Physical Applications of Geometric Algebra

Handout 9

Spacetime Dynamics

Lorentz transformations which preserve parity and causal structure can be described with rotors, and these provide the simplest way to gain insight into the structure of the Lorentz group. They quickly show, for example, that all transformations have two points on the celestial sphere which remain fixed. Dynamics in spacetime is traditionally viewed as a hard subject. This need not be the case, however. By parameterising the motion in terms of rotors many equations are considerably simplified, and can be solved in new ways. This provides a simple understanding of the Thomas precession, as well as a new formulation of the Lorentz force law for a particle in an electromagnetic field.

1 Spacetime Rotors

We saw in Handout 8 that a restricted Lorentz transformation is generated by a rotor R, $R\tilde{R} = 1$, in the usual way as $a \mapsto Ra\tilde{R}$. Every rotor in spacetime can be written in terms of a bivector as

$$R = \pm\, \mathrm{e}^{B/2}\,. \tag{1.1}$$

(The minus sign is rarely required, and does not affect the vector transformation law.) We can understand many of the features of spacetime transformations and rotors through the properties of the bivector B.

1.1 Invariant Decomposition

The rotor R can be decomposed in a Lorentz invariant way by first writing

$$B^2 = \langle B^2\rangle_0 + \langle B^2\rangle_4 = \rho\, \mathrm{e}^{I\phi}\,, \tag{1.2}$$

and we will assume that $\rho \neq 0$. (The case of a null bivector is treated slightly differently.) We now define

$$\hat{B} = \rho^{-1/2}\, \mathrm{e}^{-I\phi/2}\, B \tag{1.3}$$

so that

$$\hat{B}^2 = \rho^{-1}\,\mathrm{e}^{-I\phi}\,B^2 = 1. \tag{1.4}$$

With this we can now write

$$B = \rho^{1/2}\,\mathrm{e}^{I\phi/2}\,\hat{B} = \alpha\hat{B} + \beta I\hat{B}, \tag{1.5}$$

which decomposes B into a pair of bivector blades $\alpha\hat{B}$ and $\beta I\hat{B}$. Since

$$\hat{B}(I\hat{B}) = (I\hat{B})\hat{B} = I, \tag{1.6}$$

the separate bivector blades commute, which is possible now that we are in 4 dimensions. The rotor R now decomposes into

$$R = \mathrm{e}^{\alpha\hat{B}/2}\,\mathrm{e}^{\beta I\hat{B}/2} = \mathrm{e}^{\beta I\hat{B}/2}\,\mathrm{e}^{\alpha\hat{B}/2} \tag{1.7}$$

exhibiting an *invariant* split into a boost and a rotation. The boost is generated by $\hat{B}$ and the rotation by $I\hat{B}$.

1.2 Fixed Points

For every timelike bivector $\hat{B}$, $\hat{B}^2 = 1$, we can construct a pair of null vectors $n_\pm$ satisfying (exercise)

$$\hat{B}\cdot n_\pm = \pm n_\pm. \tag{1.8}$$

These are necessarily null, since

$$(\hat{B}\cdot n_\pm)\cdot n_\pm = 0 = \pm n_\pm^2. \tag{1.9}$$

The two null vectors can also be chosen so that

$$n_+\wedge n_- = 2\hat{B}, \tag{1.10}$$

so that they form a null basis for the timelike plane defined by $\hat{B}$ (see Fig. 1).

The null vectors $n_\pm$ anticommute with $\hat{B}$ and therefore commute with $I\hat{B}$. The effect of the Lorentz transformation on $n_\pm$ is therefore

$$\begin{aligned} Rn_\pm\tilde{R} &= \mathrm{e}^{\alpha\hat{B}/2}\,n_\pm\,\mathrm{e}^{-\alpha\hat{B}/2} = \mathrm{e}^{\alpha\hat{B}}\,n_\pm \\ &= \mathrm{ch}(\alpha)n_\pm + \mathrm{sh}(\alpha)\hat{B}\cdot n_\pm = \mathrm{e}^{\pm\alpha}\,n_\pm. \end{aligned} \tag{1.11}$$

The two null directions are therefore just scaled — their direction is unchanged. This is another example of rotors being used to describe dilations. It follows that every Lorentz transformation has two invariant null directions. The case where the bivector generator itself is null, $B^2 = 0$, corresponds to the special situation where these two null directions coincide.

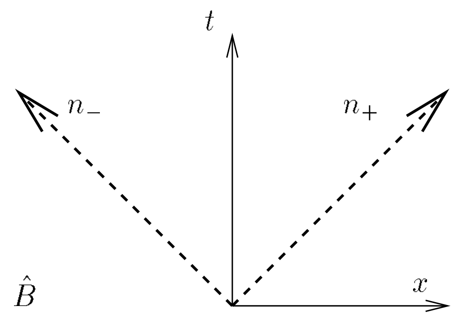

Figure 1: *A timelike plane.* Any timelike plane $\hat{B}$, $\hat{B}^2 = 1$ contains two null vectors n_+ and n_-. These can be normalised so that $n_+ \wedge n_- = 2\hat{B}$.

1.3 The Celestial Sphere

One way to visualise the effect of Lorentz transformations is through their effect on the past light cone (see Fig. 2). Each null vector on the past light cone maps to a point on the sphere S^- — the *celestial sphere* for the observer. Suppose then that light is received along the null vector n, with the observer's velocity chosen to be γ_0. The relative vector in the γ_0 frame is $n\wedge\gamma_0$. This has magnitude

$$(n\wedge\gamma_0)^2 = (n\cdot\gamma_0)^2 - n^2\gamma_0^2 = (n\cdot\gamma_0)^2. \tag{1.12}$$

We therefore define the unit relative vector $\boldsymbol{n}$ by the familiar projective formula

$$\boldsymbol{n} = \frac{n\wedge\gamma_0}{n\cdot\gamma_0}. \tag{1.13}$$

Different observers passing through the same point see different celestial spheres. If a second observer has velocity $v = R\gamma_0\tilde{R}$, the unit relative vectors in this observer's frame are formed from $n\wedge v/n\cdot v$. These can be brought to the γ_0 frame for comparison by forming

$$\boldsymbol{n}' = \tilde{R}\frac{n\wedge v}{n\cdot v}R = \frac{n'\wedge\gamma_0}{n'\cdot\gamma_0} \tag{1.14}$$

where $n' = \tilde{R}nR$. The effects of Lorentz transformations can be visualised simply by moving around points on the celestial sphere with the map $n \mapsto \tilde{R}nR$. We know immediately, then, that two points remain invariant so are the same for both observers.

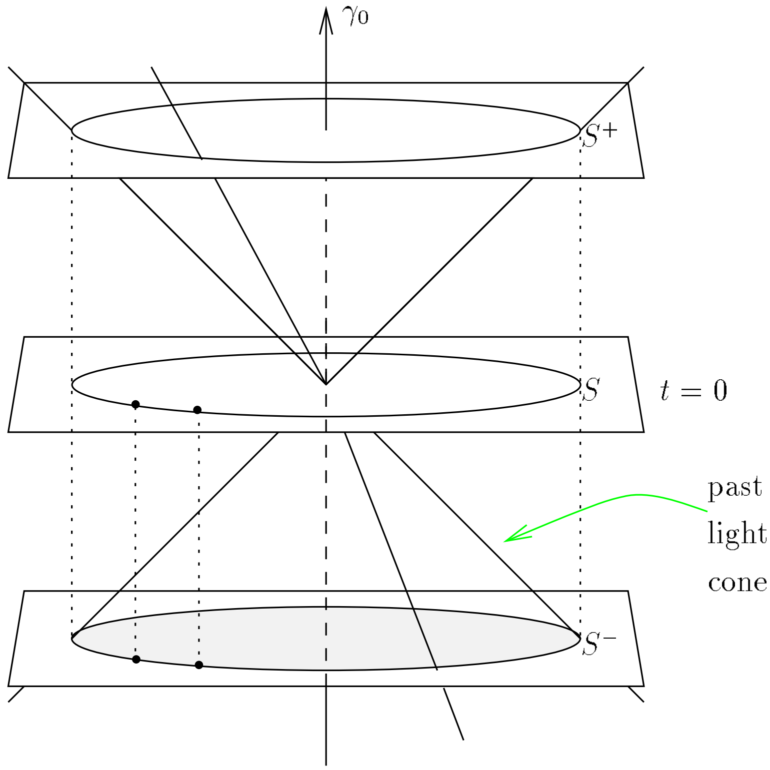

Figure 2: *The Celestial Sphere.* Each observer sees events in their past light cone, which can be viewed as defining a sphere.

1.4 Pure Boosts and Observer Splits

Suppose we are travelling with velocity u and want to boost to velocity v. We seek the rotor for this which contains no additional rotational factors. We have

$$v = Lu\tilde{L} \tag{1.15}$$

with $La_\perp \tilde{L} = a_\perp$ for any vector outside the $u\wedge v$ plane. It is clear that the appropriate bivector for the rotor is $u\wedge v$, and as this anticommutes with u and v we have

$$v = Lu\tilde{L} = L^2 u \qquad \Longrightarrow \; L^2 = vu \tag{1.16}$$

The solution to this is

$$L = \frac{1 + vu}{[2(1 + u\cdot v)]^{1/2}} = \exp\left(\frac{\alpha}{2}\frac{v\wedge u}{|v\wedge u|}\right) \tag{1.17}$$

where the angle α is defined by $\cosh(\alpha) = u\cdot v$.

Now suppose that we start in the γ_0 frame and some arbitrary rotor R takes this to $v = R\gamma_0\tilde{R}$. We know that the pure boost for this transformation is

$$L = \frac{1 + v\gamma_0}{[2(1 + v\cdot\gamma_0)]^{1/2}} = \exp\left(\frac{\alpha}{2}\frac{v\wedge\gamma_0}{|v\wedge\gamma_0|}\right), \tag{1.18}$$

where $v\cdot\gamma_0 = \mathrm{ch}(\alpha)$. Now define the further rotor U by

$$U = \tilde{L}R, \qquad U\tilde{U} = \tilde{L}R\tilde{R}L = 1. \tag{1.19}$$

This satisfies

$$U\gamma_0\tilde{U} = \tilde{L}vL = \gamma_0, \tag{1.20}$$

so $U\gamma_0 = \gamma_0 U$. We must therefore have $U = \mathrm{e}^{I\boldsymbol{b}/2}$, where $I\boldsymbol{b}$ is a relative bivector, and U generates a pure rotation in the γ_0 frame. We now have

$$R = LU \tag{1.21}$$

which decomposes R into a relative rotation and boost. Unlike earlier, this decomposition is frame dependent, and in general L and U do not commute.

2 Spacetime Rotor Equations

A spacetime trajectory $x(\tau)$ has a future-pointing velocity vector $\dot{x} = v$. This is normalised to $v^2 = 1$ by parameterising the curve in terms of the proper time. This suggests an analogy with rigid body dynamics. We write

$$v = R\gamma_0\tilde{R}, \tag{2.1}$$

which keeps v future-pointing and normalised. This moves all of the dynamics into the rotor $R = R(\tau)$, and this is the key idea which simplifies much of relativistic dynamics.

2.1 The Proper Acceleration

The first quantity we need to find is the acceleration

$$\dot{v} = \partial_\tau(R\gamma_0\tilde{R}) = \dot{R}\gamma_0\tilde{R} + R\gamma_0\dot{\tilde{R}}. \tag{2.2}$$

But we know that $\dot{R}\tilde{R} = -R\dot{\tilde{R}}$ is a bivector, so we have

$$\dot{v} = \dot{R}\tilde{R}R\gamma_0\tilde{R} + R\gamma_0\tilde{R}R\dot{\tilde{R}} = \dot{R}\tilde{R}v - v\dot{R}\tilde{R} = 2(\dot{R}\tilde{R})\cdot v. \tag{2.3}$$

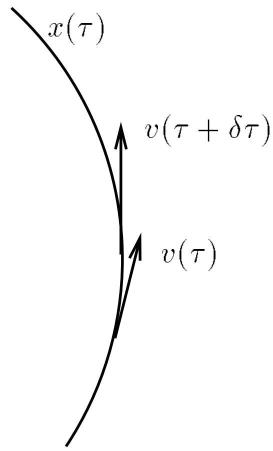

Figure 3: *The Proper Boost.* The change in velocity from τ to $\tau+\delta\tau$ should be described by a rotor solely in the $\dot{v}\wedge v$ plane.

This equation is consistent with the fact that $v\cdot\dot{v} = 0$, which follows from $v^2 = 1$.

We now have

$$\dot{v}v = 2(\dot{R}\tilde{R})\cdot v\, v. \tag{2.4}$$

But the rotor R can always carry an extra rotation with it. We need to ensure that the rotor we work with correctly describes pure boosts from one instance to the next (see Fig. 3). To first order we have

$$v(\tau+\delta\tau) = v(\tau) + \delta\tau\, \dot{v}. \tag{2.5}$$

The proper rotor between $v(\tau)$ and $v(\tau+\delta\tau)$ is

$$L = \frac{1 + v(\tau+\delta\tau)v(\tau)}{[2(1 + v(\tau+\delta\tau)\cdot v(\tau))]^{1/2}} = 1 + \tfrac{1}{2}\delta\tau\, \dot{v}v. \tag{2.6}$$

But since

$$v(\tau+\delta\tau) = R(\tau+\delta\tau)\gamma_0\tilde{R}(\tau+\delta\tau) = LR(\tau)\gamma_0\tilde{R}(\tau)\tilde{L} \tag{2.7}$$

we see that we must set

$$R(\tau+\delta\tau) = R(\tau) + \delta\tau\, \dot{R} = LR(\tau). \tag{2.8}$$

It follows that the correct expression is

$$\dot{R}\tilde{R} = \tfrac{1}{2}\dot{v}v, \tag{2.9}$$

which is sensible. The bivector describing the change in the rotor is simply the acceleration seen in the rest frame. We call this object $\dot{v}\wedge v$ the *acceleration bivector*.

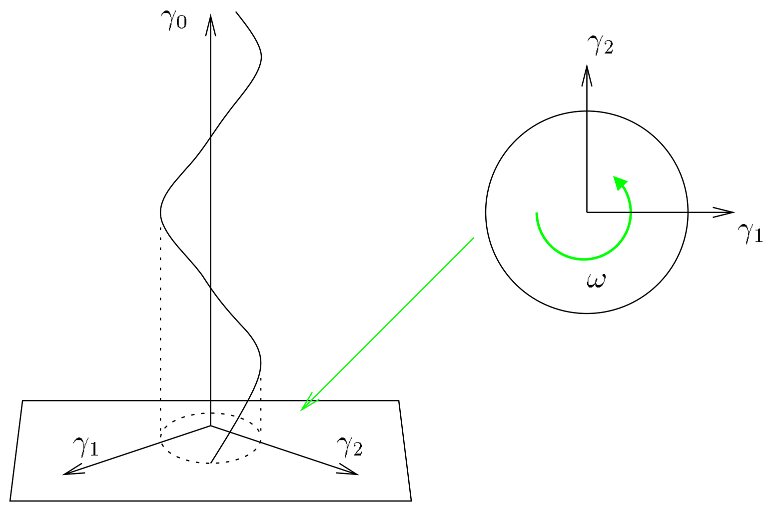

Figure 4: *Thomas Precession.* The particle follows a helical worldline, rotating at a constant rate in the γ_0 frame.

2.2 Example — Thomas Precession

As an application, consider a particle in a circular orbit (Fig. 4). The worldline is

$$x(\tau) = t(\tau)\gamma_0 + a[\cos(\omega t)\gamma_1 + \sin(\omega t)\gamma_2], \tag{2.10}$$

and the velocity is

$$v = \partial_\tau x = \dot{t}\,(\gamma_0 + aw[-\sin(\omega t)\gamma_1 + \cos(\omega t)\gamma_2])\,. \tag{2.11}$$

(Throughout we use dots to denote differentiation with respect to proper time τ). The relative velocity $\boldsymbol{v} = v\wedge\gamma_0/v\cdot\gamma_0$ has magnitude $|\boldsymbol{v}| = a\omega$. We therefore introduce the hyperbolic angle α, with

$$\tanh\alpha = a\omega, \qquad \dot{t} = \cosh\alpha. \tag{2.12}$$

The velocity is now

$$v = \mathrm{ch}(\alpha)\gamma_0 + \mathrm{sh}(\alpha)[-\sin(\omega t)\gamma_1 + \cos(\omega t)\gamma_2] = \,\mathrm{e}^{\alpha\boldsymbol{n}/2}\,\gamma_0\,\mathrm{e}^{-\alpha\boldsymbol{n}/2} \tag{2.13}$$

where

$$\boldsymbol{n} = -\sin(\omega t)\boldsymbol{\sigma}_1 + \cos(\omega t)\boldsymbol{\sigma}_2. \tag{2.14}$$

This form of time-dependence in the rotor is inconvenient to work with. To simplify, we write

$$\boldsymbol{n} = \,\mathrm{e}^{-\omega t I\boldsymbol{\sigma}_3}\,\boldsymbol{\sigma}_2 = R_\omega\boldsymbol{\sigma}_2\tilde{R}_\omega \tag{2.15}$$

where $R_\omega = \exp(-\omega t \boldsymbol{I\sigma}_3/2)$. We now have

$$\mathrm{e}^{\alpha \boldsymbol{n}/2} = \exp(\alpha R_\omega \boldsymbol{\sigma}_2 \tilde{R}_\omega/2) = R_\omega\, \mathrm{e}^{\alpha\boldsymbol{\sigma}_2/2}\, \tilde{R}_\omega = R_\omega R_\alpha \tilde{R}_\omega \tag{2.16}$$

where $R_\alpha = \exp(\alpha\boldsymbol{\sigma}_2/2)$. The velocity is now given by

$$v = R_\omega R_\alpha \tilde{R}_\omega \gamma_0 R_\omega \tilde{R}_\alpha \tilde{R}_\omega = R_\omega R_\alpha \gamma_0 \tilde{R}_\alpha \tilde{R}_\omega, \tag{2.17}$$

where the final expression follows because R_ω commutes with γ_0.

We can now see that the rotor for the motion must have the form

$$R = R_\omega R_\alpha R_T, \tag{2.18}$$

where $R_T = \exp(-\omega_T t \boldsymbol{I\sigma}_3/2)$ is some rotation in the $\boldsymbol{I\sigma}_3$ frame whose rate is to be determined. To fix this we form the acceleration bivector $\dot{v}v$. We can simplify this derivation by writing $v = R_\omega v_\alpha \tilde{R}_\omega$, where $v_\alpha = R_\alpha \gamma_0 \tilde{R}_\alpha$. We then get

$$\begin{aligned}\dot{v}v &= R_\omega[2(\tilde{R}_\omega \dot{R}_\omega)\cdot v_\alpha\, v_\alpha]\tilde{R}_\omega = -\omega\, \mathrm{ch}(\alpha) R_\omega[(\boldsymbol{I\sigma}_3)\cdot v_\alpha\, v_\alpha]\tilde{R}_\omega \\ &= \omega\, \mathrm{sh}(\alpha)\mathrm{ch}(\alpha) R_\omega[-\mathrm{ch}(\alpha)\boldsymbol{\sigma}_1 + \mathrm{sh}(\alpha)\boldsymbol{I\sigma}_3]\tilde{R}_\omega.\end{aligned} \tag{2.19}$$

We also form the rotor equivalent $2\dot{R}\tilde{R}$, which is

$$\begin{aligned}2\dot{R}\tilde{R} &= 2\dot{R}_\omega \tilde{R}_\omega + 2R_\omega R_\alpha \dot{R}_T \tilde{R}_T \tilde{R}_\alpha \tilde{R}_\omega \\ &= \mathrm{ch}(\alpha) R_\omega[-\omega \boldsymbol{I\sigma}_3 - \omega_T R_\alpha \boldsymbol{I\sigma}_3 \tilde{R}_\alpha]\tilde{R}_\omega \\ &= \mathrm{ch}(\alpha) R_\omega[-(\omega + \omega_T \mathrm{ch}(\alpha))\boldsymbol{I\sigma}_3 + \omega_T \mathrm{sh}(\alpha)\boldsymbol{\sigma}_1]\tilde{R}_\omega.\end{aligned} \tag{2.20}$$

Equating these we find that $\omega_T = -\mathrm{ch}(\alpha)\omega$, so the full rotor is

$$R = \mathrm{e}^{-\omega t \boldsymbol{I\sigma}_3/2}\, \mathrm{e}^{\alpha\boldsymbol{\sigma}_2/2}\, \mathrm{e}^{\mathrm{ch}(\alpha)\omega t \boldsymbol{I\sigma}_3/2}. \tag{2.21}$$

The fact that $\omega_T = -\mathrm{ch}(\alpha)\omega$ differs from $-\omega$ is due to the fact that the acceleration is formed in the instantaneous rest frame v and not the fixed γ_0 frame. This difference introduces a precession — the *Thomas precession*. We can see this effect by imagining the vector γ_1 being transported around the circle. We define the rotated vector by

$$e_1 = R\gamma_1 \tilde{R}. \tag{2.22}$$

In the low velocity limit $\cosh(\alpha) \mapsto 1$ the vector e_1 continues to point in the γ_1 direction. This is what we expect — there is no rotational component to the frame. At larger velocities, however, the frame starts to precess. After time $t = 2\pi/\omega$, for example, the γ_1 vector is transformed to

$$e_1(2\pi/\omega) = \mathrm{e}^{\alpha\boldsymbol{\sigma}_2/2}\, \mathrm{e}^{2\pi \mathrm{ch}(\alpha)\boldsymbol{I\sigma}_3}\, \gamma_1\, \mathrm{e}^{-\alpha\boldsymbol{\sigma}_2/2}. \tag{2.23}$$

Dotting this with the initial vector $e_1(0)$ we see that the vector has precessed through an angle

$$\theta = 2\pi(\cosh\alpha - 1). \tag{2.24}$$

This shows that the effect is of order $|\boldsymbol{v}|^2/c^2$.

3 The Lorentz Force Law

We are all familiar with the non-relativistic form of the Lorentz force law,

$$\frac{d\boldsymbol{p}}{dt} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}), \tag{3.1}$$

with all relative vectors expressed in the γ_0 frame. We seek a relativistic version of this law. The quantity $\boldsymbol{p}$ on the left-hand side is the relative vector $p\wedge\gamma_0$. We must therefore multiply through by $v\cdot\gamma_0$ to convert the derivative into one with respect to proper time. The first term on the right-hand side then becomes

$$v\cdot\gamma_0\,\boldsymbol{E} = v\cdot(\boldsymbol{E}\wedge\gamma_0) - (v\cdot\boldsymbol{E})\wedge\gamma_0 = (\boldsymbol{E}\cdot v)\wedge\gamma_0. \tag{3.2}$$

Recall at this point that $\boldsymbol{E}$ is a spacetime *bivector* and is built from the $\boldsymbol{\sigma}_k = \gamma_k\gamma_0$. The magnetic term is

$$\begin{aligned} -v\cdot\gamma_0\,\boldsymbol{v}\cdot(I\boldsymbol{B}) &= -(v\wedge\gamma_0)\times(I\boldsymbol{B}) \\ &= [(I\boldsymbol{B})\cdot v]\wedge\gamma_0 + [\gamma_0\cdot(I\boldsymbol{B})]\wedge v = [(I\boldsymbol{B})\cdot v]\wedge\gamma_0, \end{aligned} \tag{3.3}$$

where the Jacobi identity has been used in the intermediate step.

We can now write Eq. (3.1) in the form

$$\frac{d\boldsymbol{p}}{d\tau} = \dot{p}\wedge\gamma_0 = q[(\boldsymbol{E} + I\boldsymbol{B})\cdot v]\wedge\gamma_0. \tag{3.4}$$

We now define the *Faraday bivector* F by

$$F = \boldsymbol{E} + I\boldsymbol{B}. \tag{3.5}$$

This is the covariant form of the electromagnetic field strength. It unites the electric and magnetic fields into a single spacetime structure. We study this in greater detail in Handout 10. Our equation is now

$$\dot{p}\wedge\gamma_0 = q(F\cdot v)\wedge\gamma_0. \tag{3.6}$$

This must hold in all frames, so we can remove the factors of γ_0. Recalling that $p = mv$, we arrive at the relativistic form of the *Lorentz force law*,

$$m\dot{v} = qF\cdot v. \tag{3.7}$$

This is *manifestly* Lorentz covariant, because no particular frame is picked out. The acceleration bivector is

$$\dot{v}v = \frac{q}{m}F\cdot v\,v = \frac{q}{m}(F\cdot v)\wedge v = \frac{q}{m}\boldsymbol{E}_v \tag{3.8}$$

where $\boldsymbol{E}_v$ is the relative electric field in the v frame. A charged point particle only responds to the instantaneous electric field in its frame.

3.1 Rotor Form of the Lorentz Force Law

Now suppose that we parameterise the velocity with a rotor. We have

$$\dot{v} = 2\dot{R}\tilde{R}v = 2(\dot{R}\tilde{R})\cdot v = \frac{q}{m}F\cdot v. \tag{3.9}$$

We can simply equate the projected terms to get

$$\dot{R} = \frac{q}{2m}FR \tag{3.10}$$

This is not the most general possibility as we could include an extra multiple of $F\wedge v\, v$, but Eq. (3.10) is certainly the simplest equation to work with. How does this help us solve the equations of motion? One immediate advantage is that the equations are now first order:

$$\dot{x} = v = Rv_0\tilde{R}, \qquad 2m\dot{R} = qFR, \tag{3.11}$$

(we usually take $v_0 = \gamma_0$). These are numerically very robust.

3.2 Example — Constant Field

This is very easy now! We can immediately integrate the rotor equation to give

$$R = \exp(\frac{q}{2m}F\tau). \tag{3.12}$$

To proceed and recover the trajectory we form the invariant decomposition of F. We first write

$$F^2 = \langle F^2\rangle_0 + \langle F^2\rangle_4 = \rho\, \mathrm{e}^{I\beta} \tag{3.13}$$

so that

$$F = \rho^{1/2}\, \mathrm{e}^{I\beta/2}\, \hat{F} = \alpha\hat{F} + I\beta\hat{F} \tag{3.14}$$

where $\hat{F}^2 = 1$. (If F is null a slightly different procedure is followed.) We now have

$$R = \exp(\frac{q}{2m}\alpha\hat{F}\tau)\exp(\frac{q}{2m}I\beta\hat{F}\tau). \tag{3.15}$$

Next we decompose the initial velocity $v_0 = \gamma_0$ into components in and out of the $\hat{F}$ plane,

$$v_0 = \hat{F}^2v_0 = \hat{F}\,\hat{F}\cdot v_0 + \hat{F}\,\hat{F}\wedge v_0 = v_{0\|} + v_{0\perp}. \tag{3.16}$$

Now $v_{0||} = \hat{F}\,\hat{F}\cdot v_0$ anticommutes with $\hat{F}$, and $v_{0\perp}$ commutes with $\hat{F}$, so

$$\dot{x} = \exp(\frac{q}{m}\alpha\hat{F}\tau)v_{0||} + \exp(\frac{q}{m}I\beta\hat{F}\tau)v_{0\perp}. \tag{3.17}$$

This integrates immediately to give the particle history

$$x - x_0 = \frac{\mathrm{e}^{q\alpha\hat{F}\tau/m} - 1}{q\alpha/m}\hat{F}\cdot v_0 - \frac{\mathrm{e}^{q\beta I\hat{F}\tau/m} - 1}{q\beta/m}(I\hat{F})\cdot v_0 \tag{3.18}$$

The first term gives linear acceleration and the second is periodic and drives rotational motion. This is as expected, because in the $v_{0||}$ frame, $\hat{F}$ is an electric field and $I\hat{F}$ is a magnetic field.

3.3 The Gyromagnetic Moment

Suppose now that as well as setting $v = R\gamma_0\tilde{R}$ we carry round a frame with our charged particle defined by

$$e_\mu = R\gamma_\mu\tilde{R}, \qquad e_0 = v. \tag{3.19}$$

The frame vectors $\{e_i\}$, $i = 1\ldots3$ lie in the rest frame of v. The equations of motion for these frame vectors are

$$\dot{e}_\mu = 2(\dot{R}\tilde{R})\cdot e_\mu = \frac{q}{m}F\cdot e_\mu, \tag{3.20}$$

where we have used the rotor equation in its simplest form of Eq. (3.10).

We can use this idea of a frame attached to a worldline to give a classical model for a charged particle with spin. We set $e_3 = s$, where s is the (dimensionless) 'spin vector'. This satisfies

$$\dot{s} = (q/m)F\cdot s. \tag{3.21}$$

Now suppose that the particle is at rest in the γ_0 frame, so $v = \gamma_0$. We define

$$sv = s\wedge v = s\wedge\gamma_0 = \boldsymbol{s}. \tag{3.22}$$

The equation for the relative spin vector becomes

$$\dot{\boldsymbol{s}} = \frac{q}{m}(F\cdot s)\wedge\gamma_0 = \frac{q}{m}[F\cdot(\boldsymbol{s}\gamma_0)]\wedge\gamma_0. \tag{3.23}$$

Now

$$F\cdot(\boldsymbol{s}\gamma_0) = \langle(\boldsymbol{E} + I\boldsymbol{B})\boldsymbol{s}\gamma_0\rangle_1 = \boldsymbol{E}\cdot\boldsymbol{s}\,\gamma_0 + (I\boldsymbol{B})\cdot\boldsymbol{s}\,\gamma_0, \tag{3.24}$$

so the equation for $\boldsymbol{s}$ is simply

$$\dot{\boldsymbol{s}} = \frac{q}{m}(I\boldsymbol{B})\cdot\boldsymbol{s} = \frac{q}{m}\boldsymbol{s}\times\boldsymbol{B}. \tag{3.25}$$

This is the precession equation for a particle with a gyromagnetic ratio of 2! So $g = 2$ is the natural value for a relativistic frame in an electromagnetic field. The rotor equation for a frame provides a better classical model for spin than a 'current loop', and is valid for arbitrary motion. In situations where the full (Dirac) quantum theory is not required, this model gives very accurate predictions for the behaviour of fermions in electromagnetic fields.