Physical Applications of Geometric Algebra

Examples 3 — Answers

1.

$$\dot{v} = \dot{R}\gamma_0 R + R\gamma_0 \dot{\tilde{R}} = \frac{q}{2m}(Fv + v\tilde{F}) = \frac{q}{m} F\cdot v.$$

Integrates immediately for constant field to give $R = \exp(qF\tau/2m)R_0$, since

$$\partial_\tau \exp(qF\tau/2m)R_0 = \frac{q}{2m} F \exp(qF\tau/2m)R_0 = \frac{q}{2m} FR.$$

If $F^2 = 0$ as well, have

$$R = (1 + \tau\frac{q}{2m}F + \frac{\tau^2}{2!}\frac{q^2}{(2m)^2}F^2 + \cdots)R_0 = (1 + \tau\frac{q}{2m}F)R_0$$

so

$$v = (1 + \tau\frac{q}{2m}F)v_0(1 - \tau\frac{q}{2m}F) = v_0 + \tau\frac{q}{m}F\cdot v_0 - \tau^2\frac{q^2}{4m^2}Fv_0F.$$

Integrate again, get

$$x - x_0 = \tau v_0 + \tau^2\frac{q}{2m}F\cdot v_0 - \tau^3\frac{q^2}{12m^2}Fv_0F.$$

With $v_0 = \gamma_0$ and $F = \boldsymbol{\sigma}_1 + I\boldsymbol{\sigma}_2$, get

$$(x - x_0)\wedge\gamma_0 = \tau^2\frac{q}{2m}\boldsymbol{\sigma}_1 + \tau^3\frac{q^2}{6m^2}\boldsymbol{\sigma}_3.$$

Trajectory is a plot of $z = \alpha x^{3/2}$.

2. Write

$$\gamma_0\nabla F = (\partial_t + \boldsymbol{\nabla})(\boldsymbol{E} + I\boldsymbol{B}) = \gamma_0 J = \rho - \boldsymbol{J}$$

Get, in 3-d algebra,

scalar	$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho$
vector	$\boldsymbol{\nabla}\times\boldsymbol{B} = \boldsymbol{J} + \partial_t\boldsymbol{E}$
bivector	$\boldsymbol{\nabla}\times\boldsymbol{E} = -\partial_t\boldsymbol{B}$
pseudoscalar	$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$

3. $v(\tau)$ is velocity of charge on trajectory $x_0(\tau)$. X is retarded null vector from x to point on trajectory $x_0(\tau)$. For charge at rest in γ_0 frame, put at origin, so

$$x_0 = \tau\gamma_0, \qquad X = x - x_0 = (t-\tau)\gamma_0 + re_r$$

where $re_r = x\wedge\gamma_0\gamma_0$ and e_r is unit radial vector. Equation $X^2 = 0$ sets

$$r^2 = (t-\tau)^2$$

and $X\cdot v = X\cdot\gamma_0 = t-\tau = r$, hence result.

4. Use $\partial_a\, a\cdot B = 2B$ (in handout), so

$$\partial_a Ba = \partial_a(Ba - aB) + \partial_a aB = -2\partial_a a\cdot B + 4B = -4B + 4B = 0.$$

For electromagnetic stress-energy tensor get

$$\partial_a \mathsf{T}(a) = -\tfrac{1}{2}\partial_a FaF = 0,$$

from above. Hence symmetric and traceless. (Origin of $P = \frac{1}{3}\rho$ for isotropic radiation.)

5. $\hat{\sigma}_1$ done in Handout 11. For $\hat{\sigma}_2$ get

$$\hat{\sigma}_2|\psi\rangle = \begin{pmatrix} a^1 + ia^2 \\ -a^3 + ia^0 \end{pmatrix} \quad\leftrightarrow\quad a^1 + a^2 I\boldsymbol{\sigma}_3 + a^3 I\boldsymbol{\sigma}_2 + a^0 I\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_2\psi\boldsymbol{\sigma}_3.$$

$\hat{\sigma}_3$ simple to verify — just swaps signs of $I\boldsymbol{\sigma}_1$ and $I\boldsymbol{\sigma}_2$ components.

6. Pauli inner product has

$$\langle\psi|\phi\rangle \quad\leftrightarrow\quad \langle\psi^\dagger\phi\rangle - \langle\psi^\dagger\phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3.$$

Under rotations $\psi\mapsto R\psi$ and $\phi\mapsto R\phi$, so $\psi^\dagger\phi\mapsto\psi^\dagger R^\dagger R\phi = \psi^\dagger\phi$ and both parts invariant under rotations. (NB Dagger and reverse mean same thing in 3-d algebra.) Under phase changes $\psi\mapsto\psi R$ and $\phi\mapsto\phi R$. First term unchanged. Second picks up $RI\boldsymbol{\sigma}_3R^\dagger = I\boldsymbol{\sigma}_3$, since rotation entirely in $I\boldsymbol{\sigma}_3$ plane, so also unchanged. Same goes through for Dirac, with

$$\langle\psi|\phi\rangle \quad\leftrightarrow\quad \langle\tilde{\psi}\phi\rangle - \langle\tilde{\psi}\phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3.$$

7.

$$\begin{aligned}
[L_{B_1}, L_{B_2}] &= -[B_1\cdot(\boldsymbol{x}\wedge\boldsymbol{\nabla}), B_2\cdot(\boldsymbol{x}\wedge\boldsymbol{\nabla})] \\
&= (B_1\cdot\boldsymbol{x})\cdot\dot{\boldsymbol{\nabla}}\,\dot{\boldsymbol{x}}\cdot(B_2\cdot\boldsymbol{\nabla}) - (B_2\cdot\boldsymbol{x})\cdot\dot{\boldsymbol{\nabla}}\,\dot{\boldsymbol{x}}\cdot(B_1\cdot\boldsymbol{\nabla}) \\
&= [(B_1\cdot\boldsymbol{x})\cdot B_2 - (B_2\cdot\boldsymbol{x})\cdot B_1]\cdot\boldsymbol{\nabla} \\
&= [(B_1\times B_2)\cdot\boldsymbol{x}]\cdot\boldsymbol{\nabla} \\
&= (B_1\times B_2)\cdot(\boldsymbol{x}\wedge\boldsymbol{\nabla}) \\
&= -iL_{B_1\times B_2}
\end{aligned}$$

8.

$$\begin{aligned}
[B\cdot(x\wedge\nabla), \nabla] &= -\dot{\nabla}B\cdot(\dot{x}\wedge\nabla) \\
&= B\cdot\nabla = \tfrac{1}{2}(B\nabla - \nabla B)
\end{aligned}$$

and rearrange to get result.

9. Get $\partial_a\wedge\mathcal{R}(a\wedge b)=0$ from symmetry and $\partial_a\cdot\mathcal{R}(a\wedge b)=\mathcal{R}(b)=0$ from condition that have a vacuum solution. Work through $b=\gamma_0,\gamma_1,\gamma_2,\gamma_3$ get four equations. Eg $b=\gamma_1$ gives

$$\gamma^0\mathcal{R}(\gamma_0\gamma_1)+\gamma^2\mathcal{R}(\gamma_2\gamma_1)+\gamma^3\mathcal{R}(\gamma_3\gamma_1)=0$$

$\times$ by $\gamma^1=-\gamma_1$, get

$$\begin{aligned}\boldsymbol{\sigma}_1\mathcal{R}(\boldsymbol{\sigma}_1)&+\gamma_1\gamma_2\mathcal{R}(\gamma_2\gamma_1)+\gamma_1\gamma_3\mathcal{R}(\gamma_3\gamma_1)\\ &=\boldsymbol{\sigma}_1\mathcal{R}(\boldsymbol{\sigma}_1)-I\boldsymbol{\sigma}_2\mathcal{R}(I\boldsymbol{\sigma}_2)-I\boldsymbol{\sigma}_3\mathcal{R}(I\boldsymbol{\sigma}_3)=0\end{aligned}$$

Repeat for other three to get full set. Sum of final three gives

$$\boldsymbol{\sigma}_k\mathcal{R}(\boldsymbol{\sigma}_k)-2I\boldsymbol{\sigma}_k\mathcal{R}(I\boldsymbol{\sigma}_k)=0.$$

But first equation is $\boldsymbol{\sigma}_k\mathcal{R}(\boldsymbol{\sigma}_k)=0$ so get $I\boldsymbol{\sigma}_k\mathcal{R}(I\boldsymbol{\sigma}_k)=0$ as well. Substitute in final 3 equations to get

$$\mathcal{R}(I\boldsymbol{\sigma}_k)=I\mathcal{R}(\boldsymbol{\sigma}_k).$$

Can expand any bivector B in $\{\boldsymbol{\sigma}_k, I\boldsymbol{\sigma}_k\}$ basis to get required result.

10. Observer 1 has $\dot{x}=\dot{t}e_t$ and

$$v_1=\dot{t}\,\mathsf{h}^{-1}(e_t)=\dot{t}(e_t+\surd(2GM/r)e_r)$$

and second observer has $v_2=e_t$. So

$$\frac{v_1\wedge v_2}{v_1\cdot v_2}=\frac{\dot{t}\surd(2GM/r)\boldsymbol{\sigma}_r}{\dot{t}}=\left(\frac{2GM}{r}\right)^{1/2}\boldsymbol{\sigma}_r$$

just as in Newtonian physics.

11. Infalling photon has $k=\omega(e_t-e_r)$. Get

$$\frac{v_1\cdot k}{v_2\cdot k}=\frac{\omega\dot{t}(1+\surd(2GM/r))}{\omega}=\frac{1+\surd(2GM/r)}{(1-2GM/r)^{1/2}}=\left(\frac{1+\surd(2GM/r)}{1-\surd(2GM/r)}\right)^{1/2}.$$

Stationary observer sees greater frequency. Difference attributed to relativistic Doppler effect, velocity $\surd(2GM/r)$.