Physical Applications of Geometric Algebra

Handout 10

Electromagnetism

The spacetime vector derivative and the geometric product enable us to unite all four of Maxwell's equations into a single equation. This is one of the most impressive results in geometric algebra. Unlike the separate gradient and curl operators, the vector derivative is invertible and this leads to a number of simplifications. Plane waves are easily handled, as are their polarisation states. We also derive expressions for the field energy and the Poynting vector, and introduce the important idea of the field stress-energy tensor. As a final application, we look at the derivation of the fields due to a point source.

1 Maxwell's Equations

The first object we need to consider is the spacetime vector derivative

$$\nabla = \gamma^\mu \partial_\mu, \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}. \tag{1.1}$$

Here $x^0 = \gamma^0 \cdot x = t$ is the time coordinate in the γ_0 frame, and $x^i = x \cdot \gamma^i$ are the three spatial coordinates. Note that $\gamma^0 = \gamma_0$ and $\gamma^i = -\gamma_i$. If we now form the spacetime split of the vector derivative, we find that

$$\nabla \gamma_0 = (\gamma^0 \partial_t + \gamma^i \partial_i)\gamma_0 = \partial_t - \boldsymbol{\sigma}_i \partial_i = \partial_t - \boldsymbol{\nabla}. \tag{1.2}$$

The minus sign here is in contrast to $x\gamma_0 = t + \boldsymbol{x}$ and is due to the Lorentzian metric. One has to take care to remember this. It becomes obvious when forming

$$\nabla x = 4 = \gamma_0 \nabla x \gamma_0 = \gamma_0 \nabla (t + \boldsymbol{x}) \tag{1.3}$$

which tells us that we must have $\gamma_0 \nabla = \partial_t + \boldsymbol{\nabla}$. Hence $\nabla \gamma_0 = (\gamma_0 \nabla)^\sim = \partial_t - \boldsymbol{\nabla}$.

Now consider the four Maxwell equations

$$\begin{array}{ll} \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 & \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho \\ \boldsymbol{\nabla} \times \boldsymbol{E} = -\partial_t \boldsymbol{B} & \boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{J} + \partial_t \boldsymbol{E} \end{array} \tag{1.4}$$

where as usual we employ the symbol $\times$ for the vector cross product. We seek a covariant form of these equations, as we know that they are Lorentz invariant. This must involve uniting the separate time and space derivatives into the single ∇ operator.

1.1 The Source Equations

We start with the two source equations, and introduce the spacetime vector J with

$$\rho = J\cdot\gamma_0, \qquad \boldsymbol{J} = J\wedge\gamma_0. \tag{1.5}$$

We next form

$$J\gamma_0 = \rho + \boldsymbol{J} = \boldsymbol{\nabla}\cdot\boldsymbol{E} - \partial_t\boldsymbol{E} + \boldsymbol{\nabla}\times\boldsymbol{B}. \tag{1.6}$$

Now consider the following manipulation, with $\boldsymbol{b} = b\wedge\gamma_0$,

$$\gamma_0\wedge(a\cdot\boldsymbol{b}) = \gamma_0\wedge[a\cdot(b\wedge\gamma_0)] = \gamma_0\wedge(-a\cdot\gamma_0\, b) = a\cdot\gamma_0\, b\wedge\gamma_0 = a\cdot\gamma_0\,\boldsymbol{b}. \tag{1.7}$$

We can therefore write

$$-\partial_t\boldsymbol{E} = -\gamma_0\cdot\nabla\boldsymbol{E} = -\gamma_0\wedge(\nabla\cdot\boldsymbol{E}). \tag{1.8}$$

The full $\boldsymbol{E}$-field term is now

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} - \partial_t\boldsymbol{E} = (\gamma_0\wedge\nabla)\cdot\boldsymbol{E} - \gamma_0\wedge(\nabla\cdot\boldsymbol{E}) = (\nabla\cdot\boldsymbol{E})\cdot\gamma_0 + (\nabla\cdot\boldsymbol{E})\wedge\gamma_0 = \nabla\cdot\boldsymbol{E}\,\gamma_0. \tag{1.9}$$

For the $\boldsymbol{B}$ term we need the result (exercise)

$$\boldsymbol{a}\times\boldsymbol{b} = (a\wedge\gamma_0)\times(b\wedge\gamma_0) = -a\wedge b\wedge\gamma_0\,\gamma_0, \tag{1.10}$$

so that we can write

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{B} &= -I\boldsymbol{\nabla}\times\boldsymbol{B} = I(\nabla\wedge\gamma_0)\times(\boldsymbol{B}) \\ &= -I\nabla\wedge\boldsymbol{B}\,\gamma_0 = \nabla\cdot(I\boldsymbol{B})\gamma_0.\end{aligned} \tag{1.11}$$

Our 2 Maxwell equations therefore combine to give

$$J\gamma_0 = \nabla\cdot(\boldsymbol{E} + I\boldsymbol{B})\gamma_0. \tag{1.12}$$

Recalling the derivation of the relativistic form of the Lorentz force law from Handout 9, we define

$$F = \boldsymbol{E} + I\boldsymbol{B}, \tag{1.13}$$

which enables us to write down the covariant equation

$$\nabla\cdot F = J. \tag{1.14}$$

This successfully combines two equations into one.

1.2 The Electromagnetic Field Strength

The spacetime bivector $F = \boldsymbol{E} + I\boldsymbol{B}$ is the *electromagnetic field strength*, also called the Faraday bivector. It is a covariant spacetime bivector. Its components in the $\{\gamma^\mu\}$ frame give rise to the tensor

$$F^{\mu\nu} = \gamma^\nu\cdot(\gamma^\mu\cdot F) = (\gamma^\nu\wedge\gamma^\mu)\cdot F. \tag{1.15}$$

These are the components of a rank-2 antisymmetric tensor which, written out as a matrix, has entries

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}. \tag{1.16}$$

This form is often presented in textbooks on relativistic electrodynamics. The big disadvantage of this matrix form is that the the natural complex structure is hidden.

Writing $F = \boldsymbol{E} + I\boldsymbol{B}$ decomposes F into the sum of a relative vector $\boldsymbol{E}$ and a relative bivector $I\boldsymbol{B}$. The separate $\boldsymbol{E}$ and $I\boldsymbol{B}$ fields are recovered from

$$\begin{aligned} \boldsymbol{E} &= \tfrac{1}{2}(F - \gamma_0 F\gamma_0) \\ I\boldsymbol{B} &= \tfrac{1}{2}(F + \gamma_0 F\gamma_0). \end{aligned} \tag{1.17}$$

This shows clearly how the split into $\boldsymbol{E}$ and $I\boldsymbol{B}$ fields depends on the observer velocity (γ_0 here). Observers in relative motion see different fields. For example, suppose a second observer has velocity $v = R\gamma_0\tilde{R}$ and constructs the rest frame basis vectors

$$\gamma'_\mu = R\gamma_\mu\tilde{R}. \tag{1.18}$$

This observer measures components of an electric field to be

$$E'_i = (\gamma'_i\gamma'_0)\cdot F = (R\boldsymbol{\sigma}_i\tilde{R})\cdot F = \boldsymbol{\sigma}_i\cdot(\tilde{R}FR). \tag{1.19}$$

The effect of a Lorentz transformation can therefore be seen by taking F to $\tilde{R}FR$. The fact that bivectors are subject to the same rotor transformation law as vectors make it easy to recover the standard formulae.

1.3 Examples

i. Observers in Relative Motion

Suppose that in the γ_0 frame some stationary charge configuration sets up the field

$$F = \boldsymbol{E} = E_x\boldsymbol{\sigma}_1 + E_y\boldsymbol{\sigma}_2. \tag{1.20}$$

A second observer has velocity $\tanh(\alpha)$ in the γ_1 direction, so

$$R = \mathrm{e}^{\alpha\boldsymbol{\sigma}_1/2}. \tag{1.21}$$

This observer measures the $\boldsymbol{\sigma}_i$ components of

$$\tilde{R}FR = \mathrm{e}^{-\alpha\boldsymbol{\sigma}_1/2}\, F\, \mathrm{e}^{\alpha\boldsymbol{\sigma}_1/2} = E_x\boldsymbol{\sigma}_1 + E_y\,\mathrm{e}^{-\alpha\boldsymbol{\sigma}_1}\,\boldsymbol{\sigma}_2 \tag{1.22}$$

which gives

$$E'_x = E_x, \quad E'_y = \mathrm{ch}(\alpha)E_y, \quad B'_z = -\mathrm{sh}(\alpha)E'_y. \tag{1.23}$$

This approach is *much* simpler than working with tensors.

ii. Invariants

A further useful result for the F field is to construct its Lorentz invariant terms. We form the quantity

$$F^2 = \langle FF\rangle + \langle FF\rangle_4 = \alpha + I\beta. \tag{1.24}$$

But if we also form

$$(\tilde{R}FR)(\tilde{R}FR) = \tilde{R}FFR = \alpha + I\beta, \tag{1.25}$$

we see that the result is invariant. So both the scalar and pseudoscalar terms are Lorentz invariant — that is, independent of the frame in which they are measured. In the γ_0 frame these are

$$\alpha = \langle(\boldsymbol{E}+I\boldsymbol{B})(\boldsymbol{E}+I\boldsymbol{B})\rangle = \boldsymbol{E}^2 - \boldsymbol{B}^2 \tag{1.26}$$

and

$$\beta = -\langle I(\boldsymbol{E}+I\boldsymbol{B})(\boldsymbol{E}+I\boldsymbol{B})\rangle = 2\boldsymbol{E}\cdot\boldsymbol{B}. \tag{1.27}$$

The former yields the Lagrangian density for the electromagnetic field. The latter is seen less often, and at first it is quite surprising to learn that $\boldsymbol{E}\cdot\boldsymbol{B}$ is a full Lorentz invariant, rather than just being invariant under rotations.

1.4 The Remaining Equations

The remaining two Maxwell equations are

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \qquad \boldsymbol{\nabla}\times\boldsymbol{E} = -\partial_t\boldsymbol{B}. \tag{1.28}$$

The first of these can be written

$$0 = (\gamma_0 \wedge \nabla) \wedge (I\boldsymbol{B}) = \nabla \wedge (I\boldsymbol{B}) \wedge \gamma_0 = \nabla \wedge F \wedge \gamma_0, \tag{1.29}$$

since $\boldsymbol{E} \wedge \gamma_0 = 0$. This suggests forming

$$\begin{aligned}(\nabla \wedge F) \cdot \gamma_0 &= \nabla \wedge (F \cdot \gamma_0) + \partial_t F \\ &= \langle \nabla \boldsymbol{E} \gamma_0 \rangle_2 + \partial_t F \\ &= -\langle (\partial_t - \boldsymbol{\nabla}) \boldsymbol{E} \rangle_2 + \partial_t (\boldsymbol{E} + I\boldsymbol{B}) \\ &= I(\partial_t \boldsymbol{B} + \boldsymbol{\nabla} \times \boldsymbol{E}) = 0. \end{aligned} \tag{1.30}$$

The two equations combine to tell us that $\nabla \wedge F \, \gamma_0$ vanishes, from which we can extract the second covariant equation

$$\nabla \wedge F = 0. \tag{1.31}$$

We have now reduced the four Maxwell equations to two. This is as far as most mathematical systems can go. For example, in tensor language our two equations are

$$\partial_\mu F^{\mu\nu} = J^\nu, \qquad \partial_\mu \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0. \tag{1.32}$$

Much the same is true of the popular language of *differential forms*. But geometric algebra has one further simplification to offer. We can combine the vector equation $\nabla \cdot F = J$ and the trivector equation $\nabla \wedge F = 0$ by utilising the geometric product to write

$$\nabla F = J. \tag{1.33}$$

Now we have all of Maxwell's equations in one! This is more than a mere cosmetic trick — this unified equation offers a number of significant improvements. In particular, the ∇ operator is invertible — there is a Green's function for it. (This was first met in Handout 7). This simplifies diffraction theory and directly encodes Huygen's principle (outside this course). In addition, first order equations are numerically more robust than second order equations, so are preferable for numerical computation.

The wave theory of electromagnetism is recovered by introducing the *vector potential* A, defined so that

$$F = \nabla \wedge A. \tag{1.34}$$

It then follows automatically that

$$\nabla \wedge F = \nabla \wedge (\nabla \wedge A) = 0. \tag{1.35}$$

We have some *gauge* freedom in the choice of A, as we can always add the gradient of a scalar field to it (exercise). The most natural way to soak up this freedom is to impose the *Lorentz condition* $\nabla \cdot A = 0$, so that $F = \nabla A$. We then recover the familiar wave equation

$$\nabla^2 A = J. \tag{1.36}$$

2 Electromagnetic Waves

The F field is equipped with a complex structure through the pseudoscalar, so we can start to look for vacuum solutions (with $\nabla F = 0$) of the form

$$F = F_0 \, \mathrm{e}^{Ik\cdot x} \, . \tag{2.1}$$

The vacuum equation therefore reduces to

$$kF_0 = 0. \tag{2.2}$$

Pre-multiplying by k we immediately see that $k^2 = 0$, as expected of the wave vector. F_0 must therefore also contain a factor of k, as nothing else totally annihilates k. We therefore must have

$$F_0 = k\wedge n = kn, \tag{2.3}$$

with $k{\cdot}n = 0$. We can always add a further multiple of k to n, which is usually employed to ensure that n has no components in the spacetime plane containing the null vector k.

2.1 Circularly Polarised Light

Consider a wave travelling in the $+z$ direction, frequency ω, with wave vector

$$k = \omega(\gamma_0 + \gamma_3). \tag{2.4}$$

The vector n can be chosen to just contain γ_1 and γ_2 components. We can therefore write

$$\begin{aligned} F &= -(\gamma_0 + \gamma_3)(\alpha\gamma_1 + \beta\gamma_2) \, \mathrm{e}^{I\omega(t-z)} \\ &= (1 + \boldsymbol{\sigma}_3)(\alpha\boldsymbol{\sigma}_1 + \beta\boldsymbol{\sigma}_2) \, \mathrm{e}^{I\omega(t-z)} \, . \end{aligned} \tag{2.5}$$

The multivector $1+\boldsymbol{\sigma}_3$ has a number of interesting properties. In particular, it absorbs factors of $\boldsymbol{\sigma}_3$

$$\boldsymbol{\sigma}_3(1 + \boldsymbol{\sigma}_3) = 1 + \boldsymbol{\sigma}_3, \tag{2.6}$$

and it squares to give a multiple of itself back again

$$(1 + \boldsymbol{\sigma}_3)^2 = 2(1 + \boldsymbol{\sigma}_3). \tag{2.7}$$

(This latter property means that $\frac{1}{2}(1 + \boldsymbol{\sigma}_3)$ is a projection operator, and implies that $1 + \boldsymbol{\sigma}_3$ does not have an inverse.)

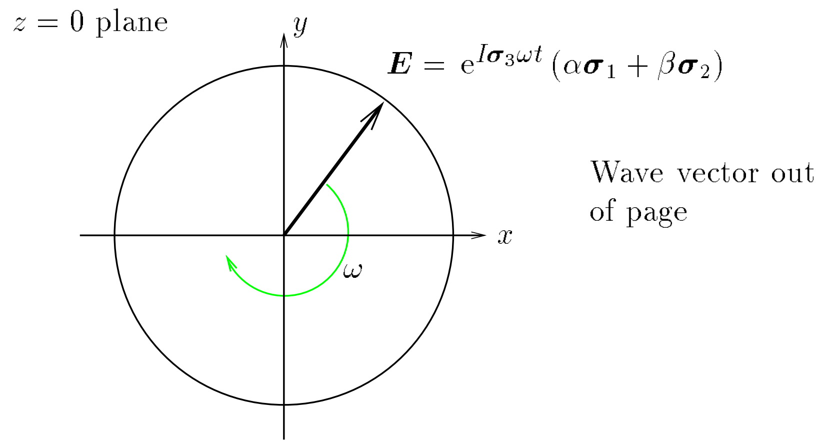

Figure 1: *Circularly Polarised Light.* In the $z = 0$ plane the $\boldsymbol{E}$ vector rotates clockwise, so the light is left-handed. The wave vector points out of the page.

Eq. (2.6) enables us to convert phase rotations with the pseudoscalar I to rotations on the bivector $\boldsymbol{I\sigma}_3$, using

$$\begin{aligned}(1+\boldsymbol{\sigma}_3)\,\mathrm{e}^{I\phi} &= (1+\boldsymbol{\sigma}_3)[\cos(\phi) + I\sin(\phi)] \\ &= (1+\boldsymbol{\sigma}_3)[\cos(\phi) + \boldsymbol{I\sigma}_3\sin(\phi)] = (1+\boldsymbol{\sigma}_3)\,\mathrm{e}^{\boldsymbol{I\sigma}_3\phi}. \end{aligned} \tag{2.8}$$

We now have

$$F = \mathrm{e}^{\boldsymbol{I\sigma}_3\omega(t-z)}(1+\boldsymbol{\sigma}_3)(\alpha\boldsymbol{\sigma}_1 + \beta\boldsymbol{\sigma}_2) \tag{2.9}$$

from which we extract

$$\begin{aligned}\boldsymbol{E} &= \mathrm{e}^{\boldsymbol{I\sigma}_3\omega(t-z)}(\alpha\boldsymbol{\sigma}_1 + \beta\boldsymbol{\sigma}_2) \\ \boldsymbol{B} &= \mathrm{e}^{\boldsymbol{I\sigma}_3\omega(t-z)}(-\beta\boldsymbol{\sigma}_1 + \alpha\boldsymbol{\sigma}_2). \end{aligned} \tag{2.10}$$

In a plane of constant z the $\boldsymbol{E}$ vector rotates in a clockwise direction. This is defined as left-hand circularly polarised light (see Fig. 1). (Note that the spiral in *space* at constant t is then right-handed.) Right-hand circularly polarised light is described by reversing the sign of the exponent $I\omega(t-z)$.

2.2 General Polarisation States

A further manipulation we can perform is to write

$$(1+\boldsymbol{\sigma}_3)(\alpha\boldsymbol{\sigma}_1 + \beta\boldsymbol{\sigma}_2) = (1+\boldsymbol{\sigma}_3)\boldsymbol{\sigma}_1(\alpha - \beta I). \tag{2.11}$$

A general decomposition into circularly polarised modes is then given by

$$F = (1+\boldsymbol{\sigma}_3)\boldsymbol{\sigma}_1[R\,\mathrm{e}^{I\omega(z-t)} + L\,\mathrm{e}^{-I\omega(z-t)}] \tag{2.12}$$

where the R and L are 'complex' (scalar + pseudoscalar) coefficients. Plane and elliptic polarised light is built from these modes. For example, setting $R = L = 1/2$ produces

$$F = (1 + \boldsymbol{\sigma}_3)\boldsymbol{\sigma}_1 \cos(\omega t - \omega z) \tag{2.13}$$

which describes light linearly polarised in the $\boldsymbol{\sigma}_1$ direction.

3 Field Momentum and the Stress-Energy Tensor

The energy contained in an electromagnetic field is

$$\mathcal{E} = \tfrac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2), \tag{3.1}$$

and the momentum is described by the Poynting vector

$$\boldsymbol{P} = \boldsymbol{E}\times\boldsymbol{B} = -\boldsymbol{E}\cdot(I\boldsymbol{B}). \tag{3.2}$$

These ought to be the components of a spacetime 4-vector P, so we form

$$\begin{aligned} P = (\mathcal{E} + \boldsymbol{P})\gamma_0 &= \tfrac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2)\gamma_0 + \tfrac{1}{2}(I\boldsymbol{B}\boldsymbol{E} - \boldsymbol{E}I\boldsymbol{B})\gamma_0 \\ &= \tfrac{1}{2}(\boldsymbol{E} + I\boldsymbol{B})(\boldsymbol{E} - I\boldsymbol{B})\gamma_0 \\ &= \tfrac{1}{2}F(-\gamma_0 F\gamma_0)\gamma_0 = -\tfrac{1}{2}F\gamma_0 F. \end{aligned} \tag{3.3}$$

But this quantity is still frame-dependent as it contains a factor of γ_0. We have in fact constructed the *stress-energy tensor* of the electromagnetic field. We write this as

$$\mathsf{T}(a) = -\tfrac{1}{2}FaF. \tag{3.4}$$

The stress-energy tensor $\mathsf{T}(a)$ returns the flux of 4-momentum across the hypersurface perpendicular to a. This is the relativistic extension of the stress tensor, and it is as fundamental to fields as momentum is to point particles. It is instructive to contrast the neat STA form of Eq. (3.4) with the tenor formula

$$\mathsf{T}^{\mu}{}_{\nu} = \tfrac{1}{4}\delta^{\mu}_{\nu}F^{\alpha\beta}F_{\alpha\beta} + F^{\mu\alpha}F_{\alpha\nu}. \tag{3.5}$$

There is little doubt which form best captures the geometric content of the tensor!

3.1 General Properties

Details non-examinable.

All relativistic fields, classical or quantum, have a stress-energy tensor which contains information about the distribution of energy in the fields (and acts as a source of gravity). We can illustrate some general properties of these using electromagnetism as an example. The first property is that the stress-energy tensor is (usually) symmetric. For example, we have

$$a \cdot \mathsf{T}(b) = -\tfrac{1}{2}\langle aFbF \rangle = -\tfrac{1}{2}\langle FaFb \rangle = \mathsf{T}(a) \cdot b. \tag{3.6}$$

The stress-energy tensor can have a non-symmetric contribution in the presence of quantum spin. The second property is that the energy density $v \cdot \mathsf{T}(v)$ is positive for any timelike vector v. Matter which does not satisfy this property is said to be 'exotic'.

The third main property of stress-energy tensors is that, in the absence of external sources, the total flux of energy-momentum over a closed hypersurface is zero:

$$\int_{\partial V} dA \, \mathsf{T}(n) = 0. \tag{3.7}$$

Here dA is the scalar measure over the closed 3-surface ∂V, and n is the normal vector to the surface. These combine into (Handout 7, Eq. 3.16)

$$n \, dA = dS \, I^{-1}. \tag{3.8}$$

The fundamental theorem of calculus now gives

$$\int_{\partial V} \mathsf{T}(n \, dA) = \int_{\partial V} \mathsf{T}(dS I^{-1}) = \int_V \dot{\mathsf{T}}(\dot{\nabla} \, dX I^{-1}) = \int_V \dot{\mathsf{T}}(\dot{\nabla}) \, dV, \tag{3.9}$$

where dV is the scalar measure. This must vanish for any hypersurface, so we must have

$$\dot{\mathsf{T}}(\dot{\nabla}) = 0. \tag{3.10}$$

Alternatively, we can use the symmetry of T to write this as

$$\nabla \cdot \mathsf{T}(a) = 0 \qquad \forall \text{ const } a. \tag{3.11}$$

Eq. (3.10) is easy to check for free field electromagnetism:

$$\dot{\mathsf{T}}(\dot{\nabla}) = -\tfrac{1}{2}[\dot{F}\dot{\nabla}F + F\nabla F] = 0, \tag{3.12}$$

since $\nabla F = \dot{F}\dot{\nabla} = 0$ in the absence of sources.

Provided all fields fall off suitably at infinity, Eq. (3.10) enables us to write down a conserved total 4-momentum,

$$P_{\text{tot}} = \int |d^3x| \, \mathsf{T}(\gamma_0), \tag{3.13}$$

where $|d^3x|$ is the scalar measure over the 3-space perpendicular to γ_0. P_{tot} is a constant vector because

$$\partial_t P_{\text{tot}} = \int |d^3x|\, \partial_t \mathsf{T}(\gamma_0) = \int |d^3x| \dot{\mathsf{T}}(\dot{\boldsymbol{\nabla}}\gamma_0), \tag{3.14}$$

and the final term is a pure boundary term which must vanish if $\mathsf{T}(a) \mapsto 0$ at infinity. It is also not hard to show that P_{tot} is independent of the chosen timelike axis. It is a covariant (nonlocal) property of the field configuration.

In the presence of additional source fields, it is only the total field stress-energy tensor that is conserved. The degree by which the separate tensors for each field are not conserved contains useful information about the flow of energy-momentum. For example, suppose that an external current is present, so that

$$\dot{\mathsf{T}}(\dot{\nabla}) = -\tfrac{1}{2}(-JF + FJ) = J\cdot F. \tag{3.15}$$

In the γ_0 frame, $J\cdot F$ decomposes into

$$J\cdot F = \langle(\rho + \boldsymbol{J})\gamma_0(\boldsymbol{E} + I\boldsymbol{B})\rangle_1 = -[\boldsymbol{J}\cdot\boldsymbol{E} + \rho\boldsymbol{E} + \boldsymbol{J}\times\boldsymbol{B}]\gamma_0 \tag{3.16}$$

The scalar term in brackets, $\boldsymbol{J}\cdot\boldsymbol{E}$, is the work done (rate of change of energy density), and the relative vector term is familiar from the Lorentz force law (rate of change of field momentum). Notice how easily one can move between covariant expressions and relative expressions in a chosen frame.

4 Fields from a Point Charge

We now give a compact formula for the fields of a radiating charge. A charge q moves along a world-line $x_0(\tau)$ (see Fig. 2). An observer at spacetime position x receives an electromagnetic influence from the point where the charge's worldline intersects the observer's past light-cone. The vector

$$X \equiv x - x_0(\tau) \tag{4.1}$$

is the separation vector down the light-cone, joining the observer to this intersection point. This vector must be null, $X^2 = 0$. For every spacetime position x there is a unique value of the proper time along the charge's world-line for which the vector connecting x to the world-line is null. We can write $\tau = \tau(x)$, and treat τ as a scalar field.

The Liénard-Wiechert potential for the retarded field from the charge is

$$A = \frac{q}{4\pi\epsilon_0}\frac{v}{X\cdot v}, \tag{4.2}$$

where $v = \dot{x}_0$ is the velocity of the charge, and X is the null vector connecting $x_0(\tau)$ to the *past* lightcone of the position x. It is simple to check that the field of Eq. (4.2) reproduces the Coulomb potential for a charge at rest (exercise).

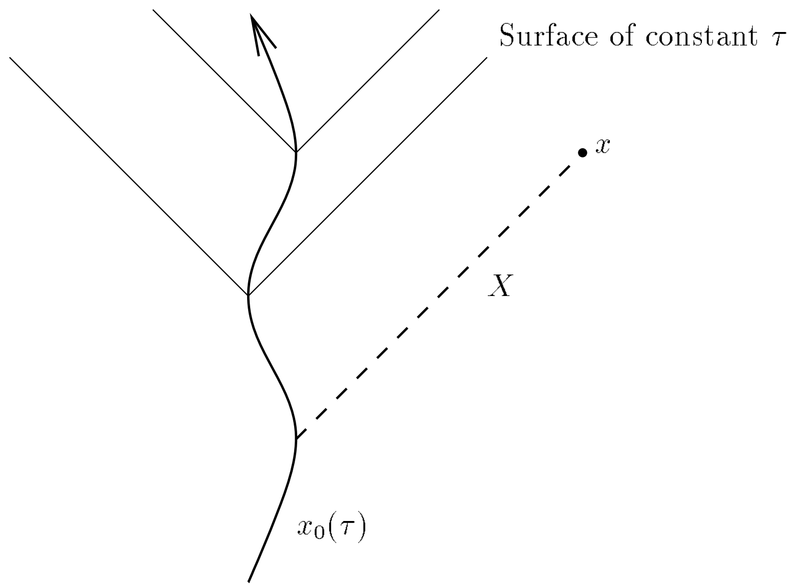

Figure 2: *Field from a moving point charge.* The charge follows the trajectory $x_0(\tau)$. $X = x - x_0(\tau)$ is the null vector connecting the point x to the worldline. The time τ can be viewed as a scalar field with each value of τ extended out over the forward null cone.

4.1 The Field Strength

We now differentiate the potential of Eq. (4.2) to find the Faraday bivector. First, we differentiate the equation $X^2 = 0$ to obtain

$$\dot{\nabla}\,\dot{X}\cdot X = \dot{\nabla}\,\dot{x}\cdot X - \nabla\tau(\partial_\tau x_0)\cdot X = X - \nabla\tau(v\cdot X) = 0. \tag{4.3}$$

It follows that

$$\nabla\tau = \frac{X}{X\cdot v}. \tag{4.4}$$

The gradient of τ points in the direction of constant τ! This is a peculiarity of null surfaces and is one reason why one has to be careful when defining the normal vector to a surface in mixed signature spaces. In finding an expression for $\nabla\tau$ we have demonstrated how the particle proper time can be treated as a spacetime scalar field. Feynman and Wheeler call this an *adjunct* field. It carries information, but does not exist in any physical sense.

To differentiate A we need $\nabla(X\cdot v)$. Using the results already established we have

$$\nabla(X\cdot v) = \dot{\nabla}\,\dot{X}\cdot v + \nabla\tau\, X\cdot(\partial_\tau v) = v - \nabla\tau + \nabla\tau\, X\cdot\dot{v} \tag{4.5}$$

where $\dot{v} = \partial_\tau v$. (This double use for overdots should not cause any confusion. In cases where there is some potential for ambiguity we often replace overdots by overstars.) We now evaluate ∇A as follows:

$$\begin{aligned}
\nabla A &= \frac{q}{4\pi\epsilon_0}\left(\frac{\nabla v}{X\cdot v} - \frac{1}{(X\cdot v)^2}\nabla(X\cdot v)v\right) \\
&= \frac{q}{4\pi\epsilon_0}\left(\frac{X\dot{v}}{(X\cdot v)^2} - \frac{1}{(X\cdot v)^2} - \frac{(X\,X\cdot\dot{v} - X)v}{(X\cdot v)^3}\right) \\
&= \frac{q}{4\pi\epsilon_0}\left(\frac{X\wedge\dot{v}}{(X\cdot v)^2} + \frac{X\wedge v - X\cdot\dot{v}\,X\wedge v}{(X\cdot v)^3}\right).
\end{aligned} \tag{4.6}$$

The bracketed term is a pure bivector, so $\nabla\cdot A = 0$ and the A field of Eq. (4.2) is in the Lorentz gauge.

We can gain some insight into the expression for F by writing

$$X\cdot v\,X\wedge\dot{v} - X\cdot\dot{v}\,X\wedge v = -X\wedge[X\cdot(\dot{v}\wedge v)] = \tfrac{1}{2}X\dot{v}\wedge vX, \tag{4.7}$$

which uses the fact that $X^2 = 0$. Writing $\Omega_v = \dot{v}\wedge v$ for the acceleration bivector of the particle, we arrive at the compact formula

$$F = \frac{q}{4\pi\epsilon_0}\frac{X\wedge v + \frac{1}{2}X\Omega_v X}{(X\cdot v)^3}. \tag{4.8}$$

This displays a clean split into a velocity term proportional to $1/(\text{distance})^2$ and a long-range radiation term proportional to $1/(\text{distance})$. The first term is exactly the Coulomb field in the rest frame of the charge, and the radiation term,

$$F_{rad} = \frac{q}{4\pi\epsilon_0}\frac{\frac{1}{2}X\Omega_v X}{(X\cdot v)^3}, \tag{4.9}$$

is proportional to the rest-frame acceleration projected down the null-vector X. One can go on now to show that, away from the worldline, F satisfies the free-field equation $\nabla F = 0$. The details are left as a (voluntary) exercise.

4.2 Example — Circular Orbits

The result of Eq. (4.8) is simple to implement numerically. For example, consider the circular orbit used in the description of the Thomas precession in Handout 9. We already have simple expressions in place for v and Ω_v. All that remains to do is to establish the value of τ for which each null vector X intersects the particle worldline, which is found numerically. One can then plot field lines for various values of the angular velocity. These are shown in Figures 3 and 4, which display many interesting features.

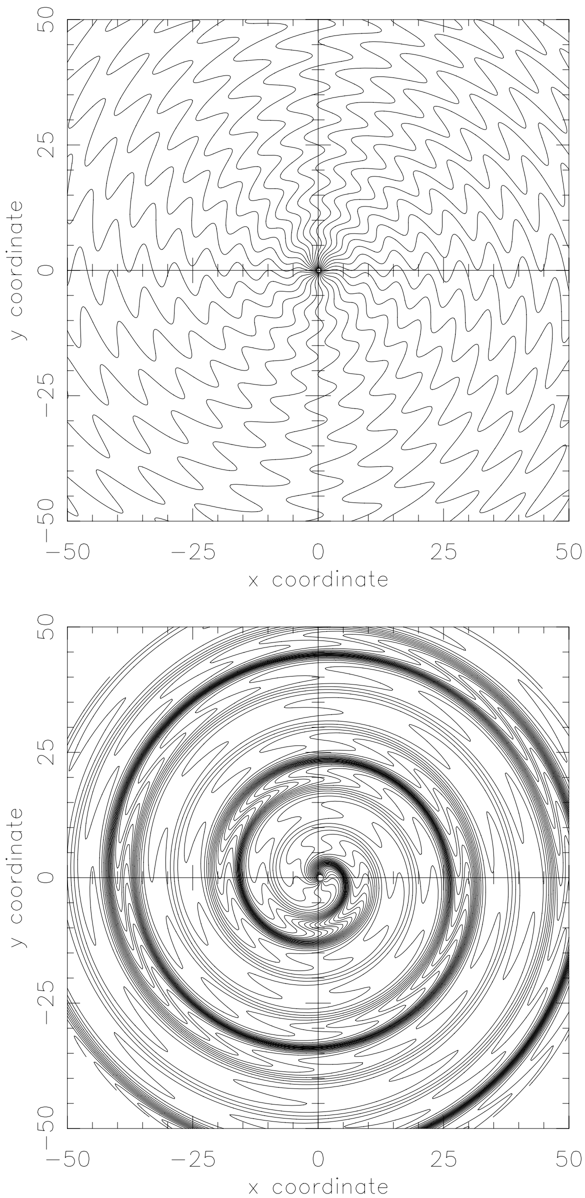

Figure 3: *Field lines of a rotating charge.* The top diagram has $\alpha = 0.1$ and the particle velocity $\tanh(\alpha)$ is low. A gentle wavy pattern of field lines is produced, characteristic of electromagnetic waves. At intermediate velocities (bottom diagram, with $\alpha = 0.4$) a complicated structure emerges as the field lines start to concentrate together.

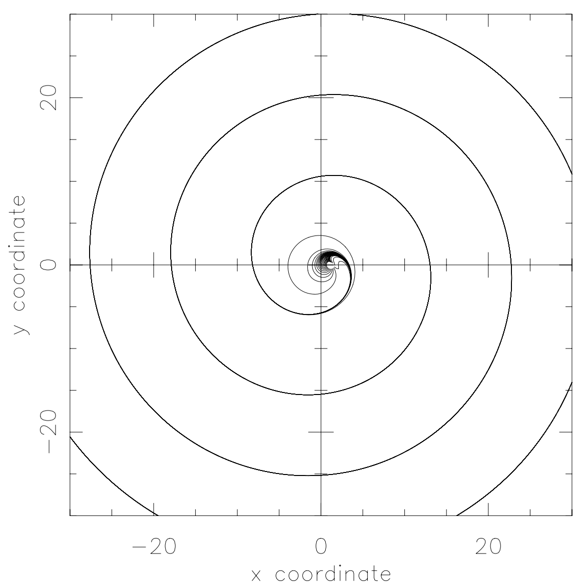

Figure 4: *Synchrotron Radiation.* By $\alpha = 1$ the field lines concentrate into pure synchrotron pulses as the radiation is focussed into the direction of motion of the particle.