Geometric Algebra 2
Quantum Theory

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Spin

- Stern-Gerlach tells us that electron wavefunction contains two terms
  \[ |\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle \]

- Describe state in terms of a spinor
  \[ |\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \]

- A 2-state system or qubit
Pauli Matrices

- Operators acting on a spinor must obey angular momentum relations

\[ \hat{l}_i = -i\hbar \epsilon_{ijk} x_j \partial_k, \quad [\hat{l}_i, \hat{l}_j] = i\hbar \epsilon_{ijk} \hat{l}_k \]

- Get spin operators

\[
\begin{align*}
\hat{\sigma}_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\hat{\sigma}_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
\hat{\sigma}_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]

- These form a Clifford algebra
- A matrix representation of the geometric algebra of 3D space
Observables

- Want to place Pauli theory in a more geometric framework with $\hat{\sigma}_k \rightarrow \sigma_k$
- Construct observables
  \[ s_k = \frac{1}{2} \hbar n_k = \langle \psi | \hat{s}_k | \psi \rangle \]
- Belong to a unit vector
- Written in terms of polar coordinates, find parameterisation
  \[ |\psi\rangle = \begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix} \]
Rotors and Spinors

• From work on Euler angles, encode degrees of freedom in the rotor

\[ R = e^{-\phi I \sigma_3/2} e^{-\theta I \sigma_2/2} \]

• Represent spinor / qubit as element of the even subalgebra:

\[ |\psi\rangle = \begin{pmatrix} a^0 + i a^3 \\ -a^2 + i a^1 \end{pmatrix} \quad \leftrightarrow \quad \psi = a^0 + a^k I \sigma_k \]

• Verify that

\[ \hat{\sigma}_k |\psi\rangle \leftrightarrow \sigma_k \psi \sigma_3 \quad (k = 1, 2, 3) \]

Keeps result in even algebra
Imaginary Structure

- Can construct imaginary action from

\[
\hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}
\]

- So find that

\[
i |\psi\rangle \leftrightarrow \sigma_1 \sigma_2 \sigma_3 \psi (\sigma_3)^3 = \psi I \sigma_3
\]

- Complex structure controlled by a \textbf{bivector}
- Acts on the right, so \textbf{commutes} with operators applied to the left of the spinor
- Hints at a geometric substructure
- Can always use \textit{i} to denote the structure
Inner Products

- The reverse operation in 3D is same as Hermitian conjugation
- Real part of inner product is
  \[ \mathbb{R} \langle \psi | \phi \rangle \leftrightarrow \langle \psi^\dagger | \phi \rangle \]
- Follows that full inner product is
  \[ \langle \psi | \phi \rangle \leftrightarrow \langle \psi^\dagger | \phi \rangle - \langle \psi^\dagger | \phi I \sigma_3 \rangle I \sigma_3 \]
- The projection onto the 1 and \( I \sigma_3 \) components,
  \[ \langle A \rangle_q = \frac{1}{2} (A + \sigma_3 A \sigma_3) \]
Observables

- Spin observables become
  \[ \langle \psi | \hat{\sigma}_k | \psi \rangle \leftrightarrow \langle \psi^\dagger \sigma_k \psi \sigma_3 \rangle = \sigma_k \cdot (\psi \sigma_3 \psi^\dagger) \]

- All information contained in the spin vector
  \[ s = \frac{1}{2} \hbar \psi \sigma_3 \psi^\dagger \]

- Now define normalised rotor
  \[ \rho = \psi \psi^\dagger, \quad \psi = \rho^{1/2} R \]

- Operation of forming an observable reduces to
  \[ s = \frac{1}{2} \hbar \rho R \sigma_3 R^\dagger \]
  Same as classical expression
Rotating Spinors

• So have a natural ‘explanation’ for 2-sided construction of quantum observables
• Now look at composite rotations

\[ R_\theta = \exp(-\hat{B}\theta/2), \quad R\sigma_3 R^\dagger \mapsto R_\theta R\sigma_3 R^\dagger R_\theta^\dagger \]

• So rotor transformation law is

\[ R \mapsto R' = R_\theta R \]

• Take angle through to \(2\pi\)

\[ R' = e^{-\hat{B}\pi} R = -R \]

Sign change for fermions
Unitary Transformations

- Spinors can transform under the full unitary group U(2)
- Decomposes into SU(2) and a U(1) term
- SU(2) term becomes a rotor on left
- U(1) term applied on the right

\[ U(\psi) = R\psi e^{i\phi I}\sigma_3 \]

- Separates out the group structure in a helpful way
- Does all generalise to multiparticle setting
Magnetic Field

- Rotor contained in $\psi = \rho^{1/2} R$
- Use this to Simplify equations
- Magnetic field $\hat{H} = -\frac{1}{2} \gamma \hbar B_k \hat{\sigma}_k$
- Schrödinger equation $\frac{d}{dt} |\psi\rangle = \frac{1}{2} \gamma i B_k \hat{\sigma}_k |\psi\rangle$
- Reduces to simple equation $\dot{\psi} = \frac{1}{2} \gamma B_k I \sigma_k \psi = \frac{1}{2} \gamma I B \psi$
- Magnitude is constant, so left with \textbf{rotor equation} $\dot{R} = \frac{1}{2} \gamma I B R$
Density Matrices

- Mixed states are described by a density matrix.
- For a pure state this is

\[ \hat{\rho} = |\psi\rangle \langle \psi| = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix} \]

- GA version is

\[ \psi \frac{1}{2} (1 + \sigma_3) \psi^\dagger = \frac{1}{2} (1 + s) \]

- Addition is fine in GA!
- General mixed state from sum

\[ \rho = \frac{1}{2n} (n + s_1 + \cdots + s_n) = \frac{1}{2} (1 + P), \quad |P| \leq 1 \]
Spacetime Algebra

• Basic tool for relativistic physics is the spacetime algebra or STA.

\[
1, \quad \gamma_\mu, \quad \gamma_\mu \gamma_\nu, \quad I \gamma_\mu, \quad I = \gamma_0 \gamma_1 \gamma_2 \gamma_3
\]

1 scalar, 4 vectors, 6 bivectors, 4 trivectors, 1 pseudoscalar

• Generators satisfy

\[
\gamma_\mu \cdot \gamma_\nu = \eta_{\mu \nu} = \text{diag}(+ - - -), \quad \mu = 0 \ldots 3
\]

• A matrix-free representation of Dirac theory

• Currently used for classical mechanics, scattering, tunnelling, supersymmetry, gravity and quantum information
Relative Space

- Determine 3D space relative for observer with velocity given by timelike vector $\gamma_0$
- Suppose event has position $x$ in natural units

\[
t = x \cdot \gamma_0 \quad x = x \land \gamma_0
\]

\[
t + x = x \gamma_0
\]

- The basis elements of relative vector are

\[
\sigma_i = \gamma_i \gamma_0
\]

- Satisfy

\[
\sigma_i \cdot \sigma_j = \frac{1}{2} (\gamma_i \gamma_0 \gamma_j \gamma_0 + \gamma_j \gamma_0 \gamma_i \gamma_0)
\]

\[
= \frac{1}{2} (-\gamma_i \gamma_j - \gamma_j \gamma_i) = \delta_{ij}
\]
Relative Split

• Split bivectors with $\gamma_0$ to determine relative split

\[
\begin{align*}
1 & \quad \{\gamma_\mu\} & \{\sigma_i, I\sigma_i\} & \quad I\gamma_\mu & I \\
1 & \quad \{\sigma_i\} & \quad \{I\sigma_i\} & & I
\end{align*}
\]

• Relative vectors generate 3D algebra with same volume element

\[
\sigma_1 \sigma_2 \sigma_3 = (\gamma_1 \gamma_0)(\gamma_2 \gamma_0)(\gamma_3 \gamma_0) = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = I
\]

• Relativistic (Dirac) spinors constructed from full 3D algebra
Lorentz Transformation

- Moving observers construct a new coordinate grid
- Both position and time coordinates change

\[
t' = \gamma(t - \beta z), \quad z' = \gamma(z - \beta t),
\]
\[
\gamma = (1 - \beta^2)^{-1/2} \quad \beta = v/c
\]

Need to re-express this in terms of vector transformations
Frames and Boosts

- Vector unaffected by coordinate system, so
  \[ x = x^\mu e_\mu = x'^{\mu'} e'_{\mu'} \]

- Frame vectors related by
  \[ e'_0 = \gamma(e_0 + \beta e_3), \quad e'_3 = \gamma(e_3 + \beta e_0) \]

- Introduce the \textbf{hyperbolic angle}
  \[ \tanh(\alpha) = \beta \]

- Transformed vectors now
  \[ e'_0 = \cosh(\alpha) e_0 + \sinh(\alpha) e_3 = \exp(\alpha e_3 e_0) e_0 \]

Exponential of a bivector
Spacetime Rotors

• Define the spacetime rotor

\[ R = e^{\alpha e_3 e_0 / 2} \]

• A Lorentz transformation can now be written in rotor form

\[ e'_\mu = R e_\mu \tilde{R} \]

• Use the tilde for reverse operation in the STA (dagger is frame-dependent)
• Same rotor description as 3D
• Far superior to 4X4 matrices!
Pure Boosts

- Rotor generates proper orthochronous transformations
- Suppose we want the pure boost from $u$ to $v$
  \[ v = L_u \tilde{L} = L^2 u \]
- Solution is
  \[ L = \frac{1 + vu}{[2(1 + u \cdot v)]^{1/2}} = \exp \left( \frac{\alpha \frac{v \wedge u}{2 |v \wedge u|}}{2} \right) \]
- Remainder of a general rotor is
  \[ R = LU, \quad U = \tilde{L} R \]

A 3D rotor
Velocity and Acceleration

• Write arbitrary 4-velocity as
  \[ v = R\gamma_0 \tilde{R} \]

• Acceleration is
  \[ \dot{v} = \frac{d}{d\tau} (R\gamma_0 \tilde{R}) = \dot{R}\gamma_0 \tilde{R} + R\gamma_0 \dot{\tilde{R}} \]

• But
  \[ \dot{R} \tilde{R} = -R \dot{\tilde{R}} \]

• So
  \[ \dot{v} = \dot{R} \tilde{R} v - v \dot{R} \tilde{R} = 2(\dot{R} \tilde{R}) \cdot v \]

Pure bivector

Acceleration bivector
The vector derivative

- Define the vector derivative operator in the standard way
  \[ \nabla = \sum_k e^k \frac{\partial}{\partial x^k} \]

- So components of this are directional derivatives

- But now the vector product terms are invertible

- Can construct Green’s functions for \( \nabla \)

- These are Feynman propagators in spacetime
2 Dimensions

- Vector derivative is
  \[ \nabla = e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} \]

- Now introduce the scalar+pseudoscalar field
  \[ \varphi = u + Iv \]

- Find that
  \[ \nabla \varphi = \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) e_1 + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) e_2 \]

- Same terms that appear in the Cauchy-Riemann equations!
Analytic Functions

• Vector derivative closely related to definition of analytic functions
• Statement that $\varphi$ is analytic is $\nabla \varphi = 0$
• Cauchy integral formula provides inverse

• This generalises to arbitrary dimensions
• Can construct power series in $z$ because

$$\nabla z = \nabla (e_1 x) = \nabla (2e_1 \cdot r - xe_1) = 0$$

• Lose the commutativity in higher dimensions
• But this does not worry us now!
Spacetime Vector Derivative

• Define spacetime vector derivative

\[ \nabla = \gamma^\mu \partial_\mu, \quad \partial_\mu = \frac{\partial}{\partial x^\mu} \]

• Has a spacetime split of the form

\[ \nabla \gamma_0 = (\gamma^0 \partial_t + \gamma^i \partial_i) \gamma_0 = \partial_t - \sigma_i \partial_i = \partial_t - \nabla \]

• First application - Maxwell equations

\[ \nabla \cdot D = \rho \quad \nabla \cdot B = 0 \]

\[ -\nabla \times E = \frac{\partial}{\partial t} B \quad \nabla \times H = \frac{\partial}{\partial t} D + J \]
Maxwell Equations

• Assume no magnetisation and polarisation effects and revert to natural units
• Maxwell equations become, in GA form

\[ \nabla \cdot \mathbf{E} = \rho \quad \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \wedge \mathbf{E} = -\partial_t (I \mathbf{B}) \quad \nabla \wedge \mathbf{B} = I (\mathbf{J} + \partial_t \mathbf{E}) \]

• Naturally assemble equations for the divergence and (bivector) curl
• Combine using geometric product

\[ \nabla (\mathbf{E} + I \mathbf{B}) + \partial_t (\mathbf{E} + I \mathbf{B}) = \rho - \mathbf{J} \]
STA Form

• Define the field strength (Faraday bivector)
  \[ F = E + IB \]

• And current
  \[ J\gamma_0 = \rho + J \]

• All 4 Maxwell equations unite into the single equation
  \[ \nabla F = J \]

• Spacetime vector derivative is invertible, can carry out first-order propagator theory
• First-order Green’s function for scattering
Application
Lorentz Force Law

• Non-relativistic form is

\[ \frac{dp}{dt} = q(E + \mathbf{v} \times \mathbf{B}) \]

• Can re-express in relativistic form as

\[ \dot{v} = 2(\dot{R}\mathbf{R}) \cdot v = \frac{q}{m} F \cdot v \]

• Simplest form is provided by rotor equation

\[ \dot{R} = \frac{q}{2m} FR \]
Spin Dynamics

• Suppose that a particle carries a spin vector $s$ along its trajectory

\[ s \cdot v = 0, \quad s = R \gamma_3 \tilde{R} \]

• Simplest form of rotor equation then has

\[ \dot{s} = \frac{q}{m} F \cdot s \]

• Non relativistic limit to this equation is

\[ \dot{s} = \frac{q}{m} s \times B \]

Equation for a particle with $g=2!$
Exercises

• 2 spin-1/2 states are represented by $\phi$ and $\psi$, with accompanying spin vectors

\[ s_1 = \phi \sigma_3 \phi, \quad s_2 = \psi \sigma_3 \psi \]

• Prove that

\[ \frac{|\langle \phi | \psi \rangle|^2}{\langle \phi | \phi \rangle \langle \psi | \psi \rangle} = \frac{1}{2} (1 + \cos \theta) \]

\[ s_1 \cdot s_2 = |s_1| |s_2| \cos(\theta) \]

• Given that

\[ L = \frac{1 + vu}{[2(1 + v \cdot u)]^{1/2}} \]

• Prove that

\[ v = Lu \tilde{L} \]