ_3 and $I\boldsymbol{\sigma}_3$. The first equation we could write down is simply

$$\nabla\psi = 0. \tag{2.1}$$

This is the generalisation to the STA of the Cauchy-Riemann equations. Remarkably, this equation does describe the behaviour of fermions — it is the wave equation for a *neutrino*. Any solution to this decomposes into two separate solutions by writing

$$\psi = \psi\tfrac{1}{2}(1 + \boldsymbol{\sigma}_3) + \psi\tfrac{1}{2}(1 - \boldsymbol{\sigma}_3) = \psi_+ + \psi_-. \tag{2.2}$$

The separate solutions ψ_+ and ψ_- are the right-handed and left-handed helicity eigenstates. For neutrinos, nature only appears to make use of the left-handed solutions. A more complete treatment of this subject involves the *electroweak* theory.

2.1 The Massive Dirac Equation

The formal operator identification of $i\partial_\mu$ with p_μ tells us that any wavefunction for a free massive particle should satisfy $\nabla^2\psi = -m^2\psi$. We therefore need to add a term to the right-hand side of Eq. (2.1) which is linear in the particle mass m and which generates $-m^2\psi$ on squaring the operator. If we think about plane-wave states with momentum p, we arrive at an equation of the form

$$p\psi = m\psi a_0 \tag{2.3}$$

where a_0 is some multivector to be determined. It is immediately clear that a_0 must have odd grade, and must square to $+1$. The obvious candidate is γ_0, so that ψ contains a rotor to transform γ_0 to the velocity p/m. We are therefore led to the equation

$$\nabla\psi I\boldsymbol{\sigma}_3 = m\psi\gamma_0 \tag{2.4}$$

or, post-multiplying by $I\boldsymbol{\sigma}_3$,

$$\nabla\psi = -m\psi I\gamma_3. \tag{2.5}$$

This is the *Dirac equation* in its STA form. The more common matrix/spinor form is recovered by converting ψ back to a column spinor, and writing

$$\nabla\psi\gamma_0 \;\leftrightarrow\; \hat{\gamma^\mu}\partial_\mu|\psi\rangle \tag{2.6}$$

for the vector derivative.

Our simple reasoning has led us to a first-order wave equation for the spinor wavefunction ψ. The observables for the wavefunction are as listed in Table 1 of Handout 11. Most important amongst these is the current $J = \psi\gamma_0\tilde{\psi}$. This satisfies

$$\begin{aligned}\nabla\cdot J &= \langle\nabla\psi\gamma_0\tilde{\psi}\rangle + \langle\psi\gamma_0\dot{\tilde{\psi}}\dot{\nabla}\rangle \\ &= \langle\nabla\psi\gamma_0\tilde{\psi}\rangle + \langle\psi(\nabla\psi\gamma_0)^\sim\rangle \\ &= -m\langle\psi I\boldsymbol{\sigma}_3\tilde{\psi} + \psi(\psi I\boldsymbol{\sigma}_3)^\sim\rangle = 0\end{aligned} \tag{2.7}$$

and so is conserved. This is important. It means that single fermions cannot be created or destroyed. Of course, fermion pairs such as an electron and a positron can annihilate one another, but that is a many-body problem and is described by *quantum field theory*. The timelike component of J in the γ_0 frame, say, is

$$J_0 = \gamma_0\cdot J = \langle\gamma_0\tilde{\psi}\gamma_0\psi\rangle = \langle\psi^\dagger\psi\rangle > 0 \tag{2.8}$$

which is *positive definite*. This is interpreted as a probability density, and localised wave functions are usually normalised such that

$$\int d^3x\, J_0 = 1 \tag{2.9}$$

Arriving at a relativistic theory with a consistent probabilistic interpretation was Dirac's original goal.

2.2 Plane-Wave States

A positive energy plane-wave state is defined by

$$\psi = \psi_0\, \mathrm{e}^{-I\boldsymbol{\sigma}_3 p\cdot x} \tag{2.10}$$

where ψ_0 is a constant spinor. The Dirac equation (2.4) tells us that ψ_0 satisfies

$$p\psi_0 = m\psi_0\gamma_0, \tag{2.11}$$

and post-multiplying by $\tilde{\psi}_0$ we see that

$$p\psi_0\tilde{\psi}_0 = mJ. \tag{2.12}$$

Recalling from Handout 11 that we can write $\psi\tilde{\psi} = \rho\, \mathrm{e}^{i\beta}$, and noting that both p and J are vectors, we see that we must have $\exp(i\beta) = \pm 1$. For positive energy states the timelike component of p is positive, as is the timelike component of J, so we take the positive solution $\beta = 0$. It follows that ψ_0 is then simply a rotor with a normalisation constant. The proper boost L taking $m\gamma_0$ onto the momentum has

$$p = mL\gamma_0\tilde{L} = mL^2\gamma_0, \tag{2.13}$$

which we know from Eq. (1.18) of Handout 9 is solved by

$$L = \frac{m + p\gamma_0}{[2m(m + p\cdot\gamma_0)]^{1/2}} = \frac{E + m + \boldsymbol{p}}{[2m(E + m)]^{1/2}}, \tag{2.14}$$

where we have employed the spacetime split $p\gamma_0 = E + \boldsymbol{p}$. The full spinor ψ_0 is LU, where U is a spatial rotor in the γ_0 frame, so is a Pauli spinor.

Negative energy solutions have a phase factor of $\mathrm{e}^{+I\boldsymbol{\sigma}_3 p\cdot x}$, with $E = \gamma_0\cdot p > 0$. For these we have $-p\psi\tilde{\psi} = mJ$ so it is clear that we now need $\beta = \pi$. Positive and negative energy plane wave states can therefore be summarised by

$$\begin{array}{ll} \text{positive energy} & \psi^{(+)}(x) = L(p)U\, \mathrm{e}^{-I\boldsymbol{\sigma}_3 p\cdot x} \\ \text{negative energy} & \psi^{(-)}(x) = L(p)UI\, \mathrm{e}^{I\boldsymbol{\sigma}_3 p\cdot x} \end{array} \tag{2.15}$$

with $L(p)$ given by Eq. (2.14). These are fundamental components in *scattering theory*.

2.3 Hamiltonian Form and Angular Operators

A useful notational device in the STA formalism is to borrow the 'i' symbol from standard quantum theory as an abbreviation for right-sided multiplication by $I\boldsymbol{\sigma}_3$. If we now pre-multiply our Dirac equation (2.4) by γ_0 we arrive at

$$i\partial_t\psi = -\boldsymbol{\nabla}\psi I\boldsymbol{\sigma}_3 + m\gamma_0\psi\gamma_0 = -i\boldsymbol{\nabla}\psi + m\bar{\psi}. \tag{2.16}$$

where $\bar{\psi} = \gamma_0\psi\gamma_0$. The right-hand side of this equation defines the *Hamiltonian*, which we denote with the symbol $\mathcal{H}$,

$$\mathcal{H}\psi = -i\boldsymbol{\nabla}\psi + \bar{\psi}. \tag{2.17}$$

The use of γ_0 in defining $\mathcal{H}$ shows that the Hamiltonian is an observer dependent concept in relativistic physics.

Quantum states are usually classified in terms of eigenstates of operators which commute with the Hamiltonian $\mathcal{H}$, because the accompanying quantum numbers are conserved in time. Of particular importance are the angular momentum operators $\hat{L}_i$, defined by

$$\hat{L}_i = -i\epsilon_{ijk}x_j\partial_k. \tag{2.18}$$

These are the components of the bivector operator $i\boldsymbol{x}\wedge\boldsymbol{\nabla}$. We therefore define the equivalent STA operators

$$L_B = iB\cdot(\boldsymbol{x}\wedge\boldsymbol{\nabla}), \tag{2.19}$$

where B is a relative bivector. Writing $B = I\boldsymbol{\sigma}_i$ recovers the component form. The L_B operators satisfy the commutation relations (exercise)

$$[L_{B_1}, L_{B_2}] = -iL_{B_1\times B_2}. \tag{2.20}$$

The angular momentum commutation relations directly encode the bivector commutation relations, and so relate back to the rotation (Lie) group. This is sensible, as rotations form a *symmetry group* of the system.

If we now form the commutator of L_B with the Hamiltonian $\mathcal{H}$ we find something slightly surprising. The scalar operator L_B commutes with the bar operator $\psi \mapsto \bar{\psi}$, but for the momentum term we get

$$[L_B, \mathcal{H}] = [B\cdot(\boldsymbol{x}\wedge\boldsymbol{\nabla}), \boldsymbol{\nabla}] = -\dot{\boldsymbol{\nabla}}B\cdot(\dot{\boldsymbol{x}}\wedge\boldsymbol{\nabla}) = B\times\boldsymbol{\nabla}. \tag{2.21}$$

The commutator does not vanish, so orbital angular momentum does not yield a conserved quantum number in relativistic physics. But, since $B\times\boldsymbol{\nabla} = \frac{1}{2}(B\boldsymbol{\nabla} - \boldsymbol{\nabla}B)$, we can write Eq. (2.21) as

$$[B\cdot(\boldsymbol{x}\wedge\boldsymbol{\nabla}) - \tfrac{1}{2}B, \mathcal{H}] = 0. \tag{2.22}$$

We therefore recover a conserved angular momentum operator by defining

$$J_B = L_B - \tfrac{1}{2}iB. \tag{2.23}$$

In conventional notation this is

$$\hat{J}_i = \hat{L}_i + \tfrac{1}{2}\hat{\Sigma}_i \tag{2.24}$$

where $\hat{\Sigma}_i = (i/2)\epsilon_{ijk}\hat{\gamma}_j\hat{\gamma}_k$. The extra term of $\frac{1}{2}B$ is conventionally viewed as defining "spin-1/2". If we look for eigenstates of the J_3 operator, we see that the spin contribution to this is

$$-\tfrac{1}{2}iI\boldsymbol{\sigma}_3\psi = \tfrac{1}{2}\boldsymbol{\sigma}_3\psi\boldsymbol{\sigma}_3. \tag{2.25}$$

In the non-relativistic Pauli theory the eigenstates of this operator are simply 1 and $-I\boldsymbol{\sigma}_2$, with eigenvalues $\pm\frac{1}{2}$. In the relativistic theory the separate spin and orbital operators are not conserved, and it is only the combined J_B operators that commute with the Hamiltonian.

The geometric algebra derivation employed here highlights some interesting features. Stripping away all of the extraneous terms, the result rests solely on the commutation properties of the $B\cdot(\boldsymbol{x}\wedge\boldsymbol{\nabla})$ and $\boldsymbol{\nabla}$ operators. The factor of $1/2$ would therefore be present in any dimension, and so has no special relation to the 3-d rotation group. Furthermore, in writing $J_B = L_B - \frac{1}{2}iB$ we are forming an explicit sum of a scalar and a bivector. The standard notation of Eq. (2.24) encourages us to view these as the sum of two vector operators!

2.4 Central Potentials

Coupling to a central potential $V(r)$ is easily achieved by adding the term $eV(r)$ to the Hamiltonian. The full justification for this term comes from *gauge invariance*, which is discussed in the following handout. The J_B operators still commute with the Hamiltonian, as $\boldsymbol{x}\wedge\boldsymbol{\nabla}V(r) = 0$. The key to solving the Dirac equation in a central potential is provided by analytic functions in 3-d. These are Pauli spinors satisfying

$$\boldsymbol{\nabla}\Psi = 0. \tag{2.26}$$

Since these functions immediately satisfy $\boldsymbol{\nabla}^2\Psi = 0$, their components are spherical harmonics. The radial dependence of these goes as r^l, where l is an integer for physical solutions. We therefore separate out the radial and angular dependence and write,

$$\Psi = r^l\psi(\theta,\phi) \tag{2.27}$$

where (r, θ, ϕ) are 3-d spherical polar coordinates. We now see that

$$0 = \boldsymbol{x}\boldsymbol{\nabla}\Psi = r\partial_r\Psi + \boldsymbol{x}\wedge\boldsymbol{\nabla}\Psi, \tag{2.28}$$

so the angular term ψ satisfies the eigenvalue equation

$$-\boldsymbol{x}\wedge\boldsymbol{\nabla}\psi = l\psi. \tag{2.29}$$

Adapting the argument showing that J_B commutes with $\boldsymbol{\nabla}$, it is not hard to show that J_B also commutes with $\boldsymbol{x}\wedge\boldsymbol{\nabla}$. The functions $\psi(\theta,\phi)$ can therefore be simultaneous eigenstates of both $\boldsymbol{x}\wedge\boldsymbol{\nabla}$ and one of the J_B. The latter is conventionally chosen to be J_3, and we write these eigenstates as ψ_l^m. They satisfy

$$\begin{array}{ll} -\boldsymbol{x}\wedge\boldsymbol{\nabla}\psi_l^m = l\psi_l^m & l \geq 0 \\ J_3\psi_l^m = (m+\frac{1}{2})\psi_l^m & -1-l \leq m \leq l. \end{array} \tag{2.30}$$

Eigenstates with negative values of l are constructed by first defining $\boldsymbol{\sigma}_r = \boldsymbol{x}/r$ and noting that

$$\boldsymbol{x}\wedge\boldsymbol{\nabla}\boldsymbol{\sigma}_r = \boldsymbol{x}\wedge\boldsymbol{\nabla}(\boldsymbol{x}/r) = \frac{1}{r}\boldsymbol{x}\wedge\boldsymbol{\nabla}\boldsymbol{x} = 2\boldsymbol{\sigma}_r. \tag{2.31}$$

It follows that

$$-\boldsymbol{x}\wedge\boldsymbol{\nabla}(\boldsymbol{\sigma}_r\psi_l^m\boldsymbol{\sigma}_3) = -2\boldsymbol{\sigma}_r\psi_l^m\boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_r\,\boldsymbol{x}\wedge\boldsymbol{\nabla}\psi_l^m\boldsymbol{\sigma}_3 = -(l+2)\boldsymbol{\sigma}_r\psi_l^m\boldsymbol{\sigma}_3. \tag{2.32}$$

We can therefore construct eigenstates with l running downwards from -2. It is convenient to introduce the label $\kappa = l+1$, so that κ is a non-zero integer, and the degeneracy for each value of κ is $|2\kappa|$.

We can use the angular eigenstates to construct eigenfunctions of the Dirac Hamiltonian

$$E\psi = \mathcal{H}\psi = -\boldsymbol{\nabla}\psi I\boldsymbol{\sigma}_3 + eV(r)\psi + m\gamma_0\psi\gamma_0. \tag{2.33}$$

It is clear that an energy eigenstate will need terms in both ψ_l^m and $\boldsymbol{\sigma}_r\psi_l^m$. We write, for positive l

$$\psi(\boldsymbol{x},\kappa) = \psi_l^m u(r) + \boldsymbol{\sigma}_r\psi_l^m v(r) I\boldsymbol{\sigma}_3 \tag{2.34}$$

where u and $v(r)$ are complex superpositions of $1, I\boldsymbol{\sigma}_3$. On substituting this into the Hamiltonian, we find that the radial equations reduce to

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} (\kappa-1)/r & -(E-eV(r)+m) \\ E-eV(r)-m & (-\kappa-1)/r \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}. \tag{2.35}$$

The same equations hold for negative κ as well, using a similar solution to the one above. This successfully separates the Dirac equation in any radially-symmetric potential.

2.5 The Hydrogen Atom

The radial equations describing the relativistic quantum theory of the Hydrogen atom are obtained simply by setting $eV = -Z\alpha/r$, where $\alpha = e^2/4\pi$ is the fine structure constant and Z is the atomic charge. The solutions to these equations are *hypergeometric* functions, which generalise the Laguerre polynomials of the non-relativistic theory. The key conclusion of this analysis is that the energy spectrum is obtained from the equation

$$E^2 = m^2 \left[1 - \frac{(Z\alpha)^2}{n'^2 + 2n'\nu + (l+1)^2} \right] \tag{2.36}$$

where n' is a non-negative integer, m is the electron mass, and

$$\nu = [(l+1)^2 - (Z\alpha)^2]^{1/2}. \tag{2.37}$$

One can recover the non-relativistic formula for the energy levels by recalling that $\alpha \approx 1/137$ is small. We can therefore approximate to

$$\nu \approx l + 1 \tag{2.38}$$

and

$$E \approx m \left[1 - \frac{(Z\alpha)^2}{2} \frac{1}{n'^2 + 2n'(l+1) + (l+1)^2} \right]. \tag{2.39}$$

Subtracting off the rest-mass energy we are left with the non-relativistic expression

$$E_{NR} = -m\frac{(Z\alpha)^2}{2}\frac{1}{(n'+l+1)^2} = -\frac{mZ^2e^4}{32\pi^2\epsilon_0^2\hbar^2}\frac{1}{n^2} \tag{2.40}$$

where $n = n' + l + 1$ and the dimensional constants have been put back in. We have recovered the familiar Bohr formula for the energy levels. Note in particular that the relativistic quantum number n' differs from the Bohr quantum number n.

Expanding to next order we find that

$$E_{NR} = -m\frac{(Z\alpha)^2}{2n^2} - m\frac{(Z\alpha)^4}{2n^4}\left(\frac{n}{l+1} - \frac{3}{4}\right). \tag{2.41}$$

The first relativistic correction shows that the binding energy is increased slightly from the non-relativistic value, and also introduces some dependence on the angular quantum number l. This lifts some degeneracies present in the non-relativistic solution. The various corrections contributing to the energy levels are shown in Fig. 1. A more complete analysis also requires replacing the electron mass m by the reduced mass of the two-body system. This introduces corrections of the same order of the relativistic corrections, but only affects the overall scale.

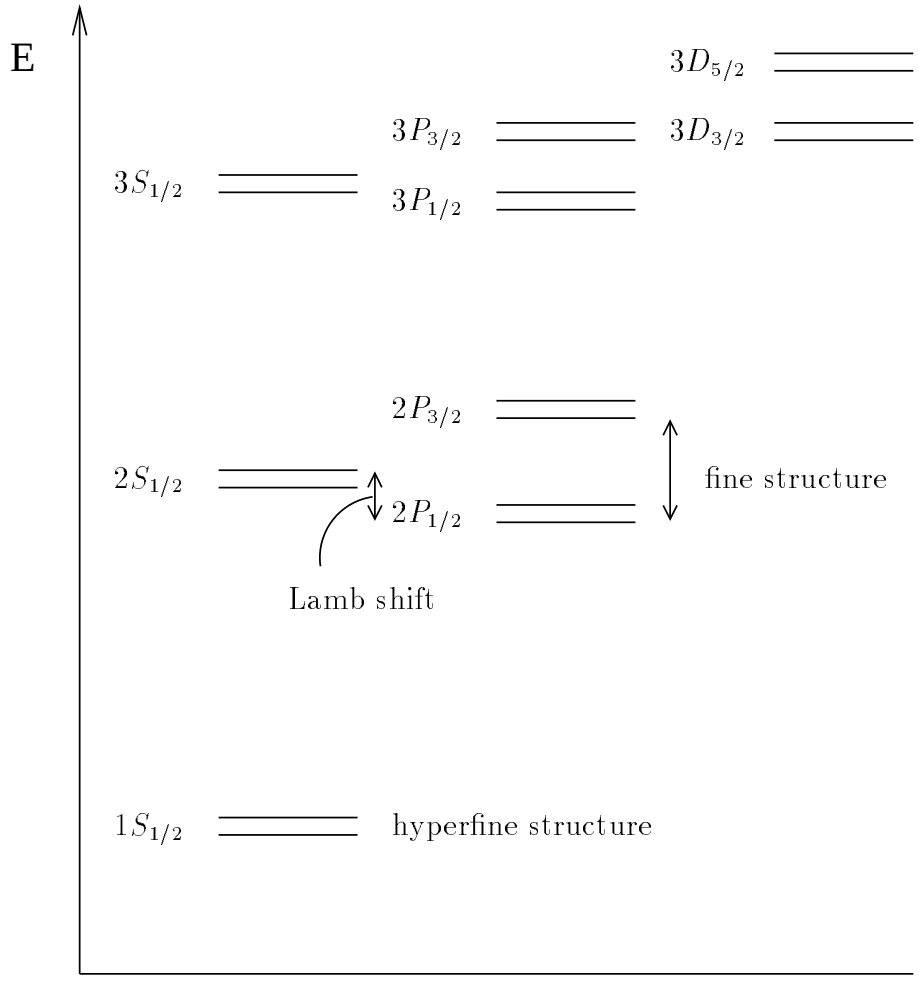

Figure 1: *Hydrogen atom energy levels.* The diagram illustrates how various degeneracies are broken by relativistic and spin effects. The Dirac equation accounts for the fine structure. The hyperfine structure is due to interaction with the magnetic moment of the nucleus. The Lamb shift is explained by quantum field theory. It lifts the degeneracy between $S_{1/2}$ and $P_{1/2}$ states.