Geometric Algebra 2
Quantum Theory

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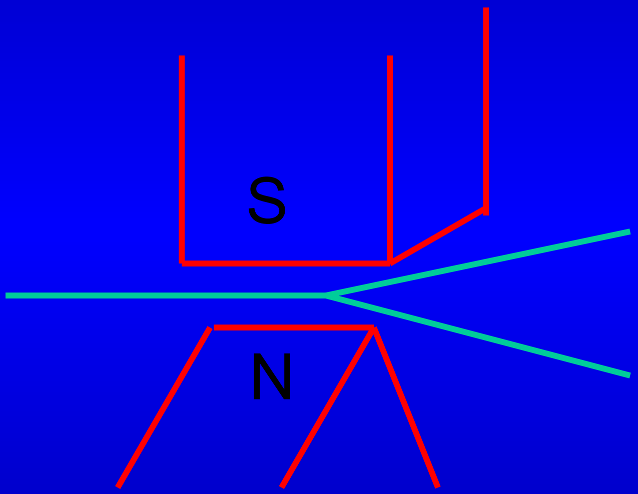
Spin

- Stern-Gerlach tells us that electron wavefunction contains two terms

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$$

- Describe state in terms of a spinor

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$



- A 2-state system or qubit

Pauli Matrices

- Operators acting on a spinor must obey angular momentum relations

$$\hat{l}_i = -i\hbar\epsilon_{ijk}x_j\partial_k, \qquad [\hat{l}_i, \hat{l}_j] = i\hbar\epsilon_{ijk}\hat{l}_k$$

- Get spin operators

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- These form a Clifford algebra
- A matrix representation of the geometric algebra of 3D space

Observables

- Want to place Pauli theory in a more geometric framework with $\hat{\sigma}_k \mapsto \boldsymbol{\sigma}_k$
- Construct observables

$$s_k = \frac{1}{2}\hbar n_k = \langle\psi|\hat{s}_k|\psi\rangle$$

- Belong to a unit vector
- Written in terms of polar coordinates, find parameterisation

$$|\psi\rangle = \begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix}$$

Rotors and Spinors

- From work on Euler angles, encode degrees of freedom in the rotor

$$R = e^{-\phi I \boldsymbol{\sigma}_3/2} e^{-\theta I \boldsymbol{\sigma}_2/2}$$

- Represent spinor / qubit as element of the even subalgebra:

$$|\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \leftrightarrow \psi = a^0 + a^k I \boldsymbol{\sigma}_k$$

- Verify that

$$\hat{\sigma}_k |\psi\rangle \leftrightarrow \boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3 \quad (k = 1, 2, 3)$$

Keeps result in even algebra

Imaginary Structure

- Can construct imaginary action from

$$\hat{\sigma}_1\hat{\sigma}_2\hat{\sigma}_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

- So find that

$$i|\psi\rangle \;\leftrightarrow\; \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3\psi(\boldsymbol{\sigma}_3)^3 = \psi I\boldsymbol{\sigma}_3$$

- Complex structure controlled by a bivector
- Acts on the right, so commutes with operators applied to the left of the spinor
- Hints at a geometric substructure
- Can always use i to denote the structure

Inner Products

- The reverse operation in 3D is same as Hermitian conjugation
- Real part of inner product is

$$\mathbb{R}\langle\psi|\phi\rangle \;\leftrightarrow\; \langle\psi^\dagger\phi\rangle$$

- Follows that full inner product is

$$\langle\psi|\phi\rangle \;\leftrightarrow\; \langle\psi^\dagger\phi\rangle - \langle\psi^\dagger\phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3$$

- The projection onto the 1 and $I\sigma_3$ components,

$$\langle A\rangle_q = \frac{1}{2}(A + \boldsymbol{\sigma}_3 A\boldsymbol{\sigma}_3)$$

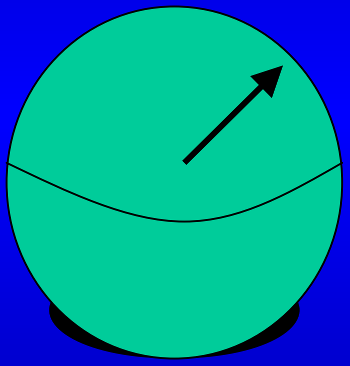
Observables

- Spin observables become

$$\langle\psi|\hat{\sigma}_k|\psi\rangle \;\leftrightarrow\; \langle\psi^\dagger\boldsymbol{\sigma}_k\psi\boldsymbol{\sigma}_3\rangle = \boldsymbol{\sigma}_k\cdot(\psi\boldsymbol{\sigma}_3\psi^\dagger)$$

- All information contained in the spin vector

$$\boldsymbol{s} = \tfrac{1}{2}\hbar\psi\boldsymbol{\sigma}_3\psi^\dagger$$



- Now define normalised rotor

$$\rho = \psi\psi^\dagger, \qquad \psi = \rho^{1/2}R$$

- Operation of forming an observable reduces to

$$\boldsymbol{s} = \tfrac{1}{2}\hbar\rho\boxed{R\boldsymbol{\sigma}_3R^\dagger}$$

Same as classical expression

Rotating Spinors

- So have a natural 'explanation' for 2-sided construction of quantum observables
- Now look at composite rotations

$$R_\theta = \exp(-\hat{B}\theta/2), \qquad R\boldsymbol{\sigma}_3 R^\dagger \mapsto R_\theta R \boldsymbol{\sigma}_3 R^\dagger R_\theta^\dagger$$

- So rotor transformation law is

$$R \mapsto R' = R_\theta R$$

- Take angle through to 2π

$$R' = e^{-\hat{B}\pi} R = -R$$

Sign change for fermions

Unitary Transformations

- Spinors can transform under the full unitary group U(2)
- Decomposes into SU(2) and a U(1) term
- SU(2) term becomes a rotor on left
- U(1) term applied on the right

$$U(\psi) = R\psi e^{\phi I \boldsymbol{\sigma}_3}$$

- Separates out the group structure in a helpful way
- Does all generalise to multiparticle setting

Magnetic Field

- Rotor contained in $\psi = \rho^{1/2} R$
- Use this to Simplify equations
- Magnetic field $\hat{H} = -\frac{1}{2}\gamma\hbar B_k \hat{\sigma}_k$
- Schrodinger equation $\frac{d|\psi\rangle}{dt} = \frac{1}{2}\gamma i B_k \hat{\sigma}_k |\psi\rangle$
- Reduces to simple equation

$$\dot{\psi} = \frac{1}{2}\gamma B_k I \boldsymbol{\sigma}_k \psi = \frac{1}{2}\gamma I \boldsymbol{B} \psi$$

- Magnitude is constant, so left with rotor equation $\dot{R} = \frac{1}{2}\gamma I \boldsymbol{B} R$

Density Matrices

- Mixed states are described by a density matrix
- For a pure state this is

$$\hat{\rho} = |\psi\rangle \langle\psi| = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix}$$

- GA version is $\psi\frac{1}{2}(1+\boldsymbol{\sigma}_3)\psi^\dagger = \frac{1}{2}(1+\boldsymbol{s})$
- Addition is fine in GA!
- General mixed state from sum

$$\rho = \frac{1}{2n}(n + \boldsymbol{s}_1 + \cdots + \boldsymbol{s}_n) = \frac{1}{2}(1+\boldsymbol{P}), \quad |\boldsymbol{P}| \leq 1$$

Spacetime Algebra

- Basic tool for relativistic physics is the spacetime algebra or STA.

1	γ_μ	$\gamma_\mu\gamma_\nu$	$I\gamma_\mu$	$I = \gamma_0\gamma_1\gamma_2\gamma_3$
1 scalar	4 vectors	6 bivectors	4 trivectors	1 pseudoscalar

- Generators satisfy

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \mathrm{diag}(+ - - -), \quad \mu = 0 \ldots 3$$

- A matrix-free representation of Dirac theory
- Currently used for classical mechanics, scattering, tunnelling, supersymmetry, gravity and quantum information

Relative Space

- Determine 3D space relative for observer with velocity given by timelike vector γ_0
- Suppose event has position x in natural units

$$t = x\cdot\gamma_0 \qquad \boldsymbol{x} = x\wedge\gamma_0$$
$$t + \boldsymbol{x} = x\gamma_0$$

- The basis elements of relative vector are

$$\boldsymbol{\sigma}_i = \gamma_i\gamma_0$$

- Satisfy

$$\begin{aligned}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j &= \tfrac{1}{2}(\gamma_i\gamma_0\gamma_j\gamma_0 + \gamma_j\gamma_0\gamma_i\gamma_0) \\ &= \tfrac{1}{2}(-\gamma_i\gamma_j - \gamma_j\gamma_i) = \delta_{ij}\end{aligned}$$

Relative Split

- Split bivectors with γ_0 to determine relative split

$$1 \qquad \{\gamma_\mu\} \qquad \{\sigma_i, I\sigma_i\} \qquad I\gamma_\mu \qquad I$$

$$1 \qquad \{\sigma_i\} \qquad \{I\sigma_i\} \qquad I$$

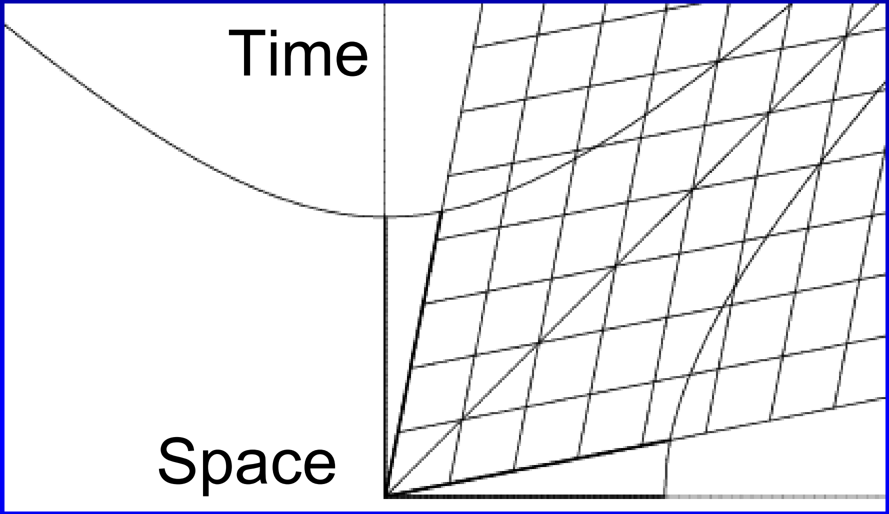
- Relative vectors generate 3D algebra with same volume element

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = (\gamma_1\gamma_0)(\gamma_2\gamma_0)(\gamma_3\gamma_0) = \gamma_0\gamma_1\gamma_2\gamma_3 = I$$

- Relativistic (Dirac) spinors constructed from full 3D algebra

Lorentz Transformation

- Moving observers construct a new coordinate grid
- Both position and time coordinates change



$$t' = \gamma(t - \beta z), \qquad z' = \gamma(z - \beta t),$$

$$\gamma = (1 - \beta^2)^{-1/2} \qquad \beta = v/c$$

Need to re-express this in terms of vector transformations

Frames and Boosts

- Vector unaffected by coordinate system, so

$$x = x^\mu e_\mu = x^{\mu\prime} e'_\mu$$

- Frame vectors related by

$$e'_0 = \gamma(e_0 + \beta e_3), \qquad e'_3 = \gamma(e_3 + \beta e_0)$$

- Introduce the hyperbolic angle

$$\tanh(\alpha) = \beta$$

- Transformed vectors now

$$e'_0 = \cosh(\alpha)\, e_0 + \sinh(\alpha)\, e_3 = \exp(\alpha\, e_3 e_0)\, e_0$$

Exponential of a bivector

Spacetime Rotors

- Define the spacetime rotor

$$R = e^{\alpha\, e_3 e_0/2}$$

- A Lorentz transformation can now be written in rotor form

$$e'_\mu = R e_\mu \tilde{R}$$

- Use the tilde for reverse operation in the STA (dagger is frame-dependent)
- Same rotor description as 3D
- Far superior to 4X4 matrices!

Pure Boosts

- Rotors generate proper orthochronous transformations
- Suppose we want the pure boost from u to v

$$v = Lu\tilde{L} = L^2 u$$

- Solution is

$$L = \frac{1 + vu}{[2(1 + u \cdot v)]^{1/2}} = \exp\left(\frac{\alpha}{2} \frac{v \wedge u}{|v \wedge u|}\right)$$

- Remainder of a general rotor is

$$R = LU, \qquad U = \tilde{L}R$$

A 3D rotor

Velocity and Acceleration

- Write arbitrary 4-velocity as

$$v = R\gamma_0\tilde{R}$$

- Acceleration is

$$\dot{v} = \frac{d}{d\tau}(R\gamma_0\tilde{R}) = \dot{R}\gamma_0\tilde{R} + R\gamma_0\dot{\tilde{R}}$$

- But $\dot{R}\tilde{R} = -R\dot{\tilde{R}}$ Pure bivector
- So

$$\dot{v} = \dot{R}\tilde{R}v - v\dot{R}\tilde{R} = 2(\dot{R}\tilde{R})\cdot v$$

Acceleration bivector

The vector derivative

- Define the vector derivative operator in the standard way

$$\nabla = \sum_k e^k \frac{\partial}{\partial x^k}$$

- So components of this are directional derivatives
- But now the vector product terms are invertible
- Can construct Green's functions for ∇
- These are Feynman propagators in spacetime

2 Dimensions

- Vector derivative is

$$\nabla = \boldsymbol{e}_1 \frac{\partial}{\partial x} + \boldsymbol{e}_2 \frac{\partial}{\partial y}$$

- Now introduce the scalar+pseudoscalar field

$$\varphi = u + Iv$$

- Find that

$$\nabla\varphi = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\boldsymbol{e}_1 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\boldsymbol{e}_2$$

- Same terms that appear in the Cauchy-Riemann equations!

Analytic Functions

- Vector derivative closely related to definition of analytic functions
- Statement that φ is analytic is $\boxed{\nabla\varphi = 0}$
- Cauchy integral formula provides inverse
- This generalises to arbitrary dimensions
- Can construct power series in z because

$$\nabla z = \nabla(e_1 x) = \nabla(2e_1 \cdot r - x e_1) = 0$$

- Lose the commutativity in higher dimensions
- But this does not worry us now!

Spacetime Vector Derivative

- Define spacetime vector derivative

$$\nabla = \gamma^\mu \partial_\mu, \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}$$

- Has a spacetime split of the form

$$\nabla \gamma_0 = (\gamma^0 \partial_t + \gamma^i \partial_i)\gamma_0 = \partial_t - \boldsymbol{\sigma}_i \partial_i = \partial_t - \boldsymbol{\nabla}$$

- First application - Maxwell equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

$$-\boldsymbol{\nabla} \times \boldsymbol{E} = \frac{\partial}{\partial t} \boldsymbol{B} \qquad \boldsymbol{\nabla} \times \boldsymbol{H} = \frac{\partial}{\partial t} \boldsymbol{D} + \boldsymbol{J}$$

Maxwell Equations

- Assume no magnetisation and polarisation effects and revert to natural units
- Maxwell equations become, in GA form

$$\nabla \cdot E = \rho \qquad \nabla \cdot B = 0$$

$$\nabla \wedge E = -\partial_t (IB) \qquad \nabla \wedge B = I(J + \partial_t E)$$

- Naturally assemble equations for the divergence and (bivector) curl
- Combine using geometric product

$$\nabla (E + IB) + \partial_t (E + IB) = \rho - J$$

STA Form

- Define the field strength (Faraday bivector)

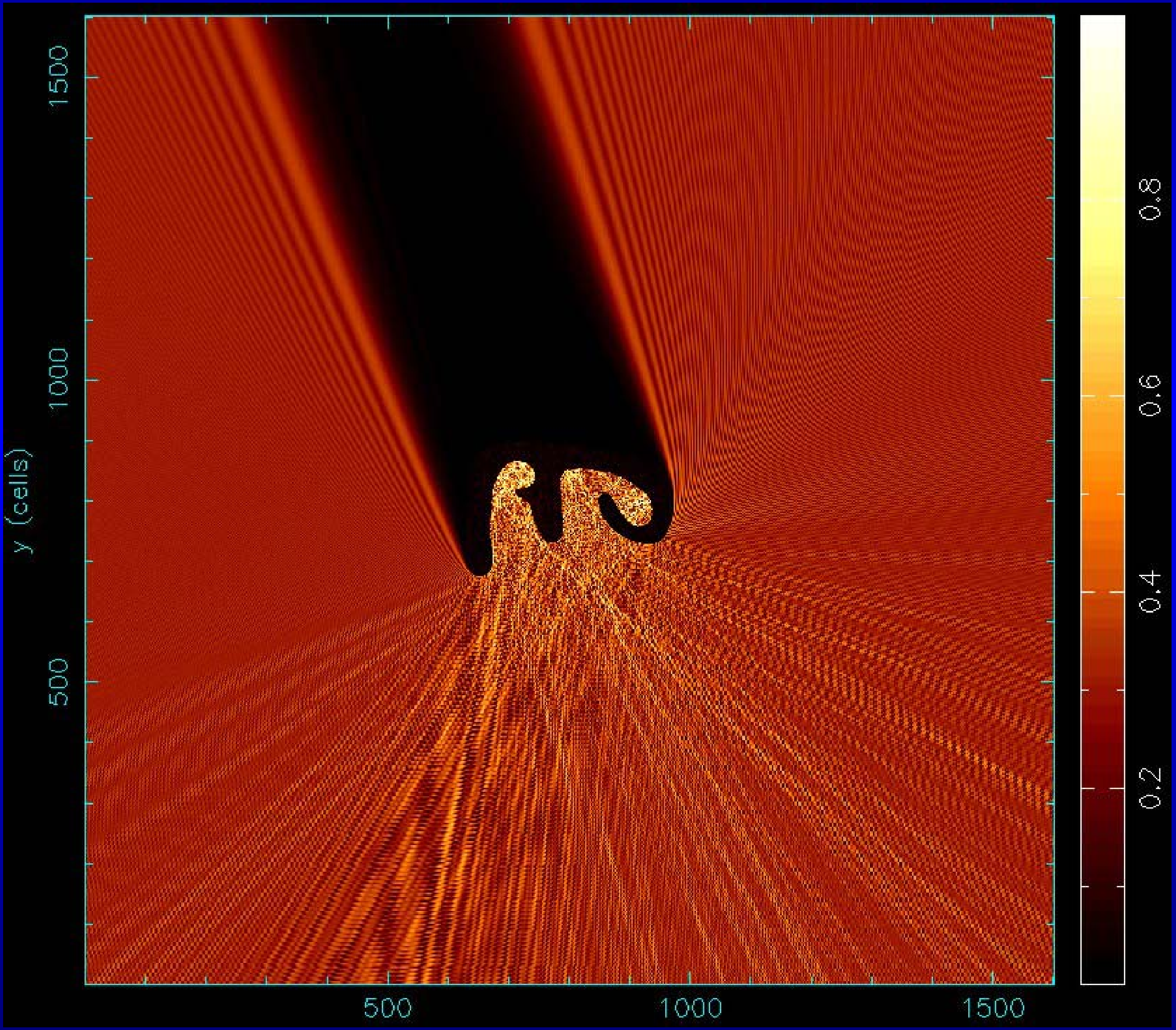
$$F = \boldsymbol{E} + I\boldsymbol{B}$$

- And current $J\gamma_0 = \rho + \boldsymbol{J}$
- All 4 Maxwell equations unite into the single equation

$$\boxed{\nabla F = J}$$

- Spacetime vector derivative is invertible, can carry out first-order propagator theory
- First-order Green's function for scattering

Application



Lorentz Force Law

- Non-relativistic form is

$$\frac{d\boldsymbol{p}}{dt} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

- Can re-express in relativistic form as

$$\dot{v} = 2(\dot{R}\tilde{R}) \cdot v = \frac{q}{m} F \cdot v$$

- Simplest form is provided by rotor equation

$$\dot{R} = \frac{q}{2m} F R$$

Spin Dynamics

- Suppose that a particle carries a spin vector s along its trajectory

$$s \cdot v = 0, \qquad s = R\gamma_3\tilde{R}$$

- Simplest form of rotor equation then has

$$\dot{s} = \frac{q}{m} F \cdot s$$

- Non relativistic limit to this equation is

$$\dot{\boldsymbol{s}} = \frac{q}{m} \boldsymbol{s} \times \boldsymbol{B}$$

Equation for a particle with g=2!

Exercises

- 2 spin-1/2 states are represented by ϕ and ψ, with accompanying spin vectors

$$\boldsymbol{s}_1 = \phi\boldsymbol{\sigma}_3\tilde{\phi}, \qquad \boldsymbol{s}_2 = \psi\boldsymbol{\sigma}_3\tilde{\psi}$$

- Prove that

$$\frac{|\langle\phi|\psi\rangle|^2}{\langle\phi|\phi\rangle\langle\psi|\psi\rangle} = \frac{1}{2}(1 + \cos\theta) \quad \boldsymbol{s}_1\cdot\boldsymbol{s}_2 = |\boldsymbol{s}_1||\boldsymbol{s}_2|\cos(\theta)$$

- Given that $$L = \frac{1 + vu}{[2(1 + v\cdot u)]^{1/2}}$$

- Prove that $$v = Lu\tilde{L}$$