Physical Applications of Geometric Algebra

Handout 1

An Introduction to Geometric Algebra

The ideas and concepts of physics are best expressed in the language of mathematics. But this language is far from unique. Many different algebraic systems exist and are in use today, all with their own advantages and disadvantages. In this course we will study the most powerful available mathematical system developed to date — Clifford's *geometric algebra*. This will be presented as a new tool to add to your existing base as either a theoretician or experimentalist. The aim will be to introduce new techniques via their applications, rather than as purely formal mathematics. These applications will be diverse, emphasising the generality and portability of geometric algebra. This will help to promote a more inter-disciplinary understanding of science.

1 A Quick Tour

This course is divided into 3 sections, looking at the applications of geometric algebra (GA) to classical physics, relativistic physics and gravitation respectively. During this course we will

- Discover a new, powerful technique for handling rotations in arbitrary dimensions, and analyse the insights this brings to the mathematics of special and general relativity.
- Uncover the links between rotations, *bivectors* and the structure of the Lie groups which underpin much of modern physics.
- Learn how to extend the concept of a complex analytic function in 2-d (*i.e.* a function satisfying the Cauchy-Riemann equations) to arbitrary dimensions, and how this is applied in quantum theory and electromagnetism.
- Unite all four Maxwell equations into a single equation ($\nabla F = J$), and develop new techniques for solving it.
- Combine many of the preceding ideas to construct a gauge theory of gravitation in (flat) Minkowski spacetime, which is still consistent with General Relativity.

- Use our new understanding of gravitation to quickly reach advanced applications such as black holes and cosmic strings.

Throughout, the emphasis will be placed on the unity of the mathematics underpinning each of these topics.

2 Some History

A central problem being tackled in the first part of the 19th Century was how best to represent 3-d rotations. *Hamilton* pondered this for many years, and eventually produced the *quaternions*, which generalize complex numbers and phase rotations to 3-d. Despite their obvious uses, confusion persisted for many years over the role of the quaternions. The algebra contains 4 elements $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, but only three of these are interpreted as specifying a vector. This confusion was only resolved after Hamilton's death.

In a separate development, *Grassmann* pioneered the introduction of the *exterior product*. This defined what we now call a *bivector* $a \wedge b$ from two vectors. The crucial features of this product are *associativity*

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c \tag{2.1}$$

Figure 1: *William Rowan Hamilton 1805–1865.* Inventor of quaternions, and one of the key scientific figures of the 19th century.

Figure 2: *Hermann Gunther Grassmann (1809–1877).* German mathematician and schoolteacher, famous for inventing the algebra which now bears his name.

and *anticommutativity*

$$a \wedge b = -b \wedge a. \tag{2.2}$$

Grassmann was a German schoolteacher and was largely ignored during his lifetime. His work was certainly hindered by his impenetrable prose style, and often poor choice of notation. Since his death, however, his work has given rise to the influential and fashionable areas of *differential forms* and *Grassmann* (anticommuting) *variables*. The latter are fundamental to the foundation of much of modern supersymmetry and superstring theory.

The crucial step was made in 1878 by *Clifford*, who appears to have been one of the select group of mathematicians who had read and understood Grassmann's work. Clifford introduced his *geometric algebra* by uniting the dot product and exterior product into a single *geometric* product. This was associative, like Grassmann's product, but had the crucial extra feature of being *invertible*, like Hamilton's. Indeed, Clifford's original motivation was to unite Grassmann's and Hamilton's algebras into a single structure. In Clifford's geometric algebra an equation of the type $ab = C$ had the solution $b = a^{-1}C$. Neither the dot or exterior products are capable of this inversion on their own.

Clifford's system combined all of the advantages of quaternions with those of vector geometry, so geometric algebra should have then gone forward as the main system for mathematical physics. However, two events conspired against this. The first was Clifford's untimely death at the age of just 34 and at the height of his powers. The second was *Gibbs'* introduction of his vector calculus. This was well suited to the theory of electromagnetism as it stood at the end of the 19th century, and Gibbs' considerable

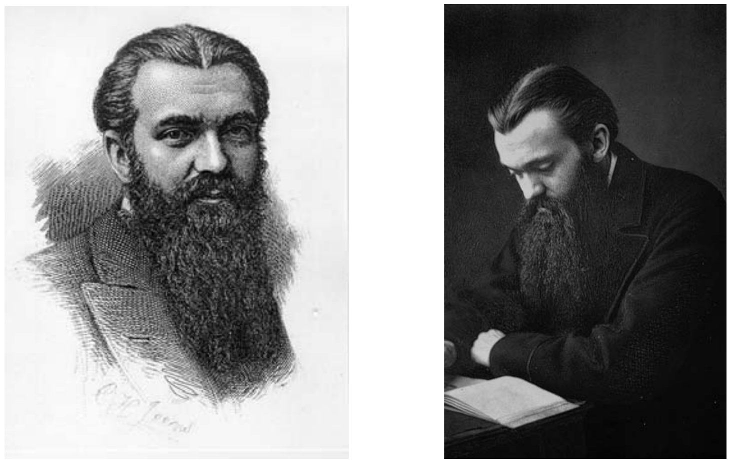


Figure 3: *William Kingdon Clifford 1845–1879.* Mathematician and philosopher. He died tragically young at the age of only 34, though he did have enough time to grow his prodigious beard.

reputation meant that his system eclipsed Clifford and Grassmann's work. By the time special relativity arrived, and physicists realised that they needed a system capable of handling 4-d space, the crucial insights of Grassmann and Clifford had been lost to a generation.

In the 1920's Clifford algebra resurfaced as the algebra underlying *quantum spin*. In particular the algebra of the *Pauli* and *Dirac* spin matrices became indispensable in quantum theory. However, these were treated just as algebras — the *geometrical* meaning was lost. For this reason we still employ the term 'Clifford algebras' when the algebra is used solely for its formal algebraic properties. When applied in its proper, geometric setting however, we prefer to use Clifford's own name of *geometric algebra*. This neatly avoids the minor historical point that Grassmann was actually the first to write down a geometric (Clifford) product!

The situation remained largely unchanged until the 1960's, when *David Hestenes* started to recover the geometrical meaning (in 3 and 4-d respectively) underlying the Pauli and Dirac algebras. His original motivation was to gain some insight into the nature of quantum mechanics, but he soon realised that, properly applied, Clifford's system was nothing less than a universal language for mathematics, physics and engineering! It has taken Hestenes many years to convince people of this fact, but interest is now gathering pace. Part of the original reluctance to accept geometric algebra was the prevailing view amongst physicists that there is something intrinsically 'quantum mechanical' in the algebra. This is quite wrong, as witnessed by the fact that Clifford predated quantum theory by 50 years, but it took a long before this was widely realised.

In Cambridge today, we routinely apply geometric algebra to topics as diverse as black

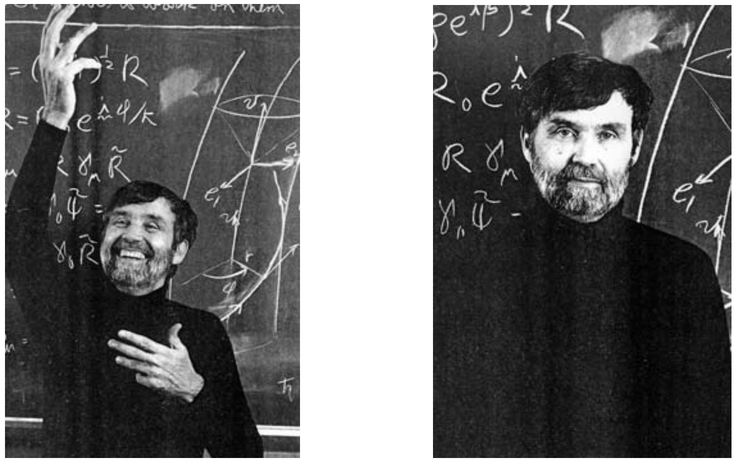


Figure 4: *David Orlin Hestenes.* Inventor of geometric calculus and first to draw attention to the universal nature of geometric algebra. He wrote the influential *Spacetime Algebra* in 1966, and followed this with a fully developed formalism in *Clifford Algebra to Geometric Calculus* (D. Hestenes & G. Sobczyk, 1984). This was followed by the (much easier!) *New Foundations for Classical Mechanics* in 1986.

holes and cosmology, quantum tunnelling and quantum field theory, beam dynamics and buckling, robotics and computer vision. Exactly the same algebraic system is used throughout, making it possible for the same people to understand and contribute to all of these different fields.

3 Multiplying Vectors

In your mathematical training so far, you will have met two products for vectors:

1. The Inner Product

The inner, or dot product, is usually written in the form $a{\cdot}b$. (Note that we do not use bold for vectors any more.) In Euclidean space the inner product is positive definite,

$$a^2 = a{\cdot}a > 0 \qquad \forall a \neq 0. \tag{3.1}$$

From this we recover the Schwarz inequality

$$\begin{aligned} (a+\lambda b)^2 &> 0 \qquad \forall \lambda \\ \implies a^2 + 2\lambda a{\cdot}b + \lambda^2 b^2 &> 0 \qquad \forall \lambda \\ \implies (a{\cdot}b)^2 &\leq a^2 b^2. \end{aligned} \tag{3.2}$$

We use this to define the cosine of the angle between a and b via

$$a \cdot b = |a||b| \cos(\theta). \tag{3.3}$$

In non-Euclidean spaces, such as Minkowski spacetime, we cannot do this. But we can still introduce an orthogonal frame and compute the dot product as $a_\mu b^\mu$ or $\eta_{\mu\nu} a^\mu b^\nu$, where $\eta_{\mu\nu}$ is the *metric tensor*.

2. The Cross Product

This only exists in 3-d space and is defined such that $a \times b$ is perpendicular to the plane defined by a and b, with magnitude $|a||b| \sin(\theta)$ and such that a, b and $a \times b$ form a right-handed set. This is sufficient to define the cross product uniquely. On introducing a right-handed orthonormal frame $\{e_i\}$ we can recover the usual definition in terms of components. We have

$$e_1 \times e_2 = e_3 \qquad \text{etc.} \tag{3.4}$$

Or, in more general index notation

$$e_i \times e_j = \epsilon_{ijk} e_k. \tag{3.5}$$

If we now expand the vectors in terms of components, $a = a_i e_i$ and $b = b_i e_i$, we find

$$\begin{aligned} a \times b &= (a_i e_i) \times (b_j e_j) \\ &= a_i b_j (e_i \times e_j) \\ &= (\epsilon_{ijk} a_i b_j) e_k. \end{aligned} \tag{3.6}$$

So the geometric definition recovers the algebraic one. One aim of GA is to extend this idea and avoid introducing frames as much as possible.

4 The Outer Product

The cross product has one major failing - it only exists in 3 dimensions. In 2-d there is nowhere else to go, whereas in 4-d the concept of a vector orthogonal to a pair of vectors is not unique. To see this, consider 4 orthonormal vectors $e_1 \ldots e_4$. If we take the pair e_1 and e_2 and attempt to find a vector perpendicular to both of these, we see that any combination of e_3 and e_4 will do.

What we need is a means of encoding a plane geometrically, without relying on the notion of a vector perpendicular to it. We define the *outer* or *exterior* product to be

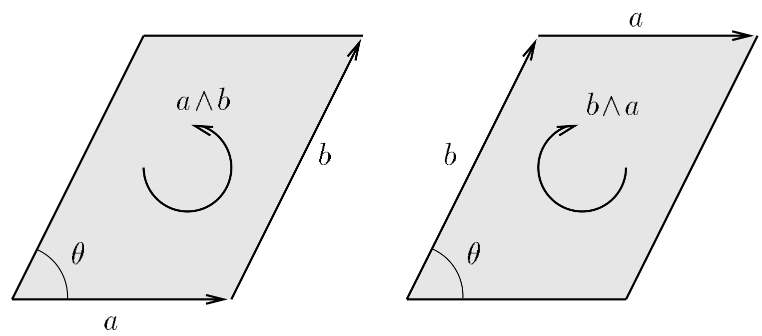


Figure 5: *The Outer Product.* The outer or wedge product of a and b returns a directed area element of area $|a||b|\sin(\theta)$.

the directed area swept out by a and b. This is denoted $a\wedge b$, or 'a wedge b'. The plane has area $|a||b|\sin(\theta)$, which is defined to be the magnitude of $a\wedge b$.

The result of the outer product is neither a scalar nor a vector. It is a new mathematical entity which encodes the notion of an oriented plane. We call this a *bivector*. It can be visualised as the parallelogram obtained by sweeping one vector along the other (Fig. 5). Changing the order of the vectors reverses the orientation of the plane.

Properties

1. The outer product of two vectors is antisymmetric,

$$a\wedge b = -b\wedge a. \tag{4.1}$$

This follows from the geometric definition.

2. Bivectors form a linear space, the same way that vectors do. In 3-d the addition of bivectors is easy to visualise (see Fig. 6). In higher dimensions this addition is not always so easy to visualise, because two planes need not share a common line. This can have some interesting consequences.

3. The outer product is distributive

$$a\wedge(b+c) = a\wedge b + a\wedge c. \tag{4.2}$$

This helps to visualise the addition of bivectors.

Note that if $a' = a + \lambda b$, we still have $a'\wedge b = a\wedge b$. There is no unique dependence on a and b. For this reason it is sometimes better to replace the directed parallelogram with a directed circle.

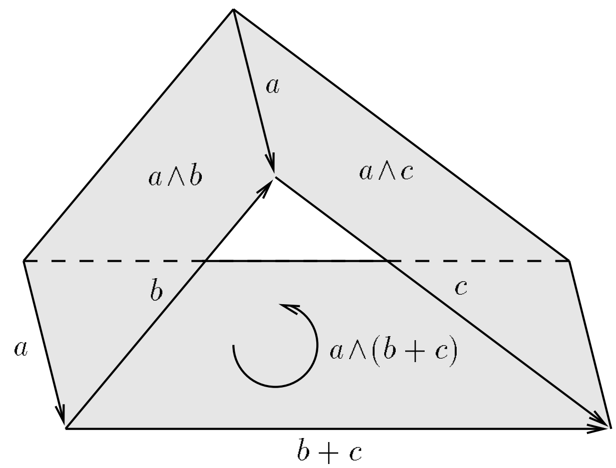


Figure 6: *Bivector Addition.* In 3-d bivector addition can be visualised like vector addition. The 'tail' of one bivector is added to the 'head' of the other.

In 3-d the space of bivectors is three dimensional. An arbitrary bivector can be decomposed in terms of an orthonormal frame of bivectors

$$\begin{aligned} a\wedge b &= (a_i e_i)\wedge(b_j e_j) \\ &= (a_2 b_3 - b_3 a_2) e_2\wedge e_3 + (a_3 b_1 - a_1 b_3) e_3\wedge e_1 \\ &\quad +(a_1 b_2 - a_2 b_1) e_1\wedge e_2. \end{aligned} \tag{4.3}$$

The components in this frame are therefore those of the cross product. In general, the components of $a\wedge b$ are $a_{[i}b_{j]}$ where the [] denotes antisymmetrisation.

5 The Geometric Product

So far we have a symmetric inner product and an antisymmetric outer product. Clifford's great idea was to introduce a new product which combines the two. This is the geometric product, written simply as ab, and satisfying

$$ab = a\cdot b + a\wedge b. \tag{5.1}$$

The right-hand side is a sum of two distinct objects - a scalar and a bivector. This looks strange, and goes against much of what you might already have been taught. The easiest way to think of the right-hand side is like a complex number, with real and imaginary parts. These are carried round in a single entity, which provides for many mathematical simplifications.

From the symmetry/antisymmetry of the terms on the right-hand side of Eq. (5.1), we see that

$$ba = b \cdot a + b \wedge a = a \cdot b - a \wedge b. \tag{5.2}$$

It follows that

$$a \cdot b = \tfrac{1}{2}(ab + ba) \qquad a \wedge b = \tfrac{1}{2}(ab - ba). \tag{5.3}$$

We can thus define the other products in terms of the geometric product. This forms the starting point for an axiomatic development of GA, which will be covered in Lecture 3. Here we summarise some of the main results.

Properties

1. General elements of a Geometric Algebra are called *multivectors* and these form a linear space - scalars can be added to bivectors, and vectors, *etc.* General multivectors are usually written in upper case, $(A, B \ldots)$.

2. The geometric product is associative

$$A(BC) = (AB)C = ABC. \tag{5.4}$$

3. The geometric product is distributive

$$A(B + C) = AB + AC. \tag{5.5}$$

(Note that nothing is assumed about the commutation properties of the geometric product. Matrix multiplication is a good picture to keep in mind.)

4. The square of any vector is a scalar.

The final property is sufficient to prove that the inner product of two vectors is a scalar. Consider the expansion

$$\begin{aligned} (a+b)^2 &= (a+b)(a+b) \\ &= a^2 + b^2 + ab + ba. \end{aligned} \tag{5.6}$$

It follows that

$$ab + ba = (a+b)^2 - a^2 - b^2 \tag{5.7}$$

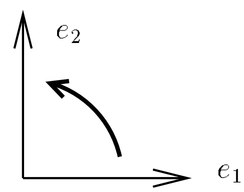
which is therefore a scalar.

6 Geometric Algebra in 2-d

The easiest way to understand the geometric product is by example, so consider a 2-d space (a plane) spanned by 2 orthonormal vectors e_1, e_2. These basis vectors satisfy

$$ {e_1}^2 = {e_2}^2 = 1, \qquad e_1 \cdot e_2 = 0. \tag{6.1} $$

The final entity present in the 2-d algebra is the bivector $e_1 \wedge e_2$. This is the highest grade element in the algebra, which is often called the *pseudoscalar*, though *directed volume element* is a more accurate description. The pseudoscalar is defined to be *right-handed*, so that e_1 sweeps onto e_2 in a right-handed sense.



The full algebra is spanned by

$$ \begin{array}{ccc} 1 & \{e_1, e_2\} & e_1 \wedge e_2 \\ \text{1 scalar} & \text{2 vectors} & \text{1 bivector.} \end{array} \tag{6.2} $$

We denote this algebra by $\mathcal{G}_2$. To study the properties of the bivector $e_1 \wedge e_2$ we first note that

$$ e_1 e_2 = e_1 \cdot e_2 + e_1 \wedge e_2 = e_1 \wedge e_2. \tag{6.3} $$

That is, for orthogonal vectors the geometric product is a pure bivector. Also note that

$$ e_2 e_1 = e_2 \wedge e_1 = -e_1 \wedge e_2 \tag{6.4} $$

from the antisymmetry of the exterior product. Another way of saying this is that in GA *orthogonal vectors anticommute.*

We can now form products when $e_1 e_2$ multiplies vectors from the left and the right. First from the left,

$$ \begin{array}{c} (e_1 \wedge e_2) e_1 = (-e_2 e_1) e_1 = -e_2 e_1 e_1 = -e_2 \\ (e_1 \wedge e_2) e_2 = (e_1 e_2) e_2 = e_1 e_2 e_2 = e_1. \end{array} \tag{6.5} $$

We see that left multiplication by the bivector rotates vectors 90^{o} clockwise (*i.e.* in a negative sense). Similarly, acting from the right

$$ e_1 (e_1 e_2) = e_2 \qquad e_2 (e_1 e_2) = -e_1. \tag{6.6} $$

So right multiplication rotates 90° anticlockwise — a positive sense.

The final product in the algebra to consider is the square of the bivector $e_1 \wedge e_2 = I$

$$I^2 = (e_1 \wedge e_2)^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -1. \tag{6.7}$$

From purely geometric considerations, we have discovered a quantity which squares to -1. This fits with the fact that 2 successive left (or right) multiplications of a vector by $e_1 e_2$ rotates the vector through 180°, which is equivalent to multiplying by -1.

6.1 Multiplying Multivectors

Suppose that we have two completely arbitrary elements of the $\mathcal{G}_2$ algebra, A and B. We can decompose these in terms of our $\{e_1, e_2\}$ frame as follows:

$$\begin{aligned} A &= a_0 + a_1 e_1 + a_2 e_2 + a_3 e_1 \wedge e_2 \\ B &= b_0 + b_1 e_1 + b_2 e_2 + b_3 e_1 \wedge e_2. \end{aligned} \tag{6.8}$$

The product of these two elements can be written

$$AB = p_0 + p_1 e_1 + p_2 e_2 + p_3 e_1 \wedge e_2. \tag{6.9}$$

We find that

$$\begin{aligned} p_0 &= a_0 b_0 + a_1 b_1 + a_2 b_2 - a_3 b_3 \\ p_1 &= a_0 b_1 + a_1 b_0 + a_3 b_2 - a_2 b_3 \\ p_2 &= a_0 b_2 + a_2 b_0 + a_1 b_3 - a_3 b_1 \\ p_3 &= a_0 b_3 + a_3 b_0 + a_1 b_2 - a_2 b_1. \end{aligned} \tag{6.10}$$

This multiplication law is easy to represent as part of a computer language (we often use Maple). The basis vectors can also be represented with matrices, though these can hide the geometry of the algebra. If we introduce the symbol $\langle AB \rangle$ to denote the scalar term in the product, we see that

$$p_0 = \langle AB \rangle = \langle BA \rangle. \tag{6.11}$$

In general, however, $AB \neq BA$

6.2 Complex Numbers and $\mathcal{G}_2$

It is clear that there is a close relationship between GA in 2-d, and the algebra of complex numbers. The unit bivector squares to -1 and generates rotations through

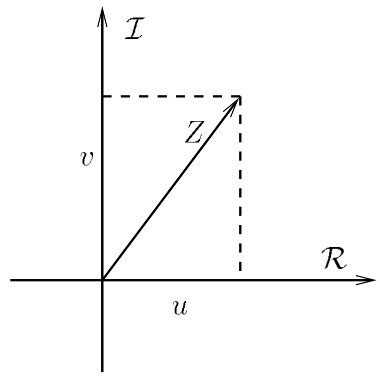


Figure 7: *The Argand Diagram.* Complex numbers can be used to represent vectors in 2-d, as well the operations of rotation and dilation applied to vectors.

90^{o}. The combination of a scalar and a bivector, which is formed naturally via the geometric product, can therefore be viewed as a complex number. We can write

$$Z = u + ve_1e_2 = u + Iv. \tag{6.12}$$

Complex numbers serve a dual purpose in 2-d. They generate rotations and dilations through their polar decomposition $r\exp(i\theta)$, and they also represent vectors as points on the argand diagram (Fig. 7). But in $\mathcal{G}_2$ our vectors are grade-1 objects.

$$x = ue_1 + ve_2. \tag{6.13}$$

Is there a natural map between this and the complex number Z? The answer is simple — pre-multiply by e_1,

$$e_1x = u + ve_1e_2 = u + Iv = Z. \tag{6.14}$$

That is all there is to it! The role of the preferred vector e_1 is clear — it is the real axis. Using this product vectors can be interchanged with complex numbers in 2-d in a natural manner.

The GA treatment shows us how complex numbers are able to play two roles, as rotations/dilations, and as position vectors. GA separates these roles, which is crucial to understanding how to generalise complex analysis to higher dimensions.

6.3 Rotations

Since we know how to rotate complex numbers, we can use this to find a formula for rotating vectors in 2-d. We know that a positive rotation through an angle ϕ for a

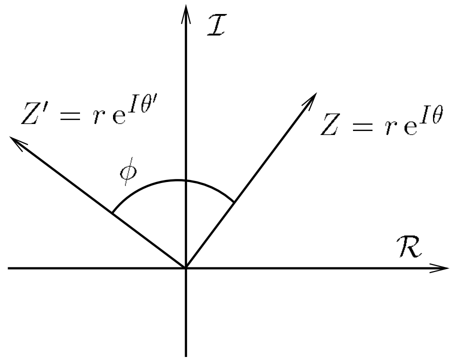


Figure 8: *A Rotation in the Complex Plane.* The overall effect is to replace θ by $\theta' = \theta + \phi$

complex number Z is achieved by

$$Z \mapsto \mathrm{e}^{I\phi}\, Z, \tag{6.15}$$

where we continue to use I for the imaginary (see Fig. 8). The exponential of a multivector is defined by power series in the normal way. We can now apply this to the vector transformation $x \mapsto x'$ as follows

$$\begin{gathered} x = e_1 Z \mapsto x' = e_1 Z' \\ x' = e_1 \mathrm{e}^{I\phi}\, Z = \mathrm{e}^{-I\phi}\, e_1 Z = \mathrm{e}^{-I\phi}\, x. \end{gathered} \tag{6.16}$$

We therefore arrive at the formulae

$$x' = \mathrm{e}^{-I\phi}\, x = x\,\mathrm{e}^{I\phi} = \mathrm{e}^{-I\phi/2}\, x\,\mathrm{e}^{I\phi/2} \tag{6.17}$$

which are all equivalent. The final form will turn out to be the most general. Note the importance of the fact that I *anticommutes* with vectors. We do not get behaviour like this with complex numbers alone.

6.4 Application — Kepler Orbits

As an application of the preceding, we will discuss an alternative formulation for 2-d motion. We start by writing the position vector x in terms of a complex number U by

$$x = U e_1 \tilde{U} = U^2 e_1, \qquad |x| = r = U\tilde{U} \tag{6.18}$$

where we have introduced the tilde symbol $\tilde{U}$ to for complex conjugation. We find

$$\begin{gathered} \dot{x} = 2\dot{U} U e_1 \\ \implies 2r\dot{U} = \dot{x} e_1 \tilde{U} = \dot{x} U e_1. \end{gathered} \tag{6.19}$$

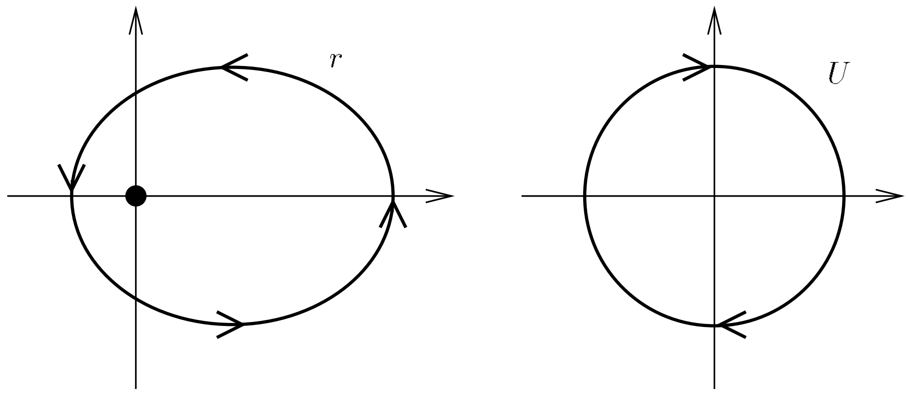


Figure 9: *Solution to the Kepler problem.* The particle completes 2 orbits every time U completes one cycle.

We now introduce the new variable s defined by

$$\frac{d}{ds} = r\frac{d}{dt}, \qquad \frac{dt}{ds} = r. \tag{6.20}$$

In terms of this

$$2\frac{dU}{ds} = \dot{x}Ue_1 \tag{6.21}$$

and

$$2\frac{d^2U}{ds^2} = r\ddot{x}Ue_1 + \dot{x}\frac{dU}{ds}e_1 = U(\ddot{x}x + \tfrac{1}{2}\dot{x}^2). \tag{6.22}$$

Now suppose we have motion in a central inverse square force:

$$m\ddot{x} = -\mu\frac{x}{r^3}. \tag{6.23}$$

The equation for U becomes

$$\frac{d^2U}{ds^2} = \frac{1}{2m}U\left(\tfrac{1}{2}m\dot{x}^2 - \frac{\mu}{r}\right) = \frac{E}{2m}U. \tag{6.24}$$

We recover the equation of simple harmonic motion! This has a number of advantages. The equation is easier to solve; it is linear, so much better for perturbation theory; there is no singularity at $r = 0$, so get better numerical stability; the equation is universal — it holds for $E > 0$ and $E < 0$; and the method extends easily to 3-d. This method is now frequently employed for computing complicated satallite motions. The motion is illustrated in Fig 9. The particle follows an ellipse, whereas U follows a circle centered on the origin. The particle completes 2 orbits for each full cycle of U.