Physical Applications of Geometric Algebra

Examples 3

1. A relativistic particle has velocity $v = R\gamma_0\tilde{R}$. Show that v satisfies the Lorentz force equation $m\dot{v} = qF\cdot v$ if R satisfies

$$\dot{R} = \frac{q}{2m}FR.$$

Show that the solution to this for a constant field is $R = \exp(qF\tau/2m)R_0$. Given that F is *null*, $F^2 = 0$, show that v is given by the polynomial

$$v = v_0 + \tau\frac{q}{m}F\cdot v_0 - \tau^2\frac{q^2}{4m^2}Fv_0F.$$

Suppose now that $F = \boldsymbol{\sigma}_1 + I\boldsymbol{\sigma}_2$ and the particle is initially at rest in the γ_0 frame. Sketch the resultant motion in the $\gamma_1\gamma_3$ plane.

2. By pre-multiplying the equation $\nabla F = J$ by the vector γ_0 and separating into scalar, relative vector, relative bivector, and pseudoscalar terms, establish the four Maxwell equations

$$\begin{array}{ll} \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 & \boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho \\ \boldsymbol{\nabla}\times\boldsymbol{E} = -\partial_t\boldsymbol{B} & \boldsymbol{\nabla}\times\boldsymbol{B} = \boldsymbol{J} + \partial_t\boldsymbol{E} \end{array}$$

3. The Liénard-Wiechert potential for a point charge is

$$A = \frac{q}{4\pi\epsilon_0}\frac{v}{X\cdot v}.$$

Explain the meaning of the symbols in this equation. If the charge is at rest in the γ_0 frame show that the potential reduces to the Coulomb potential

$$A = \frac{q}{4\pi\epsilon_0 r}\gamma_0$$

where $r = |x\wedge\gamma_0| = |\boldsymbol{x}|$.

4. Prove that in 4 dimensions

$$\partial_a B a = 0$$

where B is a bivector. The stress-energy tensor of the electromagnetic field is given by $\mathsf{T}(a) = -\frac{1}{2}FaF$. Prove that this satisfies

$$\partial_a \mathsf{T}(a) = 0$$

(The bivector part $\partial_a\wedge\mathsf{T}(a) = 0$ confirms that $\mathsf{T}(a)$ is symmetric. The scalar part $\partial_a\cdot\mathsf{T}(a) = 0$ shows that the trace vanishes.)

5. Verify that the equivalence between Pauli spinors and even multivectors of Eq. (1.5) of Handout 11 is consistent with the operator equivalences

$$\hat{\sigma}_k|\psi\rangle \quad \leftrightarrow \quad \boldsymbol{\sigma}_k\psi\boldsymbol{\sigma}_3 \quad (k = 1, 2, 3).$$

6. Verify that the Pauli inner product is invariant under both spatial rotations and gauge transformations (*i.e.* rotations in the $I\boldsymbol{\sigma}_3$ plane applied to the right of the spinor ψ). Repeat the analysis for Dirac spinors.

7. Prove that the angular momentum operators $L_B = iB\cdot(\boldsymbol{x}\wedge\boldsymbol{\nabla})$ satisfy

$$[L_{B_1}, L_{B_2}] = -iL_{B_1\times B_2}.$$

8. Prove that, in any dimension,

$$[B\cdot(x\wedge\nabla) - \tfrac{1}{2}B, \nabla] = 0$$

where B is a bivector.

9. Explain why the Riemann tensor for the gravitational fields in a vacuum satisfies

$$\partial_a\mathcal{R}(a\wedge b) = 0.$$

By expanding in the $\{\gamma_\mu\}$ frame, prove that this equation is equivalent to the set

$$\begin{aligned}
\boldsymbol{\sigma}_1\mathcal{R}(\boldsymbol{\sigma}_1) + \boldsymbol{\sigma}_2\mathcal{R}(\boldsymbol{\sigma}_2) + \boldsymbol{\sigma}_3\mathcal{R}(\boldsymbol{\sigma}_3) &= 0\\
\boldsymbol{\sigma}_1\mathcal{R}(\boldsymbol{\sigma}_1) - I\boldsymbol{\sigma}_2\mathcal{R}(I\boldsymbol{\sigma}_2) - I\boldsymbol{\sigma}_3\mathcal{R}(I\boldsymbol{\sigma}_3) &= 0\\
-I\boldsymbol{\sigma}_1\mathcal{R}(I\boldsymbol{\sigma}_1) + \boldsymbol{\sigma}_2\mathcal{R}(\boldsymbol{\sigma}_2) - I\boldsymbol{\sigma}_3\mathcal{R}(I\boldsymbol{\sigma}_3) &= 0\\
-I\boldsymbol{\sigma}_1\mathcal{R}(I\boldsymbol{\sigma}_1) - I\boldsymbol{\sigma}_2\mathcal{R}(I\boldsymbol{\sigma}_2) + \boldsymbol{\sigma}_3\mathcal{R}(\boldsymbol{\sigma}_3) &= 0.
\end{aligned}$$

Summing the final three, prove that $I\boldsymbol{\sigma}_k\mathcal{R}(I\boldsymbol{\sigma}_k) = 0$. Hence establish that

$$\mathcal{R}(IB) = I\mathcal{R}(B) \quad \forall B$$

This is a special property of the Riemann tensor for *vacuum* solutions.

10. A stationary observer, v_1 (constant r, θ and ϕ) is stationed outside a spherically symmetric source. A second observer v_2 is in radial free-fall from rest at infinity. Show that when their paths cross the relative velocity they measure is

$$\frac{v_1\wedge v_2}{v_1\cdot v_2} = \left(\frac{2GM}{r}\right)^{1/2}\boldsymbol{\sigma}_r$$

Does this differ from the Newtonian result?

11. The same two observers as in question 10, both at distance r, measure the frequency of a radially in-falling photon. Show that their answers disagree by a factor of

$$\frac{v_1\cdot k}{v_2\cdot k} = \left(\frac{1+\sqrt{(2GM/r)}}{1-\sqrt{(2GM/r)}}\right)^{1/2}.$$

Which observer sees the greater frequency. To what effect do the observers attribute this difference?