Geometric Algebra 3
Dirac Theory and Multiparticle Systems

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Dirac Algebra

- Dirac matrix operators are

$$\hat{\gamma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{\gamma}_k = \begin{pmatrix} 0 & -\hat{\sigma}_k \\ \hat{\sigma}_k & 0 \end{pmatrix}, \qquad \hat{\gamma}_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- These act on 4-component Dirac spinors

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix}$$

- These spinors satisfy a first-order wave equation

$$i\hat{\gamma}^{\mu}\partial_{\mu}|\psi\rangle = m|\psi\rangle$$

STA Form

- Adapt the map for Pauli spinors

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix} \leftrightarrow \psi = \phi + \eta \boldsymbol{\sigma}_3$$

- Action of the various operators now

$$\begin{aligned} \hat{\gamma}_\mu|\psi\rangle &\leftrightarrow \gamma_\mu \psi \gamma_0 \\ i|\psi\rangle &\leftrightarrow \psi I \boldsymbol{\sigma}_3 \\ \hat{\gamma}_5|\psi\rangle &\leftrightarrow \psi \boldsymbol{\sigma}_3 \end{aligned}$$

Imaginary structure still a bivector

Dirac equation $\nabla \psi I \boldsymbol{\sigma}_3 - eA\psi = m\psi\gamma_0$

Comments

- Dirac equation based on the spacetime vector derivative
- Same as the Maxwell equation, so similar propagator structure
- Electromagnetic coupling from gauge principal
- Plane wave states have

$$p\psi = m\psi\gamma_0$$

A boost plus a rotation

Observables

- Observables are
  1. Gauge invariant
  2. Transform covariantly under Lorentz group $\psi \mapsto R\psi$

Bilinear covariant	Standard form	STA equivalent	Frame-free form
Scalar	$\langle\bar{\psi}\vert\psi\rangle$	$\langle\psi\tilde{\psi}\rangle$	$\rho\cos(\beta)$
Vector	$\langle\bar{\psi}\vert\hat{\gamma}_\mu\vert\psi\rangle$	$\gamma_\mu\cdot(\psi\gamma_0\tilde{\psi})$	$\psi\gamma_0\tilde{\psi}=J$
Bivector	$\langle\bar{\psi}\vert i\hat{\gamma}_{\mu\nu}\vert\psi\rangle$	$(\gamma_\mu\wedge\gamma_\nu)\cdot(\psi I\boldsymbol{\sigma}_3\tilde{\psi})$	$\psi I\boldsymbol{\sigma}_3\tilde{\psi}=S$
Pseudovector	$\langle\bar{\psi}\vert\hat{\gamma}_\mu\hat{\gamma}_5\vert\psi\rangle$	$\gamma_\mu\cdot(\psi\gamma_3\tilde{\psi})$	$\psi\gamma_3\tilde{\psi}=s$
Pseudoscalar	$\langle\bar{\psi}\vert i\hat{\gamma}_5\vert\psi\rangle$	$\langle\psi\tilde{\psi}I\rangle$	$-\rho\sin(\beta)$

Current

- Main observable is the Dirac current

$$J = \psi\gamma_0\tilde{\psi}$$

- Satisfies the conservation equation

$$\nabla \cdot J = 0$$

- Understand the observable better by writing

$$\psi\tilde{\psi} = \rho e^{I\beta} \qquad \psi = \rho^{1/2} e^{I\beta/2} R$$

$$J = \rho R \gamma_0 \tilde{R}$$

A Lorentz transformation

Current 2

- The Dirac current has a wider symmetry group than U(1).
- Take $\psi \mapsto \psi M$
- Require $M\gamma_0\tilde{M} = \gamma_0$
- Four generators satisfy this requirement

$$I\boldsymbol{\sigma}_1, \quad I\boldsymbol{\sigma}_2, \quad I\boldsymbol{\sigma}_3, \quad I$$

- Arbitrary transform

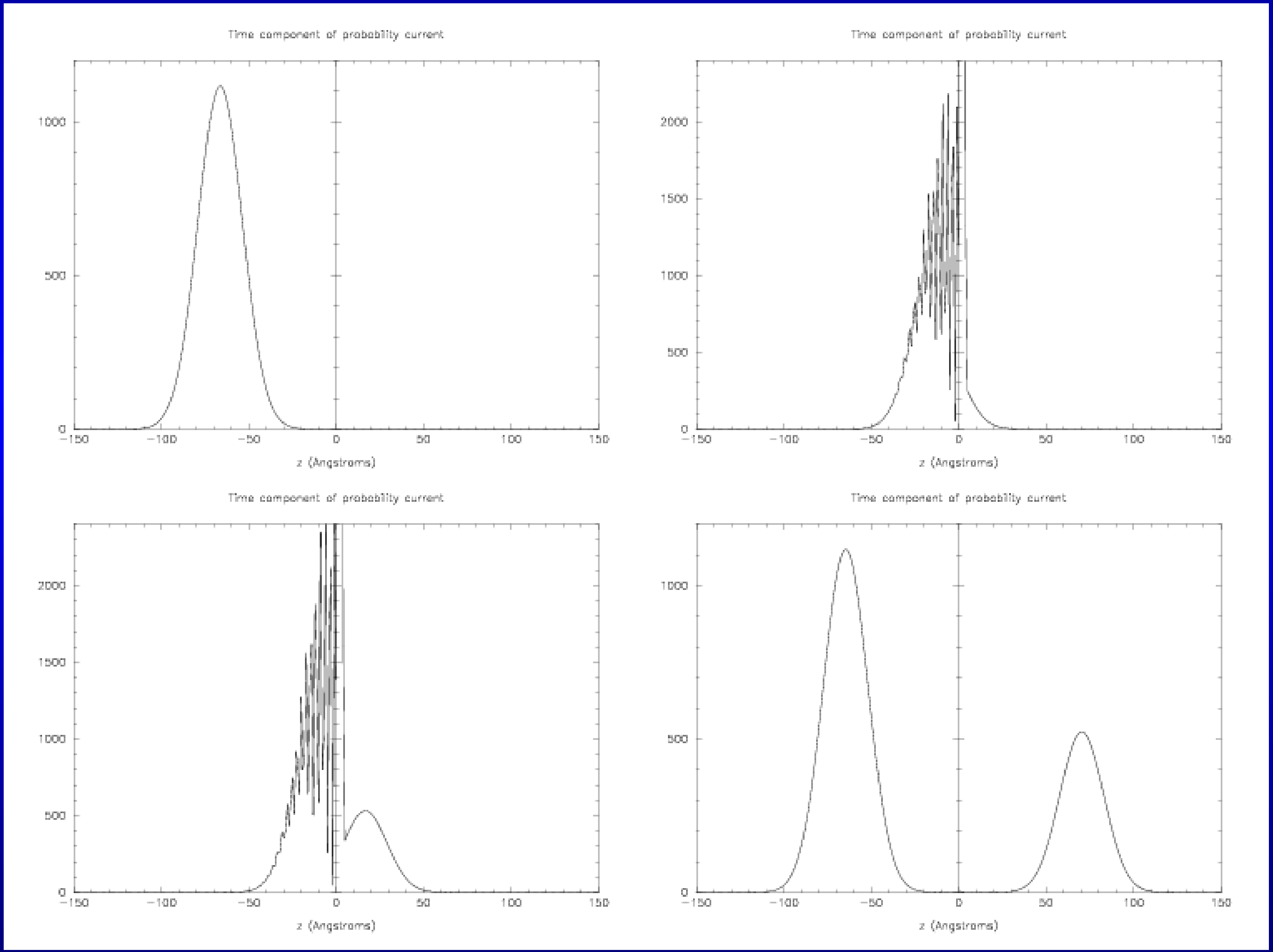
$$\psi \mapsto \psi e^{I\boldsymbol{b}} e^{I\phi}$$

SU(2) U(1)

Streamlines

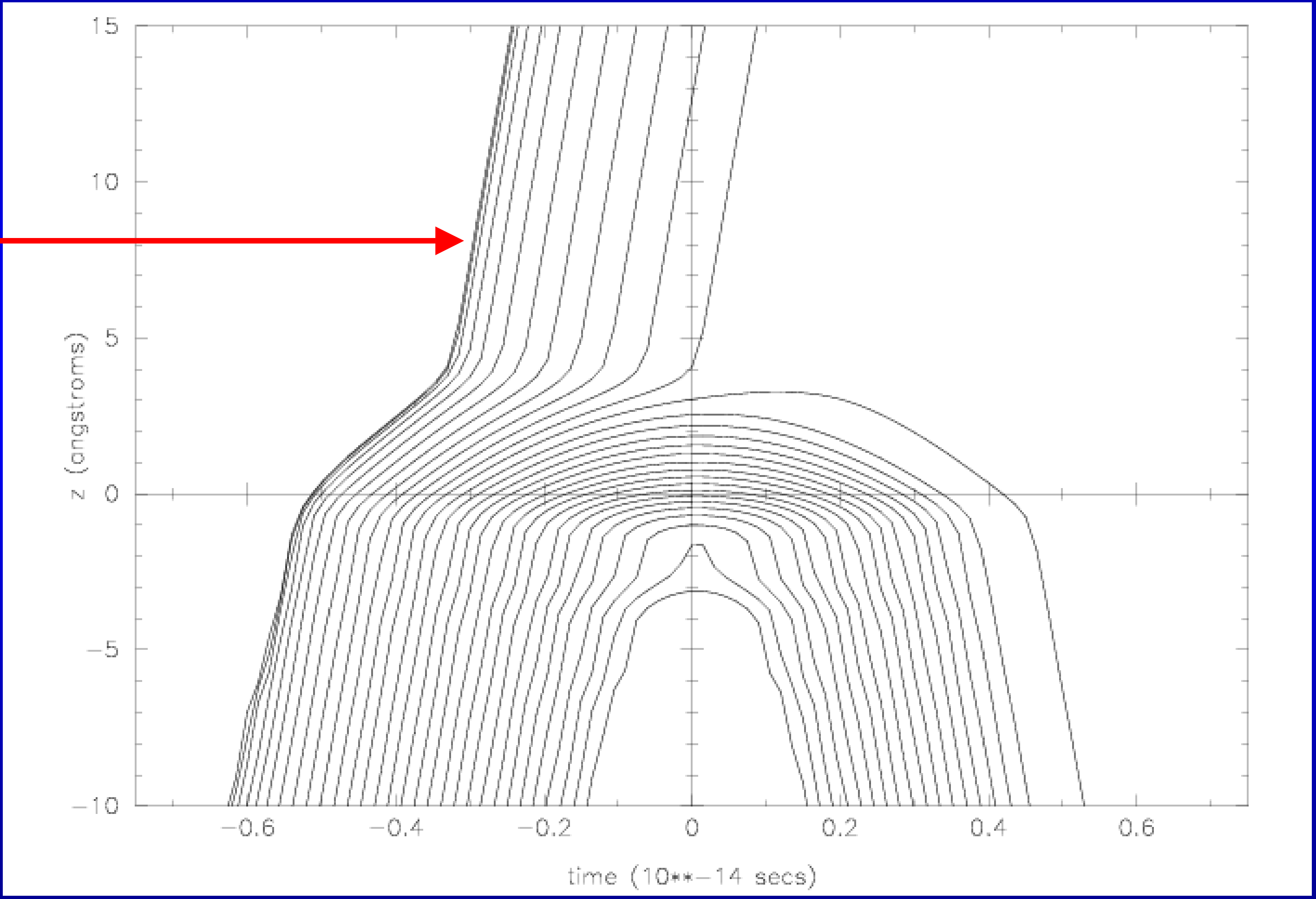
- The conserved current tells us where the probability density flows
- Makes sense to plot current streamlines
- These are genuine, local observables
- Not the same as following a Bohmian interpretation
- No need to insist that a ‘particle’ actually follows a given streamline
- Tunnelling is a good illustration

Tunnelling


Time component of probability current
z (Angstroms)
Time component of probability current
z (Angstroms)
Time component of probability current
z (Angstroms)
Time component of probability current
z (Angstroms)

Streamlines

Only front of the packet gets through



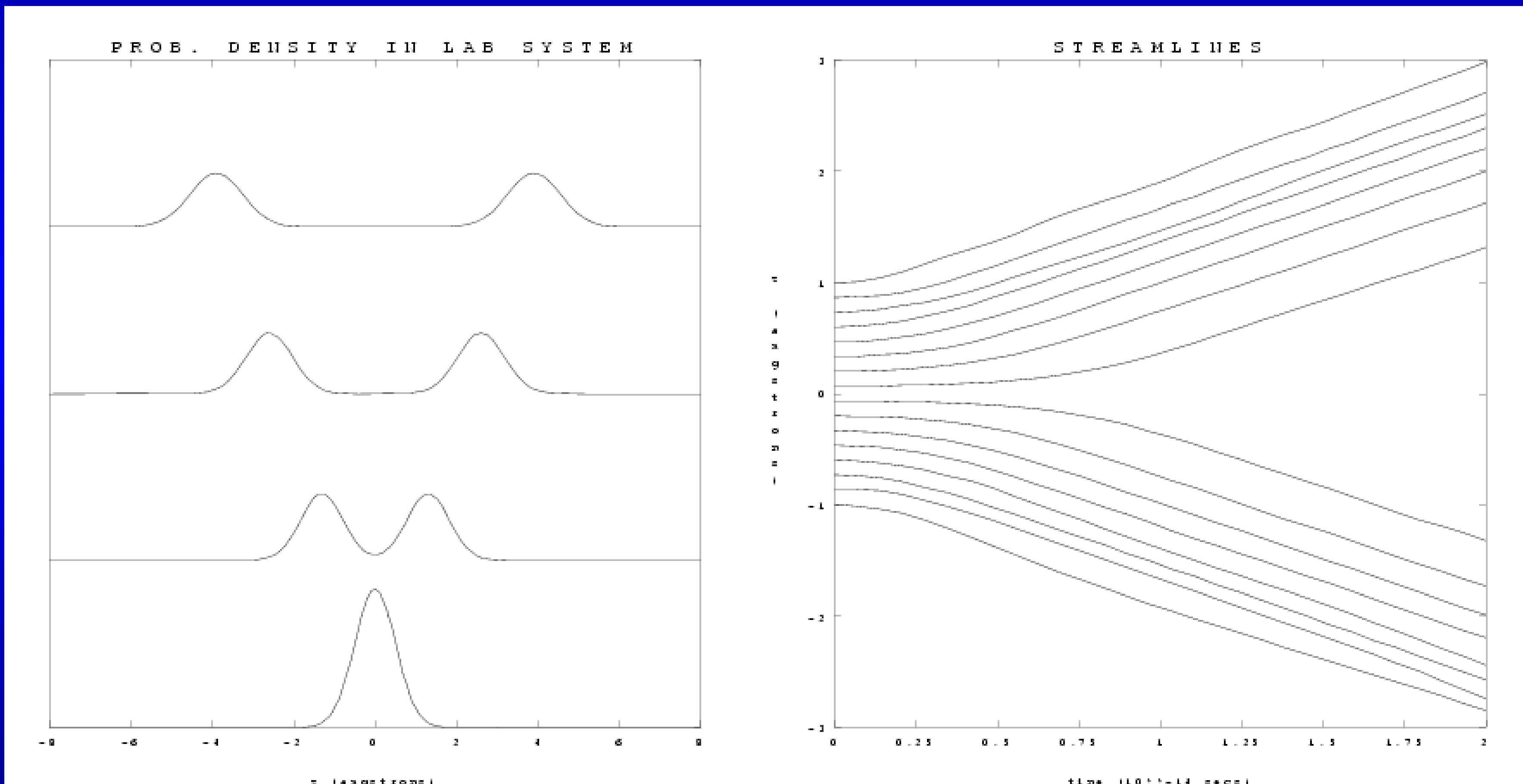
Spin Vector

- The Dirac spin observable is

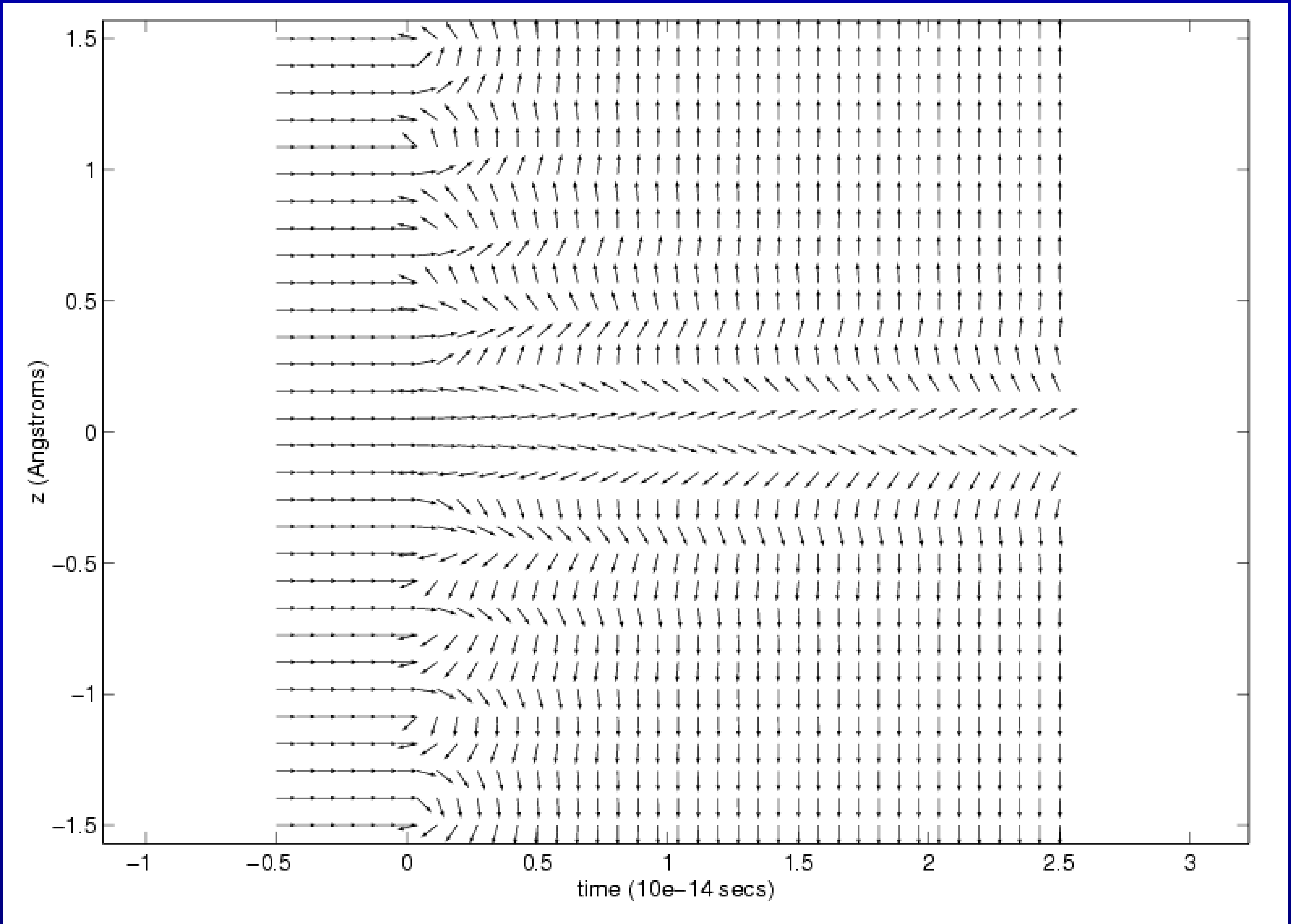
$$s = \psi \gamma_3 \tilde{\psi} = \rho R \gamma_0 \tilde{R}$$

- Same structure as used in classical model
- Use a 1D wavepacket to simulate a spin measurement
- Magnetic field simulated by a delta function shock
- Splits the initial packet into 2

Streamlines



Spin Orientation



Multiparticle Space

- Now suppose we want to describe *n* particles.
- View their trajectories as a path in 4*n* dimensional configuration space
- The vectors generators of this space satisfy

$$\gamma_\mu^a \gamma_\nu^b + \gamma_\nu^b \gamma_\mu^a = \begin{cases} 0 & a \neq b \\ 2\eta_{\mu\nu} & a = b \end{cases}$$

- Generators from different spaces anticommute
- These give a means of projecting out individual particle species

N-Particle Bivectors

- Now form the relative bivectors from separate spaces $\sigma_i^a = \gamma_i^a \gamma_0^a$
- These satisfy

$$\begin{aligned}\boldsymbol{\sigma}_i^1 \boldsymbol{\sigma}_j^2 &= \gamma_i^1 \gamma_0^1 \gamma_j^2 \gamma_0^2 = \gamma_i^1 \gamma_j^2 \gamma_0^2 \gamma_0^1 \\ &= \gamma_j^2 \gamma_0^2 \gamma_i^1 \gamma_0^1 = \boldsymbol{\sigma}_j^2 \boldsymbol{\sigma}_i^1\end{aligned}$$

- Bivectors from different spaces commute
- This is the GA implementation of the tensor product
- ‘Explains’ the nature of multiparticle Hilbert space

Complex Structure

- In quantum theory, states all share a single complex structure.
- So in GA, 2 particle quantum states must satisfy $\psi(I\boldsymbol{\sigma}_3)^1 = \psi(I\boldsymbol{\sigma}_3)^2$

$$\psi = -\psi(I\boldsymbol{\sigma}_3)^1(I\boldsymbol{\sigma}_3)^2 = \psi\frac{1}{2}\left(1-(I\boldsymbol{\sigma}_3)^1(I\boldsymbol{\sigma}_3)^2\right)$$

- Define the 2 particle correlator

$$E = \frac{1}{2}\left(1-(I\boldsymbol{\sigma}_3)^1(I\boldsymbol{\sigma}_3)^2\right)$$

$$E^2 = E$$

2-Particle States

- Correlator ensures that 2-particle states have 8 real degrees of freedom
- A 2-particle direct-product state is

$$|\psi,\phi\rangle \leftrightarrow \psi^1\phi^2 E$$

- Action of imaginary is

$$\begin{aligned} i|\psi,\phi\rangle \leftrightarrow \psi^1\phi^2 E(I\boldsymbol{\sigma}_3)^1 &= \psi^1\phi^2 E(I\boldsymbol{\sigma}_3)^2 \\ &= \psi^1\phi^2 J \end{aligned}$$

- Complex structure now generated by J

$$J^2 = -E$$

Operators

- Action of 2-particle Pauli operators

$$\begin{aligned}
\hat{\sigma}_k \otimes \hat{I}\,|\psi\rangle &\leftrightarrow -(I\boldsymbol{\sigma}_k)^1\psi J \\
\hat{\sigma}_k \otimes \hat{\sigma}_l\,|\psi\rangle &\leftrightarrow -(I\boldsymbol{\sigma}_k)^1\,(I\boldsymbol{\sigma}_l)^2\psi E \\
\hat{I} \otimes \hat{\sigma}_k\,|\psi\rangle &\leftrightarrow -(I\boldsymbol{\sigma}_k^2)\psi J
\end{aligned}$$

- Inner product

$$\langle\psi|\phi\rangle \leftrightarrow (\psi,\phi)_q = 2\langle\phi E\tilde{\psi}\rangle - 2\langle\phi J\tilde{\psi}\rangle i$$

Generic symbol for complex structure

- Examples

$$\langle\psi|\,\hat{\sigma}_k \otimes \hat{I}\,|\psi\rangle \leftrightarrow -2(I\boldsymbol{\sigma}_k)^1\cdot(\psi J\tilde{\psi})$$

$$\langle\psi|\,\hat{\sigma}_j \otimes \hat{\sigma}_k\,|\psi\rangle \leftrightarrow -2\Big((I\boldsymbol{\sigma}_j)^1\,(I\boldsymbol{\sigma}_k)^2\Big)\cdot(\psi E\tilde{\psi})$$

Density Matrices

- A normalised 2 particle density matrix can be expressed as

$$\hat{\rho} = \frac{1}{4}\left(\hat{I}\otimes\hat{I} + a_k\,\hat{\sigma}_k\otimes\hat{I} + b_k\,\hat{I}\otimes\hat{\sigma}_k + c_{jk}\,\hat{\sigma}_j\otimes\hat{\sigma}_k\right)$$

- So, for example

$$a_k = \mathrm{tr}\left(\hat{\rho}(\hat{\sigma}_k\otimes\hat{I})\right) = -2(I\boldsymbol{\sigma}_k)^1\cdot(\psi J\tilde{\psi})$$

- All of the information in the density matrix is held in the observables

$$\psi E\tilde{\psi} \qquad\qquad \psi J\tilde{\psi}$$

Inner products and traces

- Can write the overlap probability as

$$P(\psi,\phi) = |\langle\psi|\phi\rangle|^2 = \mathrm{tr}(\hat{\rho}_\psi \hat{\rho}_\phi)$$

- So have, for n-particle pure states

$$P(\psi,\phi) = 2^{n-2}\Big(\big\langle(\psi E\tilde{\psi})(\phi E\tilde{\phi})\big\rangle - \big\langle(\psi J\tilde{\psi})(\phi J\tilde{\phi})\big\rangle\Big)$$

- The partial trace operation corresponds to forming the observables, and throwing out terms
- Clearly see how this is removing information
- For mixed states, can correlate on pseudoscalar

Schmidt Decomposition

- General way to handle 2-particle states is to write as a matrix and perform an SVD

$$|\psi\rangle = e^{i\chi}\left(\cos\left(\frac{\alpha}{2}\right) e^{\frac{i\tau}{2}} \begin{pmatrix} \cos\left(\frac{\theta_1}{2}\right) e^{\frac{i\phi_1}{2}} \\ \sin\left(\frac{\theta_1}{2}\right) e^{\frac{i\phi_1}{2}} \end{pmatrix} \otimes \begin{pmatrix} \cos\left(\frac{\theta_2}{2}\right) e^{\frac{i\phi_2}{2}} \\ \sin\left(\frac{\theta_2}{2}\right) e^{\frac{i\phi_2}{2}} \end{pmatrix} + \sin\left(\frac{\alpha}{2}\right) e^{\frac{-i\tau}{2}} \begin{pmatrix} \sin\left(\frac{\theta_1}{2}\right) e^{\frac{i\phi_1}{2}} \\ -\cos\left(\frac{\theta_1}{2}\right) e^{\frac{i\phi_1}{2}} \end{pmatrix} \otimes \begin{pmatrix} \sin\left(\frac{\theta_2}{2}\right) e^{\frac{i\phi_2}{2}} \\ -\cos\left(\frac{\theta_2}{2}\right) e^{\frac{i\phi_2}{2}} \end{pmatrix}\right)$$

- Only works for bipartite states

GA Form

- Define the local states / operators

$$R = \psi(\theta_1, \phi_1) e^{I\boldsymbol{\sigma}_3 \tau/4}, \qquad S = \psi(\theta_2, \phi_2) e^{I\boldsymbol{\sigma}_3 \tau/4}$$

- Result of the Schmidt decomposition can now be written

$$\psi = \rho R^1 S^2 \left(\cos(\alpha/2) + \sin(\alpha/2) I\boldsymbol{\sigma}_2^1 I\boldsymbol{\sigma}_2^2\right) e^{J\chi} E$$

Local unitaries

Entangling term

- Now have a form which generalises to arbitrary particles

3-Particle States

- The GHZ state is $$\psi = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$$
- GA equivalent $$\psi = \exp\left(\pi/4\,(I\boldsymbol{\sigma}_2)^1\,(I\boldsymbol{\sigma}_2)^2\,(I\boldsymbol{\sigma}_2)^3\right)$$
- The W-state is more interesting

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$$

- Goes to $$W = R^1 R^2 R^3\, T_{12} T_{13} T_{23}\, T_{123}(\pi/4)$$

$$R = e^{\pi/4 I\boldsymbol{\sigma}_2}$$

$$T_{12} = \cos(\pi/12) + \sin(\pi/12)(I\boldsymbol{\sigma}_2)^1\,(I\boldsymbol{\sigma}_2)^2$$

Singlet State

- An example of an entangled,or non-local, state is the 2-particle singlet state

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \leftrightarrow \psi = \frac{1}{\sqrt{2}}(I\boldsymbol{\sigma}_2^1 - I\boldsymbol{\sigma}_2^2)E$$

- This satisfies the identity

$$M^1\varepsilon = \tilde{M}^2\varepsilon$$

- Gives straightforward proof of invariance

$$R^1R^2\varepsilon = R^1\tilde{R}^1\epsilon = \varepsilon$$

- Observables are $2\varepsilon E\tilde{\varepsilon} = 1 + (I\boldsymbol{\sigma}_k)^1 (I\boldsymbol{\sigma}_k)^2$

Relativistic States

- All of the previous considerations extend immediately to relativistic states
- Can give physical definitions of entanglement for Dirac states
- Some disagreement on these issues in current literature
- Has been suggested that relative observers disagree on entanglement and purity
- More likely that an inappropriate definition has been adopted

Relativistic Singlet

- Can extend the non-relativistic state to one invariant under boosts as well

$$\eta = \varepsilon(1 - I^1 I^2)$$

- This satisfies

$$R^1 R^2 \eta = R^1 \tilde{R}^1 \eta = \eta$$

A Lorentz rotor

- This state plays an important role in GA versions of 2-spinor calculus and twistor theory

Multiparticle Dirac Equation

- Relativistic multiparticle quantum theory is a slippery subject!
- Ultimately, most issues sorted by QFT
- Can make some progress, though, e.g. with Pauli principle

$$I_P = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3, \qquad \Gamma_\mu = \frac{1}{\sqrt{2}} \left(\gamma_\mu^1 + \gamma_\mu^2 \right)$$

- Antisymmetrised state constructed via

$$\psi_-(x) = \psi(x) + I_P \psi(I_P x I_P) I_P$$

Current

- For equal mass particles, basic equation is

$$\nabla\psi J = m\psi(\gamma_0^1 + \gamma_0^2)$$

- Get a conserved current in 8D space

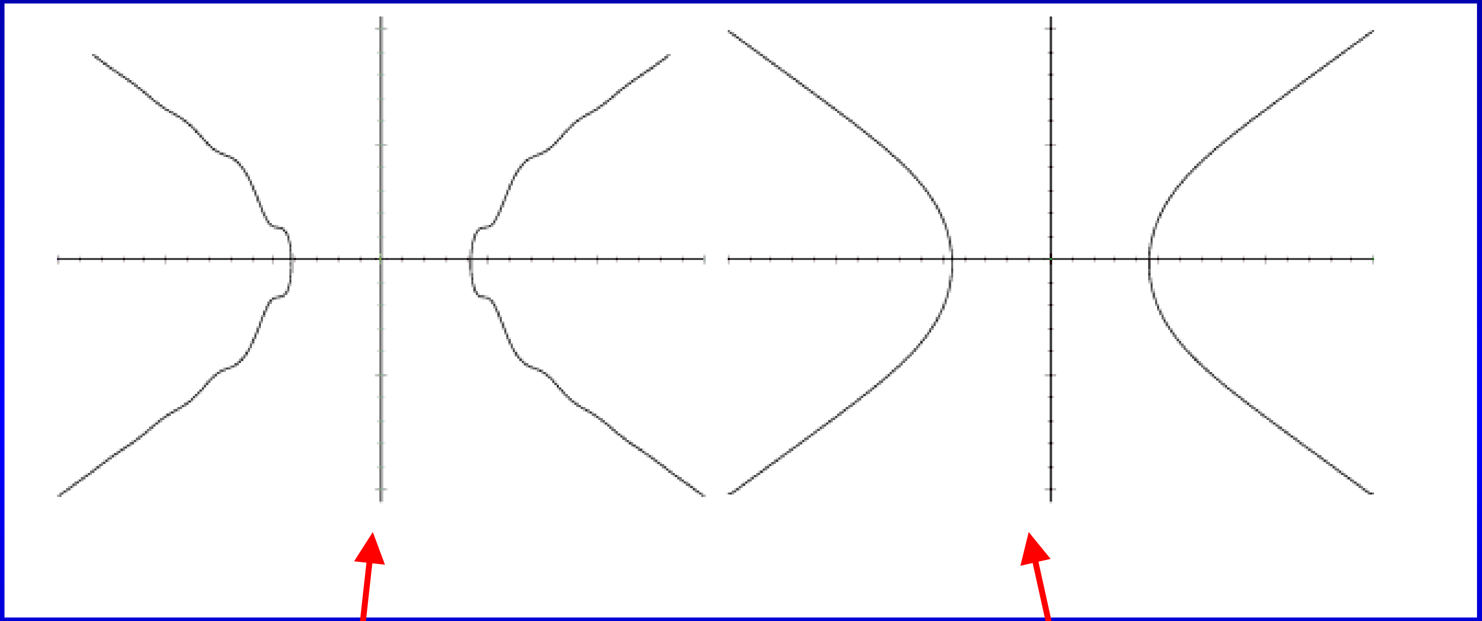
$$\mathcal{J} = \left\langle \psi(\gamma_0^1 + \gamma_0^2) \right\rangle_1$$

- Pauli principle ensures that

$$I_P \mathcal{J}(I_P x I_P) I_P = \mathcal{J}(x)$$

- Ensures that if 2 streamlines ever met, they could never separate

Plots



Spins aligned

Spins anti-aligned

In both cases the packets pass through each other

Exercises

- Verify that the overlap probability between 2 states is

$$P(\psi,\phi) = \frac{\langle(\psi E\tilde{\psi})(\phi E\tilde{\phi})\rangle - \langle(\psi J\tilde{\psi})(\phi J\tilde{\phi})\rangle}{2\langle\psi E\tilde{\psi}\rangle\langle\phi E\tilde{\phi}\rangle}$$

- Now suppose that one state is the singlet, and the other is separable. Prove that

$$P(\psi,\phi) = \langle\tfrac{1}{2}(1-P^1Q^2)\tfrac{1}{2}(1+I\sigma_k^1\, I\sigma_k^2)\rangle = \tfrac{1}{4}(1-\cos\theta)$$

Angle between the spin vectors, or between measuring apparatus