Physical Applications of Geometric Algebra

Handout 15

Motion of Point Particles

The gauging arguments we have applied to spinor fields also apply to particle trajectories, and enable us to write down a covariant equation for a point particle in a gravitational background. This equation reveals how the equivalence principle enters gauge theory gravity, and also exposes the link between the gauge theory approach and General Relativity (GR). For our first major application of gauge theory gravity we study the vacuum fields outside a spherically symmetric source. A simple argument from Newtonian physics provides a candidate for the $\bar{\mathsf{h}}$-field, which turns out to give a correct solution.

1 Trajectories and Tangents

Suppose a particle follows the trajectory $x(\lambda)$ in the STA. We know from our considerations about fields that the actual STA path taken has no relevance. All that is important is the value of the fields encountered at different points on the path. But what then are we supposed to do about the velocity? If the path is irrelevant, what relevance can be attached to the tangent vector? To resolve this we first apply a displacement to generate the new path

$$x'(\lambda) = f(x(\lambda)). \tag{1.1}$$

The new tangent vector is

$$\partial_\lambda f(x(\lambda)) = \partial_\lambda x(\lambda)\cdot\nabla f(x) = \mathsf{f}(\dot{x}) \tag{1.2}$$

where $\dot{x} = \partial_\lambda x(\lambda)$. What we see is that tangent vectors to curves pick up a factor of $\mathsf{f}(a)$ under displacements. But we now know what to with these factors — we simply introduce a suitable form of the displacement gauge field $\bar{\mathsf{h}}(a)$ in order to form a covariant vector. In this case we define

$$v = \mathsf{h}^{-1}(\dot{x}), \tag{1.3}$$

as the covariant tangent vector. This can then be equated with other covariant vectors, or acted on by covariant derivatives.

1.1 Point Particle Equations of Motion

Given that it is v that is the covariant vector, we expect that this will satisfy a covariant version of the equations of motion encountered in relativistic physics. The first thing we see is that v transforms under rotation gauge changes as

$$v \mapsto v' = Rv\tilde{R}. \tag{1.4}$$

This transformation law does not affect the underlying STA trajectory. The law is driven entirely by the transformation law of the $\bar{\mathsf{h}}$-field. The rotation gauge therefore gives us complete freedom to choose the direction of the vector v, without altering a single physical effect. A truly radical idea!

There is one restriction on v', however, which is that the norm v^2 is invariant. It is therefore this norm which determines whether a trajectory is spacelike, timelike or null. It is the presence of the $\bar{\mathsf{h}}$-field in $v = \mathsf{h}^{-1}(\dot{x})$ which means, for example, that photon trajectories ($v^2 = 0$) are no longer constrained to the light cone of the background STA. Since it is v^2 that is invariant under displacements, the proper distance along a trajectory is now

$$s = \int_{\lambda_1}^{\lambda_2} \sqrt{|v^2|}\, d\lambda. \tag{1.5}$$

It is this that gives the proper time elapsed between $x(\lambda_1)$ and $x(\lambda_2)$ along a timelike trajectory. The proper time τ is therefore the parameter along the trajectory with the property that

$$v^2 = 1, \quad \text{where} \quad v = \mathsf{h}^{-1}(\dot{x}) = \mathsf{h}^{-1}(\partial_\tau x). \tag{1.6}$$

In the absence of any fields, the velocity v satisfies $\dot{v} = 0$. This is the equation that we make covariant. Since we know that $\partial_\tau = \dot{x}\cdot\nabla$, we see that the covariant extension must be

$$\partial_\tau v + \Omega(\dot{x})\cdot v = 0. \tag{1.7}$$

We can write this more abstractly as $v\cdot\mathcal{D}\, v = 0$, where

$$v\cdot\mathcal{D}\, v = v\cdot\bar{\mathsf{h}}(\partial_a)\mathcal{D}_a v = \dot{x}\cdot\partial_a\, \mathcal{D}_a v = \partial_\tau v + \Omega(\dot{x})\cdot v = 0. \tag{1.8}$$

Writing the equation as $v\cdot\mathcal{D}v = 0$ is neat notationally, though it does not add much new information. It is immediately clear that the $\Omega(\dot{x})$ term is entering as a form of *acceleration bivector*. But one has to be careful here, because we cannot just assert that $\Omega(\dot{x})$ is the acceleration due to gravity. The reason is that this is not a gauge invariant statement. If we apply a rotation gauge transformation, so that $v' = Rv\tilde{R}$, we have

$$\partial_\tau v' + \Omega'(\dot{x})\cdot v' = \partial_\tau v' + [R\Omega(\dot{x})\tilde{R}]\cdot v' - 2(\dot{R}\tilde{R})\cdot v' = Rv\cdot\mathcal{D}v\,\tilde{R} = 0. \tag{1.9}$$

One can therefore add terms in $2\dot{R}\tilde{R}$ to the $\Omega(a)$ field without altering the physics in any way. The term $\dot{R}\tilde{R}$ is also an 'acceleration' term, so what we have is the identification

$$\text{gravity} + \text{acceleration} = \text{gravity}' \tag{1.10}$$

This is a form of the *equivalence principle*, which Einstein arrived at from thought experiments comparing an accelerating lift with one at rest in a gravitational field. The principle is neatly encoded in the gauge transformation properties of the $\Omega(a)$ field. The *weak* equivalence principle says that *the motion of a test particle in a gravitational field is independent of its mass.* Equation (1.7) clearly embodies this by making no reference to a mass. This principle ensures that we can equate gravitational and inertial masses, the equality of which was unexplained prior to the arrival of GR.

1.2 The Metric and GR

The key to understanding the link between gauge theory gravity and GR is the introduction of a coordinate frame. We take the set (x^μ) to be a set of scalar functions parameterising spacetime position as $x = x(\mathrm{x}^\mu)$. From these we define the two frames

$$e_\mu = \partial_\mu x, \qquad e^\mu = \nabla \mathrm{x}^\mu. \tag{1.11}$$

These two frames are reciprocal to one another. If we now expand out the trajectory $x(\lambda)$in this frame we form

$$\partial_\lambda x = \partial_\lambda x(\mathrm{x}^\mu) = \frac{d\mathrm{x}^\mu}{d\lambda}\partial_\mu x = \frac{d\mathrm{x}^\mu}{d\lambda} e_\mu. \tag{1.12}$$

In terms of this the proper distance along a path becomes

$$\begin{aligned} s &= \int_{\lambda_1}^{\lambda_2} \sqrt{|v^2|}\, d\lambda \\ &= \int_{\lambda_1}^{\lambda_2} \left| \frac{d\mathrm{x}^\mu}{d\lambda}\frac{d\mathrm{x}^\nu}{d\lambda} \mathsf{h}^{-1}(e_\mu)\cdot\mathsf{h}^{-1}(e_\nu)\right|^{1/2} d\lambda \\ &= \int_{x_1}^{x_2} \left|\mathsf{h}^{-1}(e_\mu)\cdot\mathsf{h}^{-1}(e_\nu) d\mathrm{x}^\mu\, d\mathrm{x}^\nu\right|^{1/2} \end{aligned} \tag{1.13}$$

A comparison with the equivalent formula in GR enables us to read off the *metric* as

$$g_{\mu\nu} = \mathsf{h}^{-1}(e_\mu)\cdot\mathsf{h}^{-1}(e_\nu). \tag{1.14}$$

In GR the metric is the fundamental object. It is viewed as giving the distance between points on a curved surface. In the gauge theory version, the metric is derived from the more fundamental displacement gauge field. The crucial result which equates the theories is that, if the $\bar{\mathsf{h}}(a)$ and $\Omega(a)$ satisfy the gauge field equations for some matter distribution, then the resultant metric (1.14) solves the GR Einstein equations for the same matter distribution. In addition, point particles moving so as to minimise the proper distance (as defined by the metric), turn out to satisfy Eq. (1.7). That is, Eq. (1.7) is the gauge theory analog of the *geodesic equation*.

A property of the metric that stands out is that it is independent of the rotation gauge. Because GR defines everything directly from the metric, the rotation gauge is never seen. This is one reason why the gauge theory nature of GR is hard to establish. A second property of the metric, which can lead to some confusion, is that displacements look just like changes of coordinates. This has led people to suggest that the crucial feature of GR is that the equations are invariant under general coordinate transformations. But this is clearly nonsense, because any sensible physical equation will remain true in any coordinate system. What makes the metric special is the fact that it cannot be transformed away by a change of coordinate system, not the fact that it transforms sensibly!

1.3 Covariant Frames

From the preceding it is clear that the frame $\{\mathsf{h}^{-1}(e_\mu)\}$ is going to useful. We therefore define

$$g_\mu = \mathsf{h}^{-1}(e_\mu), \qquad g^\mu = \bar{\mathsf{h}}(\nabla x^\mu) = \bar{\mathsf{h}}(e^\mu). \tag{1.15}$$

These are reciprocal because

$$g_\mu \cdot g^\nu = \mathsf{h}^{-1}(e_\mu)\cdot\bar{\mathsf{h}}(e^\mu) = e_\mu \cdot e^\nu = \delta^\nu_\mu. \tag{1.16}$$

In terms of the g_μ vectors the metric has the simple expression

$$g_{\mu\nu} = g_\mu \cdot g_\nu. \tag{1.17}$$

The first of our field equations also has a simple expression. We first form

$$\bar{\mathsf{h}}(\nabla)\wedge g^\mu = \bar{\mathsf{h}}(\dot{\nabla})\wedge\dot{\bar{\mathsf{h}}}(e^\mu) + \bar{\mathsf{h}}(\nabla\wedge e^\mu) = \bar{\mathsf{h}}(\dot{\nabla})\wedge\dot{\bar{\mathsf{h}}}(e^\mu). \tag{1.18}$$

It follows that we can write the first field equation $\mathcal{S}(a) = 0$ as

$$\mathcal{D}\wedge g^\mu = 0. \tag{1.19}$$

We can now complete the link with GR. We use the abbreviation

$$\mathcal{D}_\mu = e_\mu \cdot \nabla + \Omega(e_\mu)\times = \partial_\mu + \Omega(e_\mu)\times \tag{1.20}$$

for the covariant derivative in the e_μ direction. When this acts on the vector g_μ the result can be expressed back in terms of the $\{g_\mu\}$ frame. We therefore write

$$\mathcal{D}_\mu g_\nu = \Gamma^\lambda_{\mu\nu} g_\lambda \tag{1.21}$$

which defines the *Christoffel connection* $\Gamma^\lambda_{\mu\nu}$.

Vectors formed from derivatives split into two types. Those that transform like $\partial_\mu x$ and pick up factors of $\mathsf{f}(a)$ and those that transform like ∇ and pick an $\bar{\mathsf{f}}^{-1}(a)$ term. Mathematicians refer to these as vectors and 1-forms respectively, and like to view them formally as inhabiting two separate spaces. The idea then is that the metric defines a map between these two spaces, and in so doing imposes a distance scale. The gauge approach is very different. We use suitable versions of the $\bar{\mathsf{h}}$-field to map all objects into an intermediate state of covariant vectors, which do not pick up any factors of $\mathsf{f}(a)$ under displacements. This means we are then dealing with objects which only transform under rotations, whereas GR only ever works with rotation gauge scalars.

2 Spherically Symmetric Sources

As our first major application of gauge theory gravity we will find the fields around a spherically symmetric source. Those of you familiar with General Relativity will know that the equivalent problem is solved by the Schwarzschild metric. Here we will adopt a very different solution strategy, and obtain a set of fields which at first do not appear to resemble the Schwarzschild solution. Outside the horizon the two solutions are equivalent, however, and a simple gauge transformation can be found between them. Before proceeding, we first introduce some notation for spherically-symmetric coordinates. In terms of the fixed $\{\gamma_\mu\}$ frame we define:

$$\begin{aligned} t &= x\cdot\gamma_0 & \cos\theta &= x\cdot\gamma^3/r \\ r &= \sqrt{(x\wedge\gamma_0)^2} & \tan\phi &= (x\cdot\gamma^2)/(x\cdot\gamma^1). \end{aligned} \tag{2.1}$$

The associated coordinate frame is

$$\begin{aligned} e_t &= \gamma_0 \\ e_r &= x\wedge\gamma_0\,\gamma_0/r = \sin\theta(\cos\phi\,\gamma_1 + \sin\phi\,\gamma_2) + \cos\theta\,\gamma_3 \\ e_\theta &= r\cos\theta(\cos\phi\,\gamma_1 + \sin\phi\,\gamma_2) - r\sin\theta\,\gamma_3 \\ e_\phi &= r\sin\theta(-\sin\phi\,\gamma_1 + \cos\phi\,\gamma_2). \end{aligned} \tag{2.2}$$

The reciprocal frame vectors are denoted by $\{e^t, e^r, e^\theta, e^\phi\}$ and are given by

$$\begin{aligned} e^t &= e_t & e^r &= -e_r \\ e^\theta &= -e_\theta/r^2 & e^\phi &= -e_\phi/(r\sin\theta)^2. \end{aligned} \tag{2.3}$$

Notice that we have started to use the components themselves as labels for vectors. This is quite common practice. We will also make use of the unit vectors $\hat{\theta}$ and $\hat{\phi}$ defined by

$$\hat{\theta} = e_\theta / r, \qquad \hat{\phi} = e_\phi / (r \sin\theta). \tag{2.4}$$

From these we define the unit relative vectors

$$\boldsymbol{\sigma}_r = e_r e_t, \qquad \boldsymbol{\sigma}_\theta = \hat{\theta} e_t, \qquad \boldsymbol{\sigma}_\phi = \hat{\phi} e_t. \tag{2.5}$$

These satisfy

$$\boldsymbol{\sigma}_r \boldsymbol{\sigma}_\theta \boldsymbol{\sigma}_\phi = e_t e_r \hat{\theta} \hat{\phi} = I. \tag{2.6}$$

The dual spatial bivectors are given by

$$I\boldsymbol{\sigma}_r = -\hat{\theta}\hat{\phi}, \quad I\boldsymbol{\sigma}_\theta = e_r \hat{\phi}, \quad I\boldsymbol{\sigma}_\phi = -e_r \hat{\theta}. \tag{2.7}$$

Throughout the following we will include factors of the gravitational constant G, but will continue to ignore factors of c.

2.1 Newtonian Considerations

The equation for a particle accelerating towards a mass M in Newtonian physics is

$$\ddot{r} = -\frac{GM}{r^2}, \tag{2.8}$$

which integrates to give

$$\tfrac{1}{2}\dot{r}^2 = \frac{GM}{r} + \text{constant}. \tag{2.9}$$

We expect the some form of this equation must survive in the relativistic gauge theory treatment. In seeing how this could work, the first issue we must resolve is what type of derivative we should use — coordinate time or proper time? The coordinate time is a gauge-dependent concept, so it is clearly the proper time that should be used. But we have the freedom to chose the coordinate time t however we like. A natural choice, then, is to make t the proper time for freely-falling observers, since it is these that generalise the notion of inertial observers. In order that the clocks for these observers all coincide, we chose them all to be at rest at infinity. In this case we have

$$\dot{r} = -\sqrt{(2GM/r)}, \tag{2.10}$$

and the paths followed by these observers have

$$\dot{x} = \frac{dx}{dt} = e_t - \sqrt{(2GM/r)}e_r. \tag{2.11}$$

The covariant version of this vector is $v = \mathsf{h}^{-1}(\dot{x})$, with $v^2 = 1$. Again, we have some gauge freedom in the choice of this vector, this time through the choice of rotation gauge. Since we want the physics to look natural for these observers, a sensible choice is to set $v = e_t$. It is important to remember that this is a *gauge choice* — there is no physics implied in this choice. It follows that we now have $e_t = \mathsf{h}^{-1}(\dot{x})$, so

$$\mathsf{h}(e_t) = \dot{x} = e_t - \sqrt{(2GM/r)}e_r. \tag{2.12}$$

This gives us a plausible term in the $\bar{\mathsf{h}}$-field.

2.2 The Solution

We now make the simplest possible guess and assume that Eq. (2.12) is the only term in the $\bar{\mathsf{h}}$-field which differs from the identity. We therefore have

$$\mathsf{h}(a) = a - \sqrt{(2GM/r)}\, a\cdot e_t\, e_r, \tag{2.13}$$

which has the adjoint form

$$\bar{\mathsf{h}}(a) = a - \sqrt{(2GM/r)}\, a\cdot e_r\, e_t. \tag{2.14}$$

Remarkably, this is the solution we are after! To see this we first find the $\{g^\mu\}$ frame vectors

$$\begin{array}{ll} g^t = e^t & g^r = e^r - \sqrt{(2GM/r)}e^t \\ g^\theta = e^\theta & g^\phi = e^\phi \end{array} \tag{2.15}$$

The first of the field equations, $\mathcal{D}\wedge g^\mu = 0$, is solved by

$$\begin{array}{ll} \Omega(e_t) = \dfrac{GM}{r^2}\boldsymbol{\sigma}_r & \Omega(e_r) = -\dfrac{GM}{ur^2}\boldsymbol{\sigma}_r \\ \Omega(e_\theta) = u/r\; e_\theta e_t & \Omega(e_\phi) = u/r\; e_\phi e_t \end{array} \tag{2.16}$$

where

$$u = -\sqrt{(2GM/r)}. \tag{2.17}$$

Notice that the $\Omega(e_t)$ term, which governs acceleration, has picked up a factor of GM/r^2. We next compute the terms in the Riemann tensor. This is laborious, and best done with the aid of a symbolic algebra package, but the end result is strikingly simple,

$$\mathcal{R}(B) = \frac{-M}{2r^3}(B + 3\boldsymbol{\sigma}_r B\boldsymbol{\sigma}_r). \tag{2.18}$$

The immediate question, then, is why is this a solution?

2.3 The Vacuum Equations

Our second field equation is $\mathcal{G}(a) = 8\pi G\mathcal{T}(a)$, where $\mathcal{T}(a)$ is the matter stress-energy tensor. In the vacuum region outside a source we must therefore have

$$\mathcal{G}(a) = \mathcal{R}(a) - \tfrac{1}{2}a\mathcal{R} = 0. \tag{2.19}$$

Contracting this with ∂_a we see that $\mathcal{R} = 0$, so the vacuum equations are equivalent to

$$\partial_a \cdot \mathcal{R}(a\wedge b) = \mathcal{R}(a) = 0. \tag{2.20}$$

Combining this with the symmetry relation $\partial_a\wedge\mathcal{R}(a\wedge b)$ we see that the Riemann tensor for a vacuum solution must satisfy

$$\partial_a \mathcal{R}(a\wedge b) = 0. \tag{2.21}$$

Provided this is satisfied, we have a genuine vacuum solution. It is instructive to see how this is satisfied by our Riemann tensor (2.18). We first recall that

$$\partial_a a\wedge b = \partial_a(ab - a\cdot b) = (n-1)b = 3b \tag{2.22}$$

since we are working in 4-d. Next we use

$$\partial_a\, a\cdot(b\wedge c) = \partial_a(a\cdot b\, c - a\cdot c\, b) = bc - cb = 2b\wedge c, \tag{2.23}$$

from which we see that $\partial_a a\cdot B = 2B$ for any bivector B. It follows that in 4-d

$$\partial_a Ba = \partial_a(Ba - aB) + \partial_a aB = -2\partial_a a\cdot B + 4B = -4B + 4B = 0. \tag{2.24}$$

We now have assembled all of the results needed to prove that our Riemann tensor is that of a genuine vacuum solution. We form

$$\begin{aligned}\partial_a(a\wedge b + 3\boldsymbol{\sigma}_r\, a\wedge b\, \boldsymbol{\sigma}_r) &= 3b + 3\partial_a \boldsymbol{\sigma}_r(ab - a\cdot b)\boldsymbol{\sigma}_r \\ &= 3b - 3b\boldsymbol{\sigma}_r\boldsymbol{\sigma}_r \\ &= 0,\end{aligned} \tag{2.25}$$

which completes the proof. This is a massive improvement over tensor calculus, where one has no alternative but to check each component in turn.

We have found a solution for the fields outside a spherically symmetric source. It turns out that this solution is unique (up to choice of gauge). All spherically symmetric vacuum solutions have equivalent physical properties to the solution found here. In the final lecture we will look at the properties of this solution.