Geometric Algebra 4
2 Final Ideas

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Gravity

- Can construct gravity as a gauge theory
- Predictions fully consistent with general relativity
- General covariance replace by demand that observables are gauge invariant
- Equivalence principal replaced by minimal coupling
- No need for any differential geometry
- Can apply GA ideas directly

The Gauge Fields

- Remove the constraint that coordinate vectors are tied to a frame

$$e_\mu(x) \mapsto g_\mu(x)$$

- Generates the ‘metric’ via

$$g_\mu \cdot g_\nu = g_{\mu\nu}, \quad g_\mu \cdot g^\nu = \delta^\nu_\mu$$

- Theory invariant under local rotations

$$g_\mu \mapsto R g_\mu \tilde{R}$$

- Include gauge fields for Lorentz rotations
- Set of bivector fields
- Field strength is Riemann tensor

Dirac Equation

- The Dirac equation in a gravitational background in

$$g^\mu(\partial_\mu + \tfrac{1}{2}\Omega_\mu)\psi I \boldsymbol{\sigma}_3 = m\psi\gamma_0$$

Gauge fields

- Observables constructed in precisely the same way
- These are full covariant objects
- All observables are scalar combinations of these

Black Hole

- Gauge fields for a Schwarzschild black hole are extremely simple

'Flat' Minkowski vectors

$$g^0 = \gamma^0$$
$$g^i = \gamma^i - \boxed{\left(\frac{2GM}{r}\right)^{1/2} \frac{x^i}{r}\gamma^0}$$

Gravitational interaction

- Metric from this gauge choice is

$$ds^2 = dt^2 - \left(dr + \left(\frac{2GM}{r}\right)^{1/2} dt\right)^2 - r^2 d\Omega^2$$

Free-fall time

Dirac Equation

- Dirac equation now reduces to

$$\nabla \psi I \boldsymbol{\sigma}_3 + \gamma_0 \hat{H}_I \psi = m \psi \gamma_0$$

- All gravitational effects are contained in the scalar Hamiltonian

$$\hat{H}_I \psi = i\hbar (2GM/r)^{1/2} r^{-3/4} \partial_r (r^{3/4} \psi)$$

Free-fall velocity

Radial momentum

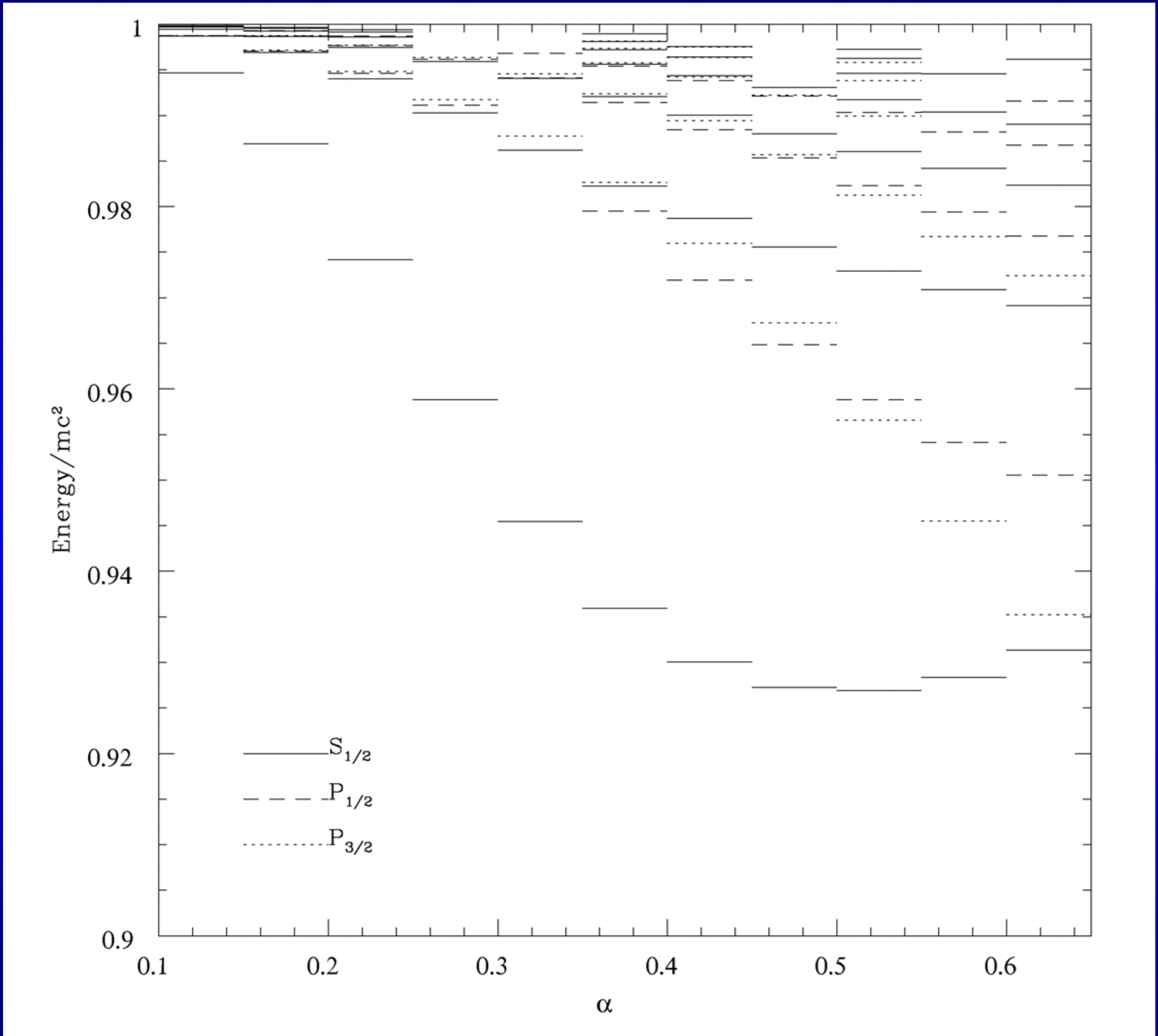
The Interaction Hamiltonian

$$\widehat{H}_I\psi = i\hbar(2GM/r)^{1/2}r^{-3/4}\partial_r(r^{3/4}\psi)$$

- All gravitational effects in a *single* term
- This is *gauge dependent*
- In all gauge theories, trick is to
  1. Find a sensible gauge
  2. Ensure that all physical predictions are gauge invariant
- Hamiltonian is scalar (no spin effects)
- Independent of particle mass
- Independent of c

Applications

- Can carry out scattering calculations using Feynman diagram techniques
- Construct the gravitational analogue of the Mott scattering formula
- Hamiltonian is non-Hermitian due to delta-function at the origin
- Describes absorption
- Compute a quantum spectrum of bound states



1
0.98
0.96
0.94
0.92
0.9
Energy/mc²
0.1
0.2
0.3
0.4
0.5
0.6
α
$S_{1/2}$
$P_{1/2}$
$P_{3/2}$

Conformal Geometry

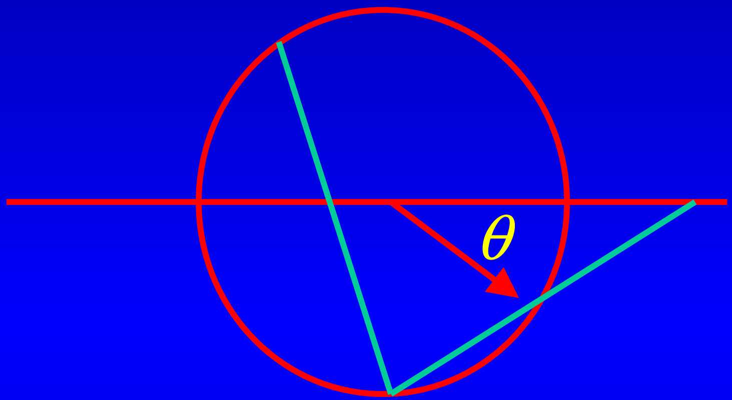
- Totally separate new application of geometric algebra
- Arose from considerations in computer graphics
- Removes deficiencies in the OpenGL use of projective geometry
- Now seen to unite themes in twistor theory, supersymmetry and cosmology

Conformal Points

- Start with the stereographic projection

$$\hat{\boldsymbol{r}} = \cos(\theta)\, e_1 + \sin(\theta)\, e_2$$

$$\cos(\theta) = \frac{2x}{1+x^2}$$



- But this representation involves a unit vector
- Seek a homogeneous representation

$$X = 2xe_1 + (1-x^2)e_2 + (1+x^2)\bar{e}$$

$$\boxed{X^2 = 0}$$

Negative norm vector

Distance Geometry

- Distance between points in conformal representation is

$$(x-y)^2 = -\frac{2X\cdot Y}{X\cdot n\, Y\cdot n}, \qquad n = (e_1 + \bar{e})$$

- Can now use rotors for general Euclidean transformations
- Construct spinors for Euclidean group
- In spacetime, these are twistors

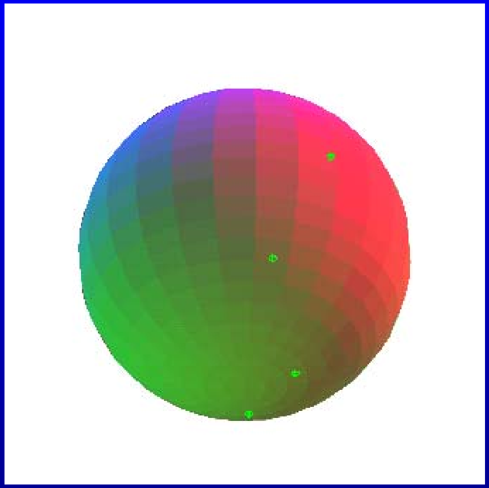
Geometric Primitives

- Take the exterior product of three points to determine a line or circle

$$\underbrace{A_1 \wedge A_2 \wedge A_3}_{\text{Trivector}} \wedge X = 0$$

- Similarly, 4 points describe a sphere

$$A_1 \wedge A_2 \wedge A_3 \wedge A_4 \wedge X = 0$$



- Can intersect lines and spheres
- Computationally very efficient
- Superior to OpenGL

Non-Euclidean Geometry

- Change the distance measure to

$$d(x,y) = 2\lambda \sin^{-1}\left(-\frac{X\cdot Y}{2X\cdot\bar{e}\,Y\cdot\bar{e}}\right)^{1/2}$$

Spherical geometry

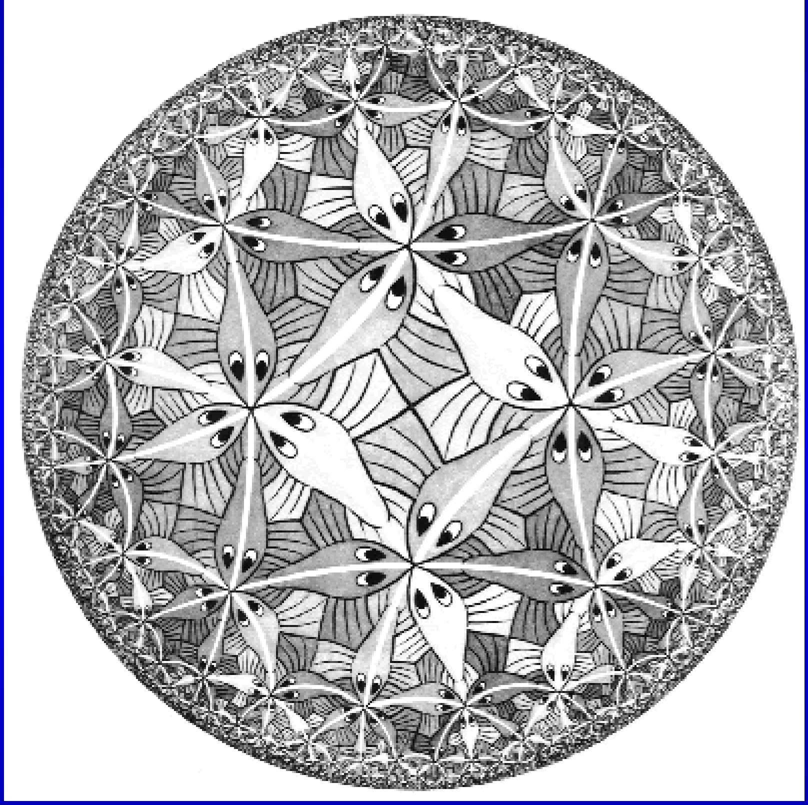
$$d(x,y) = 2\sinh^{-1}\left(-\frac{X\cdot Y}{2X\cdot e\,Y\cdot e}\right)$$

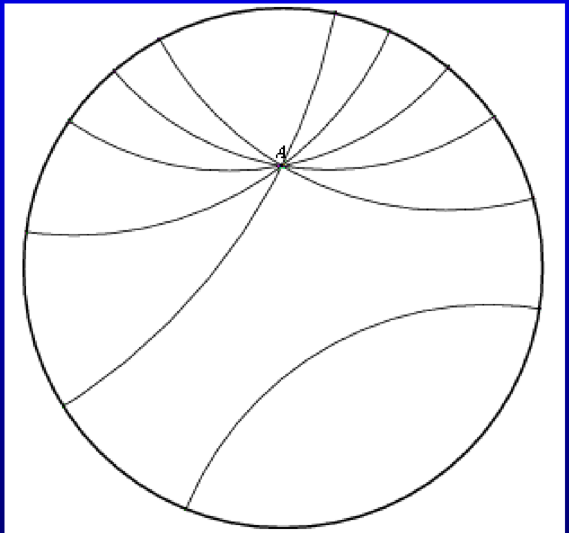
Hyperbolic Geometry

- And in spacetime get de Sitter and anti-de Sitter spaces
- All geometries united in a single framework

Hyperbolic Geometry

- Made famous by Escher prints
- Intersect points, lines in exactly the same way
- Only the distance measure changes





Applications

- Besides obvious applications to computational geometry:
- There are many links between relativistic multiparticle quantum states and conformal geometry
- Spinor representation of translations enables constructions of new quantum equations
- Strong links to cosmology and wavefunction of the universe

Resources

- A complete lecture course, including handouts, overheads and papers available from www.mrao.cam.ac.uk/~Clifford
- *Geometric Algebra for Physicists* out in March (C.U.P.)
- David Hestenes’ website modelingnts.la.asu.edu