Physical Applications of Geometric Algebra

Handout 3

The Foundations of Geometric Algebra

So far we have introduced geometric algebra in an ad hoc manner, with the geometric product defined in terms of the dot and wedge products. We must now start to place this on a firmer, axiomatic footing, with which we can uncover the properties of geometric algebras in spaces of arbitrary dimensions. In so doing we will reverse the order of presentation and define the algebra in terms of the geometric product alone. The dot and wedge products then drop out as separate terms in the full geometric product. This development will also highlight some of the new algebraic techniques we now have at our disposal. A theme running throughout this course is that access to the geometric product simplifies derivations, even if the initial and final expressions contain only dot and wedge products. As an application we look in more detail at the treatment of reflections and rotations with geometric algebra.

1 Axiomatic Development

We now have an intuitive feel for the elements of a geometric algebra — the multivectors — and some of their multiplicative properties. The next step is to define a set of axioms and conventions which enable us to efficiently manipulate them. We first discuss the structure of the linear space, and then the properties of the geometric product.

1.1 The Linear Space $\mathcal{G}_n$ and its Grading

We use the symbol $\mathcal{G}_n$ to denote the geometric algebra of n-dimensional (Euclidean) space. Elements of this algebra are called *multivectors*, and are usually written in upper case Roman, A. Lower case is reserved for vectors. This space is *linear over the real numbers* so, if λ and μ are scalars and A and B are multivectors ($A, B \in \mathcal{G}_n$), then

$$\lambda A + \mu B \in \mathcal{G}_n, \qquad \forall \lambda, \mu. \tag{1.1}$$

We only consider the algebra over the reals as most occurrences of complex numbers in physics turn out to have a geometric origin. This geometric meaning is lost if we admit a scalar unit imaginary.

The linear space $\mathcal{G}_n$ is *graded*, and every multivector can be written as a sum of pure grade terms

$$A = \langle A \rangle_0 + \langle A \rangle_1 + \cdots = \sum_r \langle A \rangle_r. \tag{1.2}$$

Each graded subspace of $\mathcal{G}_n$ is closed under addition and forms a linear subspace. The operator $\langle\ \rangle_r$ projects onto the grade-r terms in the argument, so $\langle A \rangle_r$ returns the grade-r components in A. Multivectors containing terms of only one grade are called *homogeneous*. They are often written as A_r, so

$$\langle A_r \rangle_r = A_r. \tag{1.3}$$

Take care not to confuse the grading subscript in A_r with frame indices in expressions like $\{e_k\}$. The context should always make clear which is intended.

The grade-0 terms in $\mathcal{G}_n$ are real scalars. These commute with all other elements. We employ the useful abbreviation

$$\langle A \rangle_0 = \langle A \rangle \tag{1.4}$$

for the common operation of taking the scalar part. The grade-1 objects $\langle A \rangle_1$ are vectors. These can be viewed as generating the algebra through the geometric product.

1.2 The Geometric Product

The three main axioms governing the geometric product of multivectors are were introduced in Handout 1. They are that the product is associative, $A(BC) = (AB)C = ABC$, distributive over addition, $A(B + C) = AB + AC$, and that *the square of any vector is a scalar*. We do not force this scalar to be positive, so we can incorporate Minkowski spacetime without modification of our axioms. Nothing is assumed about the commutation properties of the geometric product — matrix multiplication is one picture to keep in mind.

We also saw in Handout 1 that the symmetrised product of two vectors can be written

$$ab + ba = (a + b)^2 - a^2 - b^2, \tag{1.5}$$

and so must also be a scalar. With this we can *define* the inner product for vectors by

$$a \cdot b = \tfrac{1}{2}(ab + ba). \tag{1.6}$$

The remaining, antisymmetric contribution is the bivector part, so we can define the exterior product from the geometric product as well,

$$a \wedge b = \tfrac{1}{2}(ab - ba). \tag{1.7}$$

These definitions combine to give the by now familiar result

$$ab = a\cdot b + a\wedge b. \tag{1.8}$$

We can extend this idea to build up a host of useful results. For example, consider the product of a vector and a bivector

$$\begin{aligned} a(b\wedge c) &= \tfrac{1}{2}a(bc - cb) \\ &= (a\cdot b)c - (a\cdot c)b - \tfrac{1}{2}(bac - cab) \\ &= 2(a\cdot b)c - 2(a\cdot c)b + \tfrac{1}{2}(bc - cb)a \\ &= 2(a\cdot b)c - 2(a\cdot c)b + (b\wedge c)a. \end{aligned} \tag{1.9}$$

We therefore extend the definition of the inner product to

$$a\cdot(b\wedge c) = \tfrac{1}{2}[a(b\wedge c) - (b\wedge c)a] = (a\cdot b)c - (a\cdot c)b. \tag{1.10}$$

The remaining, symmetrised part of the product is

$$a\wedge(b\wedge c) = \tfrac{1}{2}[a(b\wedge c) + (b\wedge c)a] = a\wedge b\wedge c \tag{1.11}$$

which is a trivector (grade-3), as seen in Handout 2. The trivector part is totally antisymmetric on a, b, c (exercise). We now have

$$a(b\wedge c) = a\cdot(b\wedge c) + a\wedge(b\wedge c), \tag{1.12}$$

which was derived by an alternative argument in Handout 2.

One can already see that expressions in geometric algebra can pick up large numbers of brackets. This is the point of the operator ordering convention introduced in Handout 2: in the absence of brackets, *inner and outer products take precedence over geometric products*. This means we can write

$$(a\cdot b)c = a\cdot b\, c \tag{1.13}$$

and the right-hand side cannot be confused with $a\cdot(bc)$. We try to emphasise this typographically by writing $a\cdot b\, c$ instead of $a \cdot bc$. This is not easy with some word-processing packages!

1.3 Blades and Bases

Suppose we have an arbitrary set of vectors $a_i, i = 1 \ldots r$. The totally antisymmetrised sum of all products of these returns the outer product:

$$a_1\wedge a_2\wedge\cdots\wedge a_r = \frac{1}{r!}\sum(-1)^{\epsilon} a_{k_1}a_{k_2}\cdots a_{k_r} \tag{1.14}$$

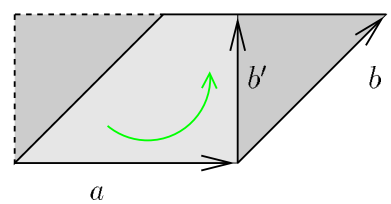
where the sum runs over every permutation of the indices $k_1 \ldots k_r$, and ϵ is $+1$ or -1 if $k_1 \ldots k_r$ is an even or odd permutation of $1 \ldots n$ respectively. So, for example,

$$a_1 \wedge a_2 = \frac{1}{2!}(a_1 a_2 - a_2 a_1) \tag{1.15}$$

as required. Any multivector which can be written purely as the outer product of a set of vectors is called a *blade*. The outer product of r vectors therefore returns a grade-r blade.

Fortunately we rarely need the full antisymmetrised expression when studying blades. Instead we can employ the result that *every blade can be written as a geometric product of orthogonal, anticommuting vectors*. The anticommutation of orthogonal vectors then takes care of the antisymmetry of the product. The proof is a form of Gram-Schmidt process. We start with the result that, with $b' = b - \lambda a$,

$$\begin{aligned} a \wedge b &= a \wedge (b - \lambda a) \\ &= a \wedge b' \end{aligned}$$



We can understand this result by recalling that a bivector encodes an oriented plane with magnitude determined by the area. Interchanging b and b' changes neither the orientation nor the magnitude, so returns the same bivector. We now form

$$a \cdot b' = a \cdot (b - \lambda a) = a \cdot b - \lambda a^2. \tag{1.16}$$

So if we set $\lambda = a \cdot b / a^2$ we have $a \cdot b' = 0$ and can write

$$a \wedge b = a \wedge b' = ab'. \tag{1.17}$$

The full proof for arbitrary grade blades continues by induction. (Some care is needed if any of the vectors are null, *i.e.* $a^2 = 0$, but the result still holds.) An alternative form for b' is also revealing,

$$\begin{aligned} b' = b - a^{-1} a \cdot b &= b - \tfrac{1}{2} a^{-1}(ab + ba) \\ &= \tfrac{1}{2}(b - a^{-1} ba) = a^{-1} \tfrac{1}{2}(ab - ba) = a^{-1} a \wedge b. \end{aligned} \tag{1.18}$$

This makes it clear why $ab' = a \wedge b$, and also gives a formula which extends to higher grades.

A natural way to view $\mathcal{G}_n$ is in terms of orthonormal basis vectors $\{e_i\}, i = 1 \ldots n$. In terms of these we build up a basis for the entire algebra as

$$1, \quad e_i, \quad e_i e_j \ (i < j), \quad e_i e_j e_k \ (i < j < k) \qquad \textit{etc.} \tag{1.19}$$

We denote each grade-r subspace of $\mathcal{G}_n$ by $\mathcal{G}_n^r$. A natural question to ask is what is the dimension of each of these graded subspaces? To answer this we imagine choosing r distinct vectors from our basis set. These have to be different because of the total antisymmetry of the exterior product. The order is irrelevant, again because of the total antisymmetry, so we just need the number of distinct combinations of r objects from a set of n. This is

$$\mathrm{Dim}\,[\mathcal{G}_n^r] = \begin{pmatrix} r \\ n \end{pmatrix}. \tag{1.20}$$

That is, the dimensions are determined by the binomial coefficients. These contain a surprising wealth of geometric information! It follows that the total dimension is

$$\mathrm{Dim}\,[\mathcal{G}_n] = \sum_{r=0}^{n} \begin{pmatrix} r \\ n \end{pmatrix} = (1+1)^n = 2^n. \tag{1.21}$$

An important feature to understand is that *not all homogeneous multivectors are pure blades.* This is confusing at first, because we have to go to four dimensions before we reach our first counter-example. Suppose that $\{e_1 \dots e_4\}$ form an orthonormal basis for $\mathcal{G}_4$. There are six independent basis bivectors in this algebra, and from these we can construct terms like

$$B = \alpha e_1 {\wedge} e_2 + \beta e_3 {\wedge} e_4, \quad \alpha, \beta \in \mathcal{R}. \tag{1.22}$$

B is a pure bivector, so is homogeneous, but it cannot be reduced to a blade. That is, we cannot find two vectors a and b such that $B = a{\wedge}b$. The reason is that $e_1{\wedge}e_2$ and $e_3{\wedge}e_4$ do not share a common line. This is not possible in 3-d, because any two planes with a common point share a common line. A 4-d bivector like (1.22) is therefore hard for us to visualise. There is a way to visualise B in 3-d, however, and it is provided by *projective geometry*. This is described in a later handout.

1.4 Further Properties of the Geometric Product

The manipulation of the geometric product of a vector and a bivector extends simply to that of a vector a and a grade-r multivector A_r. Suppose that we decompose A_r into blades, and one of these is written $a_1 a_2 \cdots a_r$. We have

$$\begin{aligned} a a_1 a_2 \cdots a_r &= 2a{\cdot}a_1\, a_2 \cdots a_r - a_1 a a_2 \cdots a_r \\ &= 2a{\cdot}a_1\, a_2 \cdots a_r - 2a{\cdot}a_2\, a_1 a_3 \cdots a_r + a_1 a_2 a a_3 \cdots a_r \\ &= 2\sum_{k=1}^{r} (-1)^{k+1} a{\cdot}a_k\, a_1 a_2 \cdots \check{a}_k \cdots a_r + (-1)^r a_1 a_2 \cdots a_r a, \end{aligned} \tag{1.23}$$

where the check on $\check{a}_k$ denotes that this term is missing from the series. Each term in the sum has grade $r-1$, so we define

$$a\cdot A_r = \langle aA_r\rangle_{r-1} = \tfrac{1}{2}(aA_r - (-1)^r A_r a). \tag{1.24}$$

The remaining term in the product aA_r is easily shown to be totally antisymmetric, so we also have

$$a\wedge A_r = \langle aA_r\rangle_{r+1} = \tfrac{1}{2}(aA_r + (-1)^r A_r a). \tag{1.25}$$

We can therefore write

$$aA_r = a\cdot A_r + a\wedge A_r. \tag{1.26}$$

Multiplication by a vector raises and lowers the grade of a multivector by 1.

In Eq. (1.23) we assumed that the $\{a_i\}$ were orthogonal. We can extend this decomposition to non-orthogonal vectors by writing

$$\begin{aligned} a\cdot(a_1\wedge a_2\wedge\cdots\wedge a_r) &= \tfrac{1}{2}[a\langle a_1a_2\cdots a_r\rangle_r - (-1)^r\langle a_1a_2\cdots a_r\rangle_r a] \\ &= \tfrac{1}{2}\langle aa_1a_2\cdots a_r - (-1)^r a_1a_2\cdots a_r a\rangle_{r-1} \end{aligned} \tag{1.27}$$

The final manipulation is possible because the geometric product $a_1a_2\cdots a_r$ only contains terms of grade r, $r-2$ and so on. Of these, only the $r-2$ grade term could give an extra, unwanted contribution, but

$$\tfrac{1}{2}(aA_{r-2} - (-1)^r A_{r-2}a) = a\cdot A_{r-2}, \tag{1.28}$$

which is grade $r-3$. We can now use Eq. (1.23) to write

$$\begin{aligned} a\cdot(a_1\wedge a_2\wedge\cdots\wedge a_r) &= \langle\sum_{k=1}^{r}(-1)^{k+1}a\cdot a_k\, a_1a_2\cdots\check{a}_k\cdots a_r\rangle_{r-1} \\ &= \sum_{k=1}^{r}(-1)^{k+1}a\cdot a_k\, a_1\wedge a_2\wedge\cdots\wedge\check{a}_k\wedge\cdots\wedge a_r \end{aligned} \tag{1.29}$$

This result is extremely useful in practice. The first two cases are sufficient to understand how the result goes:

$$\begin{aligned} a\cdot(a_1\wedge a_2) &= a\cdot a_1\, a_2 - a\cdot a_2\, a_1 \\ a\cdot(a_1\wedge a_2\wedge a_3) &= a\cdot a_1\, a_2\wedge a_3 - a\cdot a_2\, a_1\wedge a_3 + a\cdot a_3\, a_1\wedge a_2 \end{aligned} \tag{1.30}$$

Note in particularly the similarity of the first case with the double cross product of vectors in 3-d.

The general product of two homogeneous multivectors decomposes as

$$A_rB_s = \langle A_rB_s\rangle_{|r-s|} + \langle A_rB_s\rangle_{|r-s|+2} + \cdots + \langle A_rB_s\rangle_{r+s} \tag{1.31}$$

which can be seen expanding each term into sums of products of vectors. We retain the $\cdot$ and $\wedge$ symbols for the lowest and highest grade terms in this series

$$\begin{aligned} A_r \cdot B_s &= \langle A_r B_s \rangle_{|r-s|} \\ A_r \wedge B_s &= \langle A_r B_s \rangle_{r+s}. \end{aligned} \tag{1.32}$$

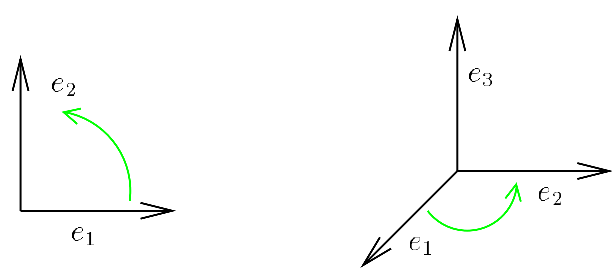
This definition ensures that the exterior product is associative (exercise).

1.5 Pseudoscalars and Duality

The exterior product of n vectors defines a grade-n blade. This must be a multiple of the unique *pseudoscalar* for $\mathcal{G}_n$. This is denoted I, and has two important properties. The first is that I is normalised to

$$|I^2| = 1. \tag{1.33}$$

The sign of I^2 depends on the size of space (and its signature). It turns out that many of the most useful algebras happen to have $I^2 = -1$, but this is by no means general. The second property is that I is formed from a *right-handed* set. The definition of right-handed works inductively. We agree that $e_1 e_2$ is right-handed if e_1 rotates onto e_2 in a positive (anti-clockwise) sense. The blade $e_1 e_2 e_3$ is then right-handed if projecting down the e_3 direction returns a right-handed plane. This carries on for each new orthogonal vector added to the product.



The product of the grade-n pseudoscalar I with a grade-r multivector A_r is a grade $n-r$ multivector. This operation is called a *duality* transformation. If A_r is a blade, IA_r returns the *orthogonal complement* of A_r. That is, the blade formed from the space of vectors not contained in A_r. It is clear why this has grade $n-r$.

In spaces of odd dimension, I commutes with all vectors, and so commutes with all multivectors. In spaces of even dimension, I anticommutes with vectors and so anticommutes with all odd-grade multivectors, and commutes with all even-grade multivectors. We can summarise this by

$$IA_r = (-1)^{r(n-1)} A_r I. \tag{1.34}$$

An important use of the pseudoscalar is for interchanging dot and wedge products. For example, we have

$$\begin{aligned} a\cdot(A_r I) &= \tfrac{1}{2}[aA_r I - (-1)^{n-r} A_r I a] \\ &= \tfrac{1}{2}[aA_r I - (-1)^{n-r}(-1)^{n-1} A_r a I] \\ &= \tfrac{1}{2}[aA_r + (-1)^r A_r a] I = a\wedge A_r\, I. \end{aligned} \tag{1.35}$$

More generally, we can take two multivectors A_r and B_s, with $r+s\leq n$, and form

$$\begin{aligned} A_r\cdot(B_s I) &= \langle A_r B_s I\rangle_{|r-(n-s)|} \\ &= \langle A_r B_s I\rangle_{n-(r+s)} = \langle A_r B_s\rangle_{r+s} I = A_r\wedge B_s\, I \end{aligned} \tag{1.36}$$

This type of interchange is very common in applications. We have already made use of it in 3-d. Note how simple this proof is made because of the application of the geometric product in the intermediate steps.

A useful idea is that every blade can act as a pseudoscalar for the space spanned by its generating vectors. So, even if we are working in 3-d, we can treat the bivector e_1e_2 as a pseudoscalar for any manipulation taking place entirely in its plane. We can then immediately apply any of the preceding results.

1.6 Further Definitions

We end this section with a look at two further operations. The first is *reversion*, first introduced in Handout 2. The reverse of a product of vectors is defined by

$$(ab\cdots c)^{\sim} = c\cdots ba. \tag{1.37}$$

For a blade the reverse can be formed by a series of swaps of anticommuting vectors, each resulting in a minus sign. The first vector has to swap pass $r-1$ vectors, the second past $r-2$, and so on. It is easy to see, then, that

$$\tilde{A}_r = (-1)^{r(r-1)/2} A_r \tag{1.38}$$

The second operation is the generalised *scalar product*, which we write as either $\langle AB\rangle$ or $A{*}B$ (the former is more common). We have

$$A{*}B = \langle AB\rangle = \sum_r \langle A_r B_r\rangle. \tag{1.39}$$

By forming

$$\langle A_r B_r\rangle = \langle A_r B_r\rangle^{\sim} = \langle \tilde{B}_r \tilde{A}_r\rangle = (-1)^{r(r-1)}\langle B_r A_r\rangle = \langle B_r A_r\rangle \tag{1.40}$$

we see that

$$\langle AB\rangle = \langle BA\rangle. \tag{1.41}$$

It follows that

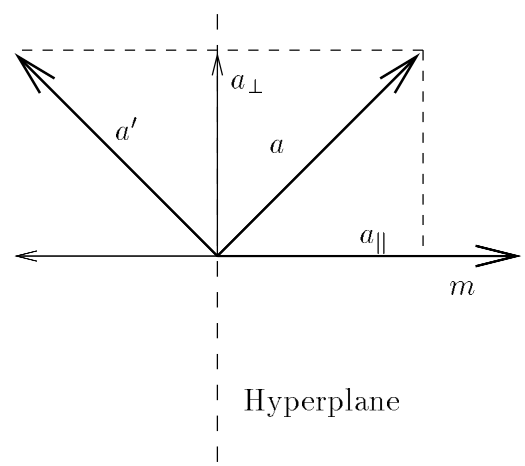
$$\langle A\cdots BC\rangle = \langle CA\cdots B\rangle. \tag{1.42}$$

This cyclic reordering property is very useful.

2 Reflections

Suppose that we reflect the vector a in the (hyper)plane orthogonal to some unit vector m ($m^2 = 1$).

$$a = a_\perp + a_{||}$$
$$a' = a_\perp - a_{||}$$



The component of a parallel to m changes sign, whereas the perpendicular component is unchanged. The parallel component is the projection onto m:

$$a_{||} = a\cdot m\, m. \tag{2.1}$$

The perpendicular component is the remainder

$$a_\perp = a - a\cdot m\, m = (am - a\cdot m)m = a\wedge m\, m \tag{2.2}$$

This shows how the wedge product projects onto the components perpendicular to a vector. The result of the reflection is therefore

$$\begin{aligned} a' = a_\perp - a_{||} &= -a\cdot m\, m + a\wedge m\, m \\ &= -(m\cdot a + m\wedge a)m = -mam. \end{aligned} \tag{2.3}$$

This remarkably compact formula only arises in geometric algebra. We can start to see now that geometric products arise naturally when *operating* on vectors.

It is simple to check that our formula has the required properties. For any vector parallel to m we have

$$-m(\lambda m)m = -\lambda mmm = -\lambda m \tag{2.4}$$

and so λm is reflected. Similarly, for any vector n perpendicular to m we have

$$-m(n)m = -mnm = nmm = n \tag{2.5}$$

and so n is unaffected. We can also give a simple proof that inner products are unchanged by reflections,

$$\begin{aligned} a'\cdot b' &= (-mam)\cdot(-mbm) = \langle mammbm \rangle \\ &= \langle mabm \rangle = \langle mmab \rangle = a\cdot b. \end{aligned} \tag{2.6}$$

We can also construct the transformation law for the bivector $a\wedge b$ under reflection of both a and b. We obtain

$$\begin{aligned} a'\wedge b' &= (-mam)\wedge(-mbm) = \langle mammbm \rangle_2 \\ &= \langle mabm \rangle_2 = m\, a\wedge b\, m. \end{aligned} \tag{2.7}$$

We recover essentially the same law, but with a crucial sign difference. Bivectors do not quite transform as vectors under reflections. This is the reason for the confusing distinction between polar and axial vectors in 3-d. Axial vectors are really bivectors, and should be treated as such.

3 Rotations

Our starting point is the result that *a rotation in the plane generated by two unit vectors m and n is achieved by successive reflections in the (hyper)planes perpendicular to m and n.* This is illustrated in Fig. 1. It is clear that any component of a outside the plane $m\wedge n$ plane is untouched. It is also a simple exercise in trigonometry to confirm that the angle between the initial vector a and the final vector a'' is twice the angle between m and n. (This is left as an exercise.) The result of the successive reflections is therefore to rotate through 2θ in the $m\wedge n$ plane, where $m\cdot n = \cos(\theta)$.

So how does this look in GA?

$$a' = -mam \tag{3.1}$$

$$a'' = -na'n = -n(-mam)n = nmamn \tag{3.2}$$

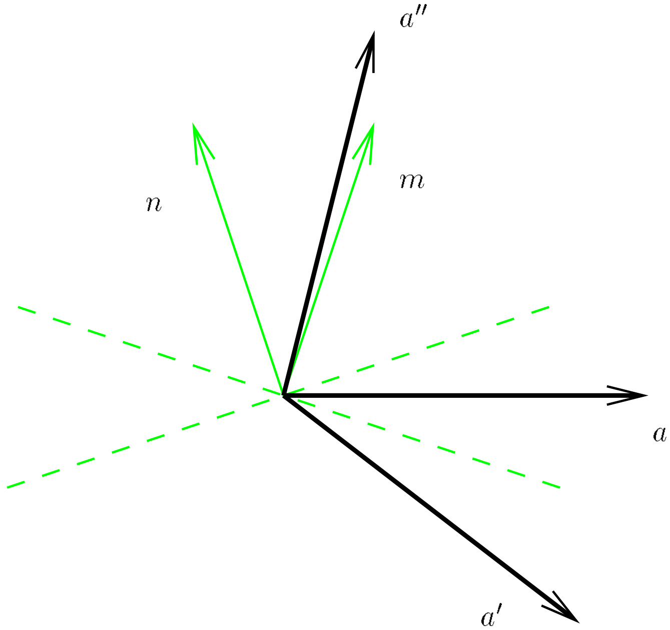


Figure 1: *A Rotation from 2 Reflections.* a' is the result of reflecting a in the plane perpendicular to m. a'' is the result of reflecting a' in the plane perpendicular to n.

This is beginning to look very simple! We define

$$R = nm \tag{3.3}$$

Note the *geometric* product here! We can now write a rotation as

$$a \mapsto Ra\tilde{R} \tag{3.4}$$

Incredibly, this formula works for any grade of multivector, in any dimension, of any signature! The quantity R is called a rotor, and we have already discovered the usefulness of these in 2-d and 3-d from a different route. To reconcile these two approaches, we note that R is the geometric product of two unit vectors n and m, so

$$R = nm = n\cdot m + n\wedge m = \cos(\theta) + n\wedge m. \tag{3.5}$$

So what is the magnitude of the bivector $n\wedge m$?

$$\begin{aligned}
(n\wedge m)\cdot(n\wedge m) &= \langle n\wedge m \, n\wedge m\rangle \\
&= \langle nm \, n\wedge m\rangle \\
&= n\cdot[m\cdot(n\wedge m)] \\
&= n\cdot(m\cos(\theta) - n) \\
&= \cos^2(\theta) - 1 = -\sin^2(\theta).
\end{aligned} \tag{3.6}$$

We therefore define a unit bivector in the $m\wedge n$ plane by

$$\hat{B} = m\wedge n/\sin(\theta), \qquad \hat{B}^2 = -1. \tag{3.7}$$

The reason for this choice of orientation ($m\wedge n$ rather than $n\wedge m$) is that the angle θ is defined as the angle between m and n in the positive sense from m to n. The definition thus ensures that $\hat{B}$ is right-handed.

In terms of the bivector $\hat{B}$ we now have

$$R = \cos(\theta) - \hat{B}\sin(\theta). \tag{3.8}$$

Look familiar? This is nothing else than the polar decomposition of a complex number, with the unit imaginary replaced by the unit bivector $\hat{B}$. We can therefore write

$$R = \exp\{-\hat{B}\theta\}. \tag{3.9}$$

The exponential here is defined in terms of its power series in the normal way. It is possible to show that this series is absolutely convergent for any multivector argument. (Exponentiating a multivector is essentially the same as exponentiating a matrix).

Now recall that our formula was for a rotation through 2θ. If we want to rotate through θ, the appropriate rotor is

$$R = \exp\{-\hat{B}\theta/2\} \tag{3.10}$$

which gives the formula

$$a \mapsto \mathrm{e}^{-\hat{B}\theta/2}\, a\, \mathrm{e}^{\hat{B}\theta/2} \tag{3.11}$$

for a positive rotation through θ in the $\hat{B}$ plane. The GA description forces us to think of rotations taking place *in a plane* as opposed to about an axis, which is an entirely 3-d concept.

Since the rotor R is a geometric product of two unit vectors, we see immediately that

$$R\tilde{R} = nm(nm)^{\sim} = nmmn = 1 = \tilde{R}R. \tag{3.12}$$

This provides a quick proof that our formula has the correct property of preserving lengths and angles,

$$a'\cdot b' = (Ra\tilde{R})\cdot(Rb\tilde{R}) = \langle Ra\tilde{R}Rb\tilde{R}\rangle = \langle Rab\tilde{R}\rangle = a\cdot b \tag{3.13}$$

Rotors are one of the fundamental concepts in geometric algebra, and we will return to their properties many times throughout this course.