Physical Applications of Geometric Algebra

Handout 5

Linear Algebra

Geometric algebra simplifies and improves our understanding of the important subject of *linear algebra*. This might appear a bit dry and formal at first, but it is is important for a number of the advanced applications treated throughout the course. In particular, it is central to the development of the gauge theory of gravitation. The geometric algebra treatment is index free, making it very tidy and compact. This enormously simplifies derivations of otherwise difficult results. This is, in fact, quite an easy topic, but it is one that traditionally causes some difficulties because it is rarely presented in a simple, coherent manner. The main application studied here is to Hamiltonian mechanics and the theory of canonical transformations.

1 Linear Functions

Geometric algebra allows for index-free manipulations, in contrast to tensors which deal with objects like a_i or T_{ij}. The index-free approach is closer to how humans understand geometric objects, whereas matrices and tensors are more closely related to the sort of manipulations performed by computers. There is a small cost in moving to an index-free notation, however. One has to construct an unambiguous notation for linear functions which maintains a clear distinction between functions and vectors. In tensor notation this is easy - we just add an extra index to the our symbols for vectors. This possibility is not available to us, however, so instead we will denote linear functions mapping vectors to vectors with expressions of the form $\mathsf{f}(a)$. Here a is the vector argument, and f is the linear function.

A linear function is defined to have the property

$$\mathsf{f}(\lambda a + \mu b) = \lambda \mathsf{f}(a) + \mu \mathsf{f}(b) \tag{1.1}$$

for all scalars λ, μ and vectors a and b. The composition of two linear functions f and g (applied first) results in a third (just like matrix multiplication). We can write

$$\mathsf{h}(a) = \mathsf{f}[\mathsf{g}(a)] = \mathsf{fg}(a) \tag{1.2}$$

The final form is consistent with our general notational principle of suppressing unnecessary brackets. The expression is unambiguous because combining linear functions is an associative operation.

1.1 The Extension to Multivectors

Linear functions are extended to define an action on arbitrary multivectors through their action on blades. We define

$$\mathsf{f}(a\wedge b\wedge\cdots\wedge c) = \mathsf{f}(a)\wedge\mathsf{f}(b)\wedge\cdots\wedge\mathsf{f}(c). \tag{1.3}$$

The right-hand side is also a blade and so must have the same grade as the original argument (unless it is zero). Extended linear functions are therefore *grade preserving*,

$$\mathsf{f}(A_r) = \langle\mathsf{f}(A_r)\rangle_r. \tag{1.4}$$

They are also *multilinear*

$$\mathsf{f}(\lambda A + \mu B) = \lambda\mathsf{f}(A) + \mu\mathsf{f}(B), \tag{1.5}$$

which holds for any scalars λ and μ and any multivectors A and B.

A good example of this extended action is provided by rotations. Given a rotation determined by a rotor R we can write

$$\mathsf{R}(a) = Ra\tilde{R}. \tag{1.6}$$

We saw in Handout 4 that the action of this rotation on a multivector, when all its component vectors are rotated, is given by the same law. The extended action is therefore simply

$$\mathsf{R}(A) = RA\tilde{R}. \tag{1.7}$$

1.2 Extended Compound Functions

A key result for linear functions concerns the extension to the entire algebra of the product of two linear functions. Suppose that $\mathsf{h}(a) = \mathsf{f}\,\mathsf{g}(a)$. We see that

$$\begin{aligned}\mathsf{h}(a\wedge b\wedge\cdots\wedge c) &= \mathsf{fg}(a)\wedge\mathsf{fg}(b)\wedge\cdots\wedge\mathsf{fg}(c)\\ &= \mathsf{f}[\mathsf{g}(a)\wedge\mathsf{g}(b)\wedge\cdots\wedge\mathsf{g}(c)]\\ &= \mathsf{f}[\mathsf{g}(a\wedge b\wedge\cdots\wedge c)].\end{aligned} \tag{1.8}$$

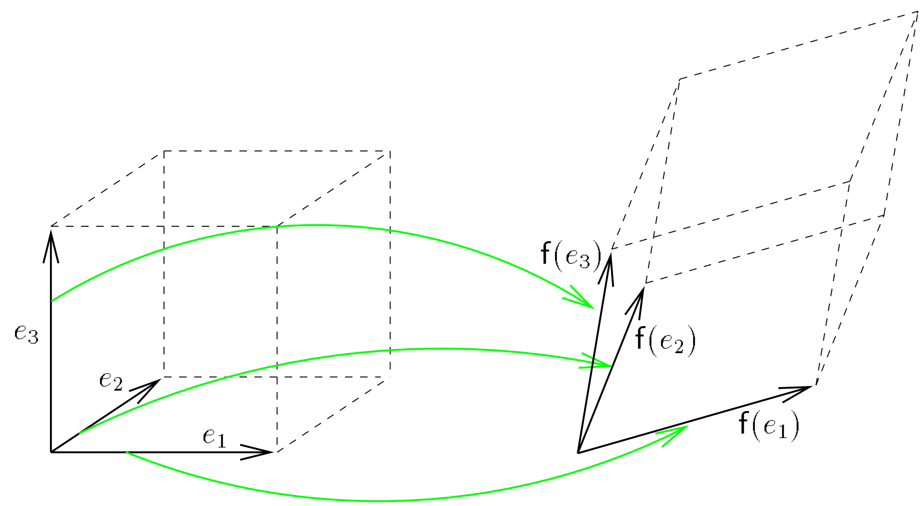

Figure 1: *The Determinant.* The unit cube is transformed to a parallelepiped with sides $\mathsf{f}(e_1)$, $\mathsf{f}(e_2)$ and $\mathsf{f}(e_3)$. The determinant is the volume scale factor, so is given by the volume of the parallelepiped, $\mathsf{f}(e_1)\wedge\mathsf{f}(e_2)\wedge\mathsf{f}(e_3) = \mathsf{f}(I)$.

This simple manipulation shows that the extended action of the product of two linear functions on a multivector is also obtained by letting the first function act on the multivector, and following this by the action of the second. In dealing with combinations of linear functions we can therefore still write

$$\mathsf{h}(A) = \mathsf{fg}(A), \tag{1.9}$$

and the meaning of the right-hand side remains unambiguous.

1.3 The Determinant

Now that we know how to extend linear functions into a grade-preserving action over the entire multivector algebra, we can proceed immediately to a definition of the determinant. The highest-grade multivector I (the pseudoscalar or directed volume element) for any space is unique up to scaling. We therefore define

$$\mathsf{f}(I) = \det(\mathsf{f})\, I. \tag{1.10}$$

This fully embodies the notion of the determinant as a *volume scale factor*. The definition tells us how n-dimensional volume elements transform as each of their basis elements are transformed. This definition agrees with the tensor one, but is rather more intuitive (see Fig. 1).

One very useful feature of our definition is the ease with which we can now one of the key results for determinants. Consider the product function $\mathsf{h} = \mathsf{f}\,\mathsf{g}$. For this

$$\mathsf{f}(\mathsf{g}(I)) = \mathsf{f}(\det(\mathsf{g})I) = \det(\mathsf{g})\mathsf{f}(I) = \det(\mathsf{f})\det(\mathsf{g})I \tag{1.11}$$

where we have just used the results for multilinearity and the extension of the product of two functions. We have therefore now proved

$$\det(\mathsf{fg}) = \det(\mathsf{f})\det(\mathsf{g}) \tag{1.12}$$

That is, the determinant of the product of two linear functions is the product of their determinants. You are unlikely to find a quicker proof of this fundamental result using any other technique!

2 Non-Orthonormal Frames

This is a useful topic which is strangely neglected in undergraduate teaching. Suppose that we have a set of n linearly independent vectors $\{e_k\}$. No other restrictions are enforced on these vectors; in particular they are *not orthonormal*. Any vector a can be expressed uniquely in terms of this frame,

$$a = a^k e_k \tag{2.1}$$

where we use superscripts to denote the component (a^k) in the $\{e_k\}$ frame. But how do we find these components? We need a second set of vectors $\{e^k\}$ related to our initial set by

$$e^i \cdot e_j = \delta^i_j. \tag{2.2}$$

The set $\{e^k\}$ is called the *reciprocal frame*. The upper and lower indices provide a useful device to record the relation between the two frames. Equipped with the reciprocal vectors, we can immediately find the components of a by forming

$$e^k \cdot a = e^k \cdot (a^j e_j) = e^k \cdot e_j \, a^j = a^j \delta^k_j = a^k. \tag{2.3}$$

2.1 Constructing the Reciprocal Frame

To construct the reciprocal frame, we note that e^1 must be orthogonal to each of $\{e_2 \cdots e_n\}$. It therefore lies entirely outside the hyperplane $e_2 \wedge e_3 \wedge \cdots e_n$. The vector perpendicular to this hyperplane is found by dualisation — multiplication by the pseudoscalar. We can therefore write

$$e^1 = \alpha\, e_2 \wedge e_3 \wedge \cdots \wedge e_n \, I, \tag{2.4}$$

where α is some constant. We fix this constant by dotting with e_1

$$\begin{aligned} 1 = e_1\cdot e^1 &= \alpha\, e_1\cdot(e_2\wedge e_3\wedge\cdots\wedge e_n\, I) \\ &= \alpha\, e_1\wedge e_2\wedge\cdots\wedge e_n\, I \end{aligned} \tag{2.5}$$

If we now define the volume element

$$E_n = e_1\wedge e_2\wedge\cdots\wedge e_n \tag{2.6}$$

we see that $\alpha E_n I = 1$, so that $\alpha = I^{-1}E_n^{-1}$. We therefore have

$$e^1 = e_2\wedge e_3\wedge\cdots\wedge e_n\, E_n^{-1}. \tag{2.7}$$

This extends to the useful formula

$$e^k = (-1)^{k+1} e_1\wedge\ldots\wedge\check{e}_k\wedge\cdots\wedge e_n E_n^{-1} \tag{2.8}$$

where again the check denotes that the e_k term is missing from the product. Our purely geometric reasoning has led quickly to an algebraic formula which can be directly applied. In 3-d this immediately recovers the formula for finding the reciprocal lattice vectors which are important in crystallography.

2.2 Some Useful Results

From the basic identity

$$a = a^k e_k = a\cdot e^k\, e_k = a\cdot e_k\, e^k \tag{2.9}$$

we can build up a series of useful results. First consider

$$\begin{aligned} e_k\, e^k\cdot(a\wedge b) &= e_k(e^k\cdot a\, b - e^k\cdot b\, a) \\ &= ab - ba = 2a\wedge b. \end{aligned} \tag{2.10}$$

This result extends inductively to yield

$$e_k\, e^k\cdot A_r = rA_r \tag{2.11}$$

for an r-grade multivector A_r. Next we note that

$$e^k e_k = e^k(e_j\cdot e_k\, e^j) = e_j\cdot e_k e^k e^j. \tag{2.12}$$

But $e_j\cdot e_k$ is symmetric on j,k, so we can only pick up the symmetric component of $e^k e^j$. This is also a scalar., so we only get a scalar contribution to the sum,

$$e_k e^k = e_k\cdot e^k = n \tag{2.13}$$

where n is the dimension of the space. It follows that

$$e_k\, e^k\wedge A_r = e_k(e^k A_r - a^k\cdot A_r) = (n-r)A_r. \tag{2.14}$$

Finally, we combine the above to give

$$e_k A_r e^k = (-1)^r e_k(e^k\wedge A_r - e^k\cdot A_r) = (-1)^r(n-2r)A_r. \tag{2.15}$$

2.3 Application — Recovering a Rotor

As an application of the preceding results, suppose that we have two sets of vectors in 3-d (not necessarily orthonormal) $\{e_k\}$ and $\{f_k\}$ which we know are related by a rotation. We know that

$$f_k = R e_k \tilde{R} \tag{2.16}$$

and we seek a simple expression for the rotor R. As we are in 3-d, we can write

$$R = \mathrm{e}^{-B/2} \quad \text{and} \quad \tilde{R} = \mathrm{e}^{B/2} = \cos(|B|/2) + \sin(|B|/2)B/|B|. \tag{2.17}$$

We therefore find that

$$\begin{aligned} e_k \tilde{R} e^k &= e_k[\cos(|B|/2) + \sin(|B|/2)B/|B|]e^k \\ &= 3\cos(|B|/2) - \sin(|B|/2)B/|B|] \\ &= 4\cos(|B|/2) - \tilde{R}. \end{aligned} \tag{2.18}$$

We now form

$$f_k e^k = R e_k \tilde{R} e^k = 4\cos(|B|/2)R - 1. \tag{2.19}$$

It follows that R is a scalar multiple of $1 + f_k e^k$. We therefore establish the simple formula

$$R = \frac{1 + f_k e^k}{|1 + f_k e^k|} = \frac{\psi}{\sqrt{(\psi\tilde{\psi})}} \tag{2.20}$$

where $\psi = 1 + f_k e^k$. This neat formula recovers the rotor directly from the frame vectors.

3 Adjoints and Inverses

Two important concepts in linear algebra are those of the adjoint and the inverse of a linear function. These arise naturally in the geometric algebra setting.

3.1 The adjoint

The adjoint, or transpose, reverses the action of a linear operator. This is particularly clear if a linear function is viewed as a map between separate spaces. If f is a map from space 1 to space 2, then the adjoint is a map in the other direction. In geometric algebra we denote the adjoint of f with a bar, $\bar{\mathsf{f}}$. It is defined by the condition that

$$a\cdot\mathsf{f}(b) = \bar{\mathsf{f}}(a)\cdot b, \qquad \forall a, b. \tag{3.1}$$

We can derive an explicit formula by decomposing $\bar{\mathsf{f}}(a)$ in an arbitrary frame

$$\bar{\mathsf{f}}(a) = \bar{\mathsf{f}}(a)\cdot e_k\, e^k = a\cdot\mathsf{f}(e_k)\, e^k \tag{3.2}$$

We next construct the extension of the adjoint

$$\begin{aligned} \bar{\mathsf{f}}(a\wedge b) &= (a\cdot\mathsf{f}(e^i)\, e_i)\wedge(b\cdot\mathsf{f}(e^j)\, e_j) \\ &= e_i\wedge e_j\, a\cdot\mathsf{f}(e^i)\, b\cdot\mathsf{f}(e^j) \\ &= \tfrac{1}{2}e_i\wedge e_j\, [a\cdot\mathsf{f}(e^i)\, b\cdot\mathsf{f}(e^j) - a\cdot\mathsf{f}(e^j)\, b\cdot\mathsf{f}(e^i)] \\ &= \tfrac{1}{2}e_i\wedge e_j\, (a\wedge b)\cdot\mathsf{f}(e^j\wedge e^i). \end{aligned} \tag{3.3}$$

This shows that the extension of the adjoint is equal to the adjoint of the extended function. We can therefore write

$$A_r\cdot\bar{\mathsf{f}}(B_r) = \mathsf{f}(A_r)\cdot B_r. \tag{3.4}$$

This formula extends to encompass the situation where the two multivectors have different grades. To see this, consider the following decomposition

$$\begin{aligned} a\cdot\mathsf{f}(b\wedge c) &= a\cdot\mathsf{f}(b)\,\mathsf{f}(c) - a\cdot\mathsf{f}(c)\,\mathsf{f}(b) \\ &= \mathsf{f}[\bar{\mathsf{f}}(a)\cdot b\, c - \bar{\mathsf{f}}(a)\cdot c\, b] \\ &= \mathsf{f}[\bar{\mathsf{f}}(a)\cdot(b\wedge c)] \end{aligned} \tag{3.5}$$

With a simple extension of this argument, we arrive at the following formulae

$$\mathsf{f}(A_r)\cdot B_s = \mathsf{f}[A_r\cdot\bar{\mathsf{f}}(B_s)] \qquad r \geq s \tag{3.6}$$

$$A_r\cdot\bar{\mathsf{f}}(B_s) = \bar{\mathsf{f}}[\mathsf{f}(A_r)\cdot B_s] \qquad r \leq s \tag{3.7}$$

These are remarkably useful in practice!

3.2 The Inverse

The preceding formulae enable us to quickly derive a formula for the inverse. If we set $B_s = I$ in the Eq. (3.7), we arrive at

$$A_r\,\det(\mathsf{f}) I = \bar{\mathsf{f}}[\mathsf{f}(A_r) I] \tag{3.8}$$

which we can write as

$$A_r = \bar{\mathsf{f}}[\mathsf{f}(A_r)I]I^{-1}\det(\mathsf{f})^{-1}. \tag{3.9}$$

Here the various terms acting on $\mathsf{f}(A_r)$ return A_r, so they must represent the inverse function. We therefore have

$$\begin{aligned} \mathsf{f}^{-1}(A) &= \det(\mathsf{f})^{-1} I\bar{\mathsf{f}}(I^{-1}A) & (3.10) \\ \bar{\mathsf{f}}^{-1}(A) &= \det(\mathsf{f})^{-1} I\mathsf{f}(I^{-1}A), & (3.11) \end{aligned}$$

which holds for *any* multivector A, in spaces of *any* signature. Again, these are very useful formulae in practice. This is especially true in relativity, which contains a few surprises! The expressions can also be easily coded up on a computer in a symbolic algebra package (such as Maple).

As an example of the inverse formula, consider the rotation $\mathsf{R}(a) = Ra\tilde{R}$. The adjoint is found from

$$\bar{\mathsf{R}}(a) = e_k\, a\cdot\mathsf{R}(e^k) = e_k\langle aRe^k\tilde{R}\rangle = e_k e^k\cdot(\tilde{R}aR) = \tilde{R}aR, \tag{3.12}$$

which extends simply to arbitrary multivectors,

$$\bar{\mathsf{R}}(A) = \tilde{R}AR. \tag{3.13}$$

Since $\det(\mathsf{R}) = 1$, the inverse is given by

$$\mathsf{R}^{-1}(A) = \det(\mathsf{R})^{-1} I\tilde{R}I^{-1}AR = \tilde{R}aR. \tag{3.14}$$

So, as expected, we see that the inverse is equal to the adjoint for rotations. This is true for any orthonormal transformation.

4 Canonical Transformations

In Handout 4 we established that Hamilton's equations could be expressed geometrically in $2n$-dimensional phase space as

$$\dot{x} = \nabla H\cdot J. \tag{4.1}$$

Here x is the position vector in phase space, the Hamiltonian H is a scalar function, and ∇ is the gradient operator (the vector derivative). We are interested in transformations of these equations which leave the Hamiltonian intact. There are two types of transformation one can consider. The first is achieved by changing the coordinate system in which the vector is expressed. This can give rise to very different looking sets of scalar equations. But in one sense these transformations are rather uninteresting.

Any genuinely geometric equation can be decomposed in different coordinate systems to get a different set of scalar equations. But the underlying geometric equation does not change. The transformation does not tell us anything about symmetries in the underlying system. These are called *passive* transformations. Their main significance is that the freedom to choose an arbitrary coordinate system is something we can often exploit to our advantage to simplify the analysis of the equations.

Passive transformations do not capture the full symmetry of Hamilton's equations. To see what is missing, we must return to the point where we first introduced the vector x in the form

$$x = p_i e_i + q_i f_i. \tag{4.2}$$

(Handout 4, Eq. 5.5). Suppose instead that we had decided to work with a different set of coordinates and canonical momenta, P_i, Q_i. With these we would form a different vector

$$x' = P_i e_i + Q_i f_i. \tag{4.3}$$

The $\{P_i, Q_i\}$ are functions of the original $\{p_i, q_i\}$, so we can view the new vector x' as a function of the old vector x. We write this as

$$x' = f(x). \tag{4.4}$$

This is now an *active* transformations. The points x are moved to some new position in phase space x' by the *displacement* $f(x)$. This transformation can be non-linear, which is why it is written $f(x)$ and not $\mathsf{f}(x)$. The only restriction that is placed on $f(x)$ is that it is an invertible map. We will also simplify the treatment slightly by assuming that $f(x)$ is time-independent.

After applying the transformation to $f(x)$, the time derivative of the transformed variable is

$$\frac{d}{dt}x' = \frac{d}{dt}f(x) = \frac{dx}{dt}\cdot\nabla f(x) = \dot{x}\cdot\nabla f(x). \tag{4.5}$$

This is a compact way of writing the chain rule. It is easily verified by expanding out x and ∇ in an arbitrary $2n$-dimensional coordinate frame $\{e_k\}$. We write

$$x = x^k e_k \qquad \text{where} \qquad x^k = e^k\cdot x, \tag{4.6}$$

and we also have

$$\nabla = e^k \frac{\partial}{\partial x^k} = e^k\, e_k\cdot\nabla. \tag{4.7}$$

It follows that

$$\frac{d}{dt} = \frac{\partial x^k}{\partial t}\frac{\partial}{\partial x^k} = \frac{\partial x}{\partial t}\cdot e^k\, e_k\cdot\nabla = \dot{x}\cdot\nabla. \tag{4.8}$$

Eq. (4.5) naturally forces us to look at derivatives of $f(x)$. We accordingly define the *differential* of $f(x)$ by

$$\mathsf{f}(a) = a \cdot \nabla f(x). \tag{4.9}$$

This is a linear function of a. There may also be some position dependence in f, which we can make explicit by writing

$$\mathsf{f}(a) = \mathsf{f}(a\,;x). \tag{4.10}$$

This separates the linear argument a from the potentially arbitrary (nonlinear) dependence on position x. Where possible, we will suppress the x-dependence. In terms of the differential, we now have

$$\frac{d}{dt}x' = \dot{x} \cdot \nabla f(x) = \mathsf{f}(\dot{x}). \tag{4.11}$$

Next we need to establish the relationship between the gradient with respect to x and x'. With x' decomposed in the $\{e_k\}$ frame as

$$x' = x^{k'} e_k, \qquad x^{k'} = e^k \cdot x' \tag{4.12}$$

we have

$$\nabla' = e^k \frac{\partial}{\partial x^{k'}} = e^k e_k \cdot \nabla'. \tag{4.13}$$

We now find that

$$\begin{aligned} \nabla = e^k\, e_k \cdot \nabla &= e^k\, e_k \cdot \nabla x^{j'} \frac{\partial}{\partial x^{j'}} \\ &= e^k\, (e_k \cdot \nabla x') \cdot \nabla' = e^k\, \mathsf{f}(e_k) \cdot \nabla' = \bar{\mathsf{f}}(\nabla'). \end{aligned} \tag{4.14}$$

Again, we can appreciate how compact the final formula $\nabla = \bar{\mathsf{f}}(\nabla')$ is once the fully index-free approach is adopted.

The equation of motion satisfied by x' is

$$\frac{d}{dt}x' = \mathsf{f}(\dot{x}) = \mathsf{f}[\nabla H \cdot J] = \mathsf{f}[\bar{\mathsf{f}}\bar{\mathsf{f}}^{-1}(\nabla H) \cdot J] = \bar{\mathsf{f}}^{-1}(\nabla H) \cdot \mathsf{f}(J). \tag{4.15}$$

But the transformed Hamiltonian is defined by $H'(x') = H(x)$, so

$$\bar{\mathsf{f}}^{-1}(\nabla H) = \nabla' H(x) = \nabla' H'. \tag{4.16}$$

The equations of motion for a trajectory in phase space after the transformation are therefore

$$\frac{d}{dt}x' = (\nabla' H') \cdot \mathsf{f}(J) \tag{4.17}$$

and these will remain Hamiltonian in form if

$$\mathsf{f}(J) = J. \tag{4.18}$$

This is the definition of a *canonical transformation*. In linear algebra terms, a linear function satisfying the condition $\mathsf{f}(J) = J$ is called a *symplectic transformation*. These form a group — the symplectic group. A canonical transformation is now seen geometrically as a displacement whose differential is a symplectic transformation. Some examples should provide a more concrete understanding of how these abstract ideas are applied in practice.

4.1 Examples

Unitary Transformations

The simplest canonical transformation to consider is one where the differential $\mathsf{f}(a)$ is constant. In this case the underlying displacement is easily found by integration,

$$f(x) = \mathsf{f}(x) + n, \tag{4.19}$$

where a possible constant translation is also included. We have already encountered a set of transformations which leave J invariant. They are the unitary transformations, which are now seen as the subgroup of the symplectic group which also preserves the inner product. A look at the bivector generators of Handout 4, Eq. 4.18 shows that the F_{ij} bivectors couple together the position and momentum components of x. The existence of canonical transformations forces us to view phase space as a single geometric entity, with no natural split between position and momentum.

The 1-d Harmonic Oscillator

As a simple illustration, consider the harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \tfrac{1}{2}kq^2. \tag{4.20}$$

We can always apply a constant rescaling to our coordinates of the form $P = \alpha p$, $Q = q/\alpha$ (exercise). This is often employed to fix up the units correctly, so that the vector x is constructed form terms of the same dimension. For our present case we see that $\alpha^2 = (mk)^{-1/2}$ is the appropriate choice, leaving

$$H = \tfrac{1}{2}\omega(P^2 + Q^2), \tag{4.21}$$

where $\omega^2 = k/m$. We now introduce the vector $x = Pe_1 + Qf_1$ so that

$$H = \tfrac{1}{2}\omega x^2 \tag{4.22}$$

and

$$\dot{x} = \omega\, x \cdot J. \tag{4.23}$$

In this simple case J is the pseudoscalar. The motion keeps x^2 constant, so describes circles in phase space. The unitary group invariance of this system is simply constant rotations in phase space. This is always a symmetry in 2-d phase space, so the transformation

$$P = p\cos\alpha + q\sin\alpha, \qquad Q = q\cos\alpha - p\sin\alpha \tag{4.24}$$

is always canonical.

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Show that the system with Hamiltonian

$$H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right) \tag{4.25}$$

can be reduced to the form of a Harmonic oscillator.

With $x = pe_1 + qf_1$ we need to establish that the map

$$x' = f(x) = q^2 p\, e_1 - \frac{1}{q} f_1 \tag{4.26}$$

is canonical, as this transformation brings the Hamiltonian to SHO form. We form

$$\begin{aligned} \mathsf{f}(e_1) &= \frac{\partial x'}{\partial p} = q^2 e_1 && (4.27) \\ \mathsf{f}(f_1) &= \frac{\partial x'}{\partial q} = 2pqe_1 + \frac{1}{q^2} f_1. && (4.28) \end{aligned}$$

We can now see that

$$\mathsf{f}(J) = \mathsf{f}(e_1)\wedge\mathsf{f}(f_1) = (q^2 e_1)\wedge(2pqe_1 + \frac{1}{q^2} f_1) = e_1\wedge f_1, \tag{4.29}$$

so the map is indeed canonical.