Physical Applications of Geometric Algebra

Examples 2 — Answers

1.

$$
\begin{aligned}
B\times(MN) &= \tfrac{1}{2}(BMN - MNB) \\
&= \tfrac{1}{2}(BMN - MBN + MBN - MNB) \\
&= B\times M\, N + MB\times N
\end{aligned}
$$

Hence

$$
\begin{aligned}
B\times(a\wedge A_r) &= \tfrac{1}{2}B\times[aA_r + (-1)^r A_r a] \\
&= \tfrac{1}{2}[B\cdot a\, A_r + a\, B\times A_r + (-1)^r B\times A_r\, a + (-1)^r A_r\, B\cdot a] \\
&= (B\cdot a)\wedge A_r + a\wedge(B\times A_r),
\end{aligned}
$$

since $B\times A_r$ could only contain terms of grade $r-2$, r and $r+2$. This proves that if $B\times A_r$ has grade r, then $B\times A_{r+1}$ has grade $r+1$. But for $r=1$ know that $B\times a_1 = B\cdot a_1$ is a vector, grade 1. Hence result true for all grades.

2. Expanding to second order in ϵ, get

$$
\begin{aligned}
R_3 &= 1 - \frac{\epsilon}{2}B_3 + \frac{\epsilon^2}{8}{B_3}^2 \\
&= 1 - \frac{\epsilon}{2}[(B_1 + B_2 - \frac{\epsilon}{2}B_2\times B_1) + \frac{\epsilon^2}{8}({B_1}^2 + B_1B_2 + B_2B_1 + {B_2}^2) \\
&= 1 - \frac{\epsilon}{2}(B_1 + B_2) + \frac{\epsilon^2}{8}({B_1}^2 + 2B_2B_1 + {B_2}^2) \\
&= \left(1 - \frac{\epsilon}{2}B_2 + \frac{\epsilon^2}{8}{B_2}^2\right)\left(1 - \frac{\epsilon}{2}B_1 + \frac{\epsilon^2}{8}{B_1}^2\right) = R_2R_1.
\end{aligned}
$$

3. Any associative algebra satisfies the Jacobi identity,

$$
\begin{aligned}
&(A\times B)\times C + (C\times A)\times B + (B\times C)\times A \\
&\quad = \tfrac{1}{4}[(AB - BA)C - C(AB - BA) + (CA - AC)B - B(CA - AC) \\
&\qquad + (BC - CB)A - A(BC - CB)] = 0.
\end{aligned}
$$

4. Have $B_j\times B_k = C^i_{jk}B_i$. For 3-d rotations take $B_i = Ie_i$. Have

$$
B_j\times B_k = -e_j\wedge e_k = -I\epsilon_{ijk}e_i = -\epsilon_{ijk}B_i
$$

Hence $C^i_{jk} = -\epsilon_{ijk}$.

5. Easiest to form geometric products and pick off bivector parts. To get a non-zero commutator between blades, they must share a common vector. Form $(i, j \neq 1)$

$$
E_{1i}\times E_{1j} = \langle E_{1i}E_{1j}\rangle_2 = \langle(e_1e_i + f_1f_i)(e_1e_j + f_1f_j)\rangle_2 = -\langle e_ie_j + f_if_j\rangle_2 = -E_{ij}
$$

Also find that

$$E_{1i} \times F_{ij} = -F_{ij}, \qquad F_{1i} \times F_{1j} = -E_{ij}$$

and

$$E_{1i} \times J_1 = -F_{1i}, \quad F_{1i} \times J_1 = E_{1i} \quad J_i \times J_j = 0.$$

6. (Use q_i instead of x_i). Have

$$x = p_i e_i + q_i f_i, \quad x' = \frac{1}{\alpha} p_i e_i + \alpha q_i f_i.$$

Now $a \cdot \nabla p_i = a \cdot (\nabla p_i) = a \cdot e_i$, with similar for f_i, so

$$\mathsf{f}(a) = a \cdot \nabla x' = \frac{1}{\alpha} a \cdot e_i \, e_i + \alpha a \cdot f_i f_i.$$

Hence

$$\mathsf{f}(e_1 \wedge f_1) = \frac{1}{\alpha} e_1 \wedge (\alpha f_1) = e_1 f_1, \qquad etc.$$

Since ∇ has same dimensions as $1/x$, the equation $\dot{x} = \nabla H \cdot J$ tells us that x^2 has dimensions of energy $\times$ time. Hence x has dimensions of $\mathrm{kg}^{1/2}\,\mathrm{m}\,\mathrm{s}^{-1/2}$.

7. Extension of reflection to grade r is

$$\mathsf{n}(A_r) = (-n a_1 n) \wedge (-n a_2 n) \wedge \cdots \wedge (-n a_r n) = (-1)^r n A_r n$$

In even dimensions, get $\mathsf{n}(I) = nIn = -I$, since all vectors anticommute with I. In odd dimensions, get $\mathsf{n}(I) = -nIn = -I$, since vectors commute with I.

8. $\det(\mathsf{f}) = \langle \mathsf{f}(I) I^{-1} \rangle = \langle I \bar{\mathsf{f}}(I^{-1}) \rangle = \det(\bar{\mathsf{f}})$.

9. $\bar{\mathsf{f}}_{ij} = e_i \cdot \bar{\mathsf{f}}(e_j) = e_j \cdot \mathsf{f}(e_i) = \mathsf{f}_{ji}$, hence components found by matrix transposition. Also have

$$\begin{aligned} \mathsf{h}_{ij} &= e_i \cdot \mathsf{h}(e_j) = e_i \cdot [\mathsf{f}\mathsf{g}(e_j)] \\ &= e_i \cdot \mathsf{f}[e_k \, e_k \cdot \mathsf{g}(e_j)] = \mathsf{g}_{kj} e_i \cdot \mathsf{f}(e_k) = \mathsf{f}_{ik} \mathsf{g}_{kj} \end{aligned}$$

recovering matrix multiplication law.

10. Suppose that we write

$$\mathsf{f}(e_1) = \mathsf{f}_{11} e_1 + \mathsf{f}_{21} e_2, \quad \mathsf{f}(e_2) = \mathsf{f}_{12} e_1 + \mathsf{f}_{22} e_2.$$

The matrix is

$$\mathsf{f}_{ij} = \begin{pmatrix} \mathsf{f}_{11} & \mathsf{f}_{12} \\ \mathsf{f}_{21} & \mathsf{f}_{22} \end{pmatrix}$$

and

$$\mathsf{f}(e_1\wedge e_2) = (\mathsf{f}_{11}e_1 + \mathsf{f}_{21}e_2)\wedge(\mathsf{f}_{12}e_1 + \mathsf{f}_{22}e_2) = (\mathsf{f}_{11}\mathsf{f}_{22} - \mathsf{f}_{12}\mathsf{f}_{21})e_1\wedge e_2$$

so determinants agree. Now have

$$\det(\mathsf{f}) = \mathsf{f}(e_1)\wedge\mathsf{f}(e_2)\wedge\cdots\wedge\mathsf{f}(e_n)I^{-1} = a_1\wedge a_2\wedge\cdots\wedge a_n I^{-1}.$$

The components of $a_k = \mathsf{f}(e_k)$ in the $\{e_k\}$ frame form the kth column of the matrix f_{ij}. Swap columns by swapping vectors, hence a minus sign by total antisymmetry. Also have

$$a_1\wedge a_2\wedge\cdots\wedge a_i\wedge\cdots\wedge a_n = a_1\wedge a_2\wedge\cdots\wedge(a_i + \lambda a_j)\wedge\cdots\wedge a_n, \quad (i \neq j)$$

so can add a multiple of one column to another.

11. K generates a global dilation:

$$\mathrm{e}^{\alpha K/2}\, n_+\, \mathrm{e}^{-\alpha K/2} = \mathrm{e}^{\alpha K}\, n_+ = \mathrm{ch}(\alpha)n_+ + \mathrm{sh}(\alpha)K\cdot n_+ = [\mathrm{ch}(\alpha) - \mathrm{sh}(\alpha)]n_+ = \mathrm{e}^{-\alpha}\, n_+$$

for *any* vector $n_+ = n_+\cdot K$. This corresponds to a global change of scale for all vectors. The subgroup obtained when this is factored out is the *special linear* group sl(R) — the group of matrices with determinant 1.

12. Along the side x_0–x_1 write $x = x_0 + \lambda(x_1 - x_0)$, so that $x - x_0 = \lambda e_1$. Also get $dS = d\lambda\, e_1$, so contribution from this side is

$$\int_0^1 d\lambda\, e_1[M_0 + \lambda(M_1 - M_0)] = \tfrac{1}{2}e_1(M_1 + M_0).$$

Similar for two remaining sides. Gives

$$\begin{aligned}\oint dS\, m(x) &= \tfrac{1}{2}e_1(M_1 + M_0) + \tfrac{1}{2}(e_2 - e_1)(M_2 + M_1) - \tfrac{1}{2}e_2(M_2 + M_0)\\ &= -\tfrac{1}{2}e_1(M_2 - M_0) + \tfrac{1}{2}e_2(M_1 - M_0)\\ &= \tfrac{1}{2}e_2\wedge e_1[e^1(M_1 - M_0) + e^2(M_2 - M_0)]\end{aligned}$$

No dependence on signature. Factor $e_2\wedge e_1/2$ has area V equal to that of the triangle, but reverse orientation. Eliminate this by replacing m by $I^{-1}m$,

$$\begin{aligned}\oint dS\, I^{-1}m(x) &= -VI[e^1I^{-1}(M_1 - M_0) + e^2I^{-1}V(M_2 - M_0)\\ &= V[e^1(M_1 - M_0) + e^2(M_2 - M_0)]\end{aligned}$$

Which is the result that generalises. In 3-d, for example, construct a tetrahedra. Four sides, so four terms in total surface integral. Factor of 1/3! now, because 6 tetrahedra in a parallelepiped. Bracketed term is ∇M in limit.

13. Write $x = x^0\gamma_0 + x^1\gamma_1$. Have

$$\langle x\gamma_0 x\gamma_0\rangle = \langle(x^0\gamma_0 + x^1\gamma_1)((x^0\gamma_0 - x^1\gamma_1)\rangle = (x^0)^2 + (x^1)^2$$

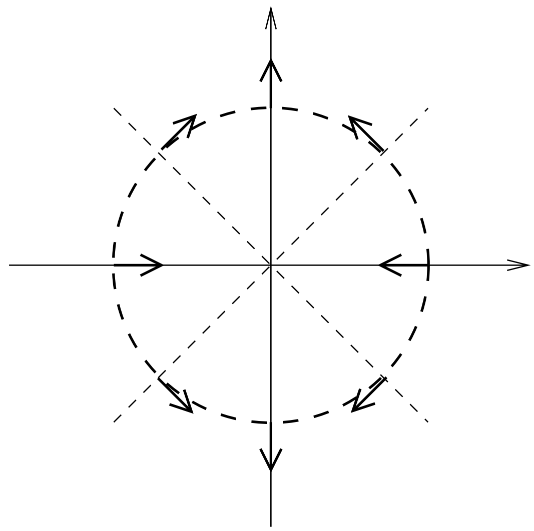

Figure 1: Normal vectors to the unit circle in 2-d spacetime. The normals all have vanishing inner product with tangent vectors.

So surface of constant $\langle x\gamma_0 v\gamma_0\rangle$ is a circle in spacetime. Gradient is

$$\nabla\langle x\gamma_0 x\gamma_0\rangle = \dot{\nabla}\langle \dot{x}\gamma_0 x\gamma_0\rangle + \dot{\nabla}\langle x\gamma_0 \dot{x}\gamma_0\rangle = 2\dot{\nabla}\,\dot{x}\cdot(\gamma_0 x\gamma_0) = 2\gamma_0 x\gamma_0 = 2(x^0\gamma_0 - x^1\gamma_1).$$

This vector points outwards along timelike axis, but inwards along spacelike! (See Fig.). 'Normal' is tangential at intersection with null directions.

14. $\hat{B}$ is unit bivector, so

$$\hat{B}^2 = (\boldsymbol{a} + I\boldsymbol{b})(\boldsymbol{a} + I\boldsymbol{b}) = \boldsymbol{a}^2 - \boldsymbol{b}^2 + 2I\boldsymbol{a}\cdot\boldsymbol{b} = 1$$

So must have $\boldsymbol{a}^2 - \boldsymbol{b}^2 = 1$ and $\boldsymbol{a}\cdot\boldsymbol{b} = 1$. Write $|\boldsymbol{b}| = \sinh(u)$, so $|\boldsymbol{a}| = \cosh(u)$, and

$$\hat{B} = \cosh(u)\hat{\boldsymbol{a}} + \sinh(u)I\hat{\boldsymbol{b}} = (\cosh(u) + \sinh(u)I\hat{\boldsymbol{b}}\hat{\boldsymbol{a}})\hat{\boldsymbol{a}}$$

using over-hats for unit vectors. Now $(I\hat{\boldsymbol{b}}\hat{\boldsymbol{a}})^2 = +1$, so

$$\hat{B} = \mathrm{e}^{uI\hat{\boldsymbol{b}}\hat{\boldsymbol{a}}}\,\hat{\boldsymbol{a}} = R_u\hat{\boldsymbol{a}}\tilde{R}_u, \qquad R_u = \mathrm{e}^{uI\hat{\boldsymbol{b}}\hat{\boldsymbol{a}}/2}$$

But in 3-d can always rotate $\boldsymbol{\sigma}_3$ onto $\hat{\boldsymbol{a}}$ with rotation in the $\boldsymbol{\sigma}_3\wedge\hat{\boldsymbol{a}}$ plane. Call this rotor R_3. Have

$$\hat{B} = R_u R_3\boldsymbol{\sigma}_3\tilde{R}_3\tilde{R}_u = R\boldsymbol{\sigma}_3\tilde{R}, \qquad R = R_u R_3$$

Now $\boldsymbol{\sigma}_3\cdot(\gamma_0 \pm \gamma_3) = \pm(\gamma_0 \pm \gamma_3)$, so $n_\pm = R(\gamma_0 \pm \gamma_3)\tilde{R}$ are required vectors.

15. $R\tilde{R} = -(\gamma_0 + \gamma_1 - \gamma_2)(\gamma_0 + \gamma_1 - \gamma_2) = 1$, so R is a rotor. Have

$$H = \frac{R+1}{R-1} = \frac{(\gamma_0+\gamma_1)\gamma_2}{(\gamma_0+\gamma_1)\gamma_2 + 2} = \tfrac{1}{4}(\gamma_0+\gamma_1)\gamma_2(2 - (\gamma_0+\gamma_1)\gamma_2) = \tfrac{1}{2}(\gamma_0+\gamma_1)\gamma_2$$

This is a null bivector, so generator is just $(\gamma_0+\gamma_1)\gamma_2$. If repeat for $-R$ get a factor $(\gamma_0+\gamma_1)\gamma_2$ in denominator of H. This has no inverse, so cannot find a bivector generator.