# Physical Applications of Geometric Algebra

## Examples 1

1. In 2-d two multivectors $A$ and $B$ are given by

$$A = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_1 \wedge e_2 \qquad B = b_0 + b_1 e_1 + b_2 e_2 + b_3 e_1 \wedge e_2$$

where $\{e_1, e_2\}$ are a pair of orthonormal basis vectors. The product of these is

$$AB = p_0 + p_1 e_1 + p_2 e_2 + p_3 e_1 \wedge e_2.$$

Find explicit formulae for $p_0$, $p_1$, $p_2$ and $p_3$ and establish that $\langle AB \rangle = \langle BA \rangle$.

2. An elliptical orbit in an inverse square force law is parameterised in terms of a scalar + pseudoscalar quantity $U$ by $x = U^2 e_1$. Prove that $U$ can be written

$$U = A_0 \, \mathrm{e}^{I\omega s} + B_0 \, \mathrm{e}^{-I\omega s}, \quad \text{where} \quad \frac{dt}{ds} = r \quad \text{and} \quad r = |x| = U\tilde{U}$$

What is the value of $\omega$? Find the conditions on $A_0$ and $B_0$ such that at time $t = 0$, $s = 0$ and the particle lies on the positive $e_1$ axis with velocity in the positive $e_2$ direction. For which value of $s$ does the velocity point in the $-e_1$ direction? Find the values for the shortest and longest diameters of the ellipse, and verify that we can write

$$U = \sqrt{a(1+\epsilon)} \cos(\omega s) - \sqrt{a(1-\epsilon)} I \sin(\omega s)$$

where $\epsilon$ is the eccentricity and $a$ is the semi-major axis.

3. By expanding the bivector $a \wedge b$ in terms of geometric products, prove that it anticommutes with both $a$ and $b$, but commutes with any vector outside the plane.

4. Prove that the bivector $a \wedge b$ has magnitude $|a||b| \sin(\theta)$.

5. Prove that any of the following forms are equivalent expressions of the vector cross product in 3-d:

$$a \times b = -I a \wedge b = b \cdot (Ia) = -a \cdot (Ib).$$

Interpret each form geometrically. Hence establish that

$$a \times (b \times c) = -a \cdot (b \wedge c) = -(a \cdot b \, c - a \cdot c \, b)$$

and

$$a \cdot (b \times c) = [a, b, c] = a \wedge b \wedge c \, I^{-1}.$$

6. Expand out $a \wedge (b \wedge c)$ in terms of geometric products. Is the result antisymmetric on $a$ and $b$? Explain why we can also write

$$a \wedge b \wedge c = \tfrac{1}{2}(abc - cba).$$

Hence prove that we can also write

$$a\wedge b\wedge c = \tfrac{1}{6}(abc + cab + bca - acb - bac - cba).$$

7. A particle in 3-d moves along a curve x(t) such that $|v|$=constant. Show that there exists a bivector $\Omega$ such that

$$\dot{v} = \Omega\cdot v,$$

and give an explicit formula for $\Omega$. Is $\Omega$ unique in 4-d, with the dot now denoting differentiation with respect to proper time?

8. Prove that the inertia tensor $\mathcal{I}(B)$ for a solid cylinder of height $h$ and radius $a$ can be written

$$\mathcal{I}(B) = \frac{Mh^2}{12}(B - B\wedge e_3\, e_3) + \frac{Ma^2}{4}(B + B\wedge e_3\, e_3)$$

where $e_3$ is the symmetry axis.

9. Prove that the kinetic energy of a rigid body can be written (following the notation of the lectures)

$$\mathrm{KE} = \tfrac{1}{2}mv_0^2 - \tfrac{1}{2}\Omega_B\cdot\mathcal{I}(\Omega_B).$$

Why is the minus sign required?

10. For a rigid body, introduce the 'reference body' angular momentum $\Pi = \mathcal{I}(\Omega_B)$. Prove that for torque-free motion both $\Pi^2$ and $\Pi\cdot\mathcal{I}^{-1}(\Pi)$ are conserved. (The inverse tensor $\mathcal{I}^{-1}(B)$ satisfies $\mathcal{I}^{-1}[\mathcal{I}(B)] = B$.)

By expanding $\Pi$ in terms of the principal axes, explain why rigid-body motion viewed in $\Pi$ space must describe orbits formed from the intersection of a sphere and an ellipse.

11. In 4-d prove that $a\wedge b + c\wedge d$ is a blade iff

$$a\wedge b\wedge c\wedge d = 0.$$

12. From the axioms of geometric algebra prove that the exterior product is associative.

13. Prove that

$$a\cdot(a_1\wedge a_2\wedge\cdots\wedge a_r) = \sum_{k=1}^{r}(-1)^{k+1} a\cdot a_k\, a_1\wedge a_2\wedge\cdots\wedge\check{a}_k\wedge\cdots\wedge a_r$$

where the check on $\check{a}_k$ denotes that this term is missing from the series.

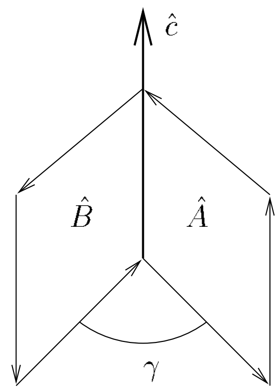
14. Given two unit vectors $m$ and $n$, prove that successive reflections in the planes perpendicular to $m$ and $n$ result in a rotation through $2\theta$, where $m\cdot n = \cos(\theta)$. Hence show that the rotor $R$ which takes the unit vector $a$ onto the unit vector $b$, leaving vectors outside the plane unchanged, is

$$R = \frac{1 + ba}{(2(1 + b\cdot a))^{1/2}}$$

15. This is a longer question to cover the application of 3-d geometric algebra to *spherical trigonometry*. It is not examinable, but should provide some good practice in manipulating 3-d multivectors.

Given two planes $A$ and $B$, the *dihedral angle* between the planes is defined by the following diagram:

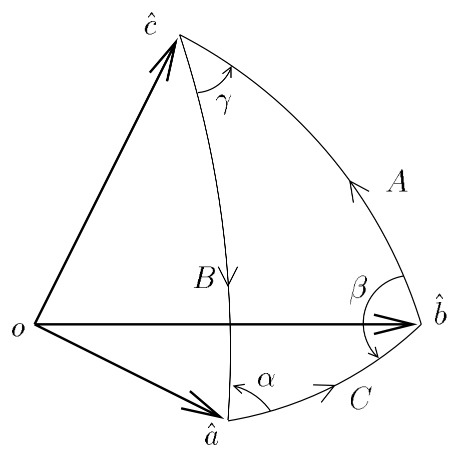
*The Dihedral Angle.* $\hat{A}$ and $\hat{B}$ are unit bivectors, and $\hat{c}$ is the unit vector along their line of intersection. The dihedral angle $\gamma$ is the angle formed between the $A$ and $B$ planes in the plane perpendicular to $\hat{c}$. Note the orientations.




Prove that

$$\hat{B}\hat{A} = \mathrm{e}^{Ic}, \quad \text{where} \quad c = \gamma\hat{c}.$$

Now consider the spherical triangle with corners described by three unit vectors from the origin to the surface of a unit sphere.




The lengths of the three arcs are given by $|A|$, $|B|$ and $|C|$. Prove that we can write

$$\begin{aligned}\hat{a}\hat{b} &= \mathrm{e}^{C}\\ \hat{b}\hat{c} &= \mathrm{e}^{A}\\ \hat{c}\hat{a} &= \mathrm{e}^{B}.\end{aligned}$$

The angles $\alpha$, $\beta$ and $\gamma$ are dihedral angles between planes. Prove that we can write

$$\begin{aligned} \hat{B}\hat{A} &= \mathrm{e}^{Ic}, & \qquad |c| &= \gamma \\ \hat{C}\hat{B} &= \mathrm{e}^{Ia}, & \qquad |a| &= \alpha \\ \hat{A}\hat{C} &= \mathrm{e}^{Ib}, & \qquad |b| &= \beta. \end{aligned}$$

Hence prove that

$$\mathrm{e}^{C}\,\mathrm{e}^{A}\,\mathrm{e}^{B} = 1, \qquad \mathrm{e}^{Ic}\,\mathrm{e}^{Ib}\,\mathrm{e}^{Ia} = -1.$$

Take the scalar part of the equation $\mathrm{e}^{-Ic} = -\,\mathrm{e}^{Ib}\,\mathrm{e}^{Ia}$ to prove the *cosine law for angles* in spherical trigonometry,

$$\cos(\gamma) = -\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\cos(|C|)$$

These formulae offer a major improvement over traditional formulations of spherical trigonometry. What applications could you envisage for this?