A Multivector Derivative Approach to Lagrangian Field Theory

AUTHORS
Anthony Lasenby
Chris Doran
Stephen Gull

Found. Phys. **23**(10), 1295-1327 (1993)

Abstract

A new calculus, based upon the multivector derivative, is developed for Lagrangian mechanics and field theory, providing streamlined and rigorous derivations of the Euler-Lagrange equations. A more general form of Noether's theorem is found which is appropriate to both discrete and continuous symmetries. This is used to find the conjugate currents of the Dirac theory, where it improves on techniques previously used for analyses of local observables. General formulae for the canonical stress-energy and angular-momentum tensors are derived, with spinors and vectors treated in a unified way. It is demonstrated that the antisymmetric terms in the stress-energy tensor are crucial to the correct treatment of angular momentum. The multivector derivative is extended to provide a functional calculus for linear functions which is more compact and more powerful than previous formalisms. This is demonstrated in a reformulation of the functional derivative with respect to the metric, which is then used to recover the full canonical stress-energy tensor. Unlike conventional formalisms, which result in a symmetric stress-energy tensor, our reformulation retains the potentially important antisymmetric contribution.

1 Introduction

'Clifford Algebra to Geometric Calculus [1]' is one of the most stimulating modern textbooks of applied mathematics, full of powerful formulae waiting for physical application. In this paper we concentrate on one aspect of this book, multivector differentiation, with the aim of demonstrating that it provides the natural framework for Lagrangian field theory. In doing so we will demonstrate that the multivector derivative simplifies proofs of a number of well-known formulae, and, in the case of Dirac theory, leads to new results and insights.

The multivector derivative not only provides a systematic and rigorous method of formulating the variational principle; it is also very powerful for performing the manipulations of tensor analysis in a coordinate-free way. This power is exploited in derivations of the conserved tensors for Poincaré and conformal symmetries. While the results are not new, the clarity which geometric calculus brings to their derivations compares very favourably with the traditional techniques of tensor analysis.

A summary of geometric algebra and the multivector derivative is provided in Section 2, with some applications to point-particle mechanics given in Section 3. Section 4 then develops the main content of the paper, dealing with the application of the multivector derivative to field theory. New results include the identification of currents conjugate to continuous extensions of discrete symmetries in the Dirac equation. The derivation of their conservation equations is easier than by any previous method. We also find a bivector generalisation of the Euler homogeneity property, valid for any Poincaré-invariant theory. Derivations of the canonical stress-energy and angular-momentum tensors lead to a clear understanding of the significance of antisymmetric terms in the stress-energy tensor, which are shown to be related to the divergence of the spin bivector. Conformal transformations are also considered, and we show how non-conservation of their conjugate tensors is related to the mass term in coupled Maxwell-Dirac theory.

Finally, Section 5 introduces a generalisation of the multivector derivative, appropriate for finding the derivative with respect to a multilinear function. Some simple results are derived and are used to formalise the technique of finding the stress-energy tensor by 'functional differentiation with respect to the metric'. The new formulation clarifies the role of reparameterisation invariance in this derivation and also provides a simple proof of the equivalence (up to a total derivative) of the canonical and functional stress-energy tensors. The artificially imposed symmetry of the metric differentiation approach is seen to be unnecessary, and some implications are discussed.

2 The Multivector Derivative

In this section we provide a brief summary of geometric algebra. We will adopt the conventions of the other papers in this series (henceforth known as Paper I [2], Paper II [3] and Paper IV [4]) which are also close to those of Hestenes & Sobczyk [1].

We write (Clifford) vectors in lower case (a) and general multivectors in upper case (A) or, in the case of fields, as Greek (ψ). The space of multivectors is graded and multivectors containing elements of a single grade, r, are termed *homogeneous* and written A_r. The geometric (Clifford) product is written by simply juxtaposing multivectors AB.

We use the symbol $\langle A\rangle_r$ to denote the projection of the grade-r components of A, and write the scalar (grade-0) part simply as $\langle A\rangle$. The interior and exterior products are defined as

$$\begin{aligned} A_r\cdot B_s &= \langle A_r B_s\rangle_{|r-s|} \\ A_r\wedge B_s &= \langle A_r B_s\rangle_{r+s} \end{aligned} \tag{2.1}$$

respectively, to which we add the scalar and commutator products

$$\begin{aligned} A{*}B &= \langle AB\rangle \\ A\times B &= \tfrac{1}{2}(AB-BA). \end{aligned} \tag{2.2}$$

The operation of taking the commutator product with a bivector (a grade-2 multivector) is grade-preserving. Reversion is defined by

$$\begin{aligned} (AB)\tilde{} &= \tilde{B}\tilde{A} \\ \tilde{a} &= a \qquad \text{for any vector } a, \end{aligned} \tag{2.3}$$

and reverses the order of vectors in any given expression.

Most of this paper is concerned with relativistic field theory and uses the spacetime algebra (STA). This is the geometric algebra of spacetime, and is generated by a set of four orthonormal vectors $\{\gamma_\mu\}$, where

$$\gamma_\mu\cdot\gamma_\nu = g_{\mu\nu} = \mathrm{diag}(+\,-\,-\,-). \tag{2.4}$$

The full STA is 16-dimensional and is spanned by

$$1, \qquad \{\gamma_\mu\} \qquad \{\sigma_k,\ i\sigma_k\}, \qquad \{i\gamma_\mu\}, \qquad i, \tag{2.5}$$

where

$$i \equiv \gamma_0\gamma_1\gamma_2\gamma_3 \tag{2.6}$$

is the pseudoscalar for spacetime and

$$\sigma_k \equiv \gamma_k\gamma_0 \tag{2.7}$$

are relativistic bivectors, representing an orthonormal frame of vectors in the space relative to the time-like γ_0 direction. To distinguish between relative and spacetime vectors, we write the former in bold type.

One of the main aims of this paper is to demonstrate the generality and power of the multivector derivative [1], which we now define. The derivative with respect to a general multivector X is written as ∂_X, and is introduced by first defining the derivative in a fixed direction A as

$$A{*}\partial_X F(X) = \frac{\partial}{\partial\tau} F(X+\tau A)\bigg|_{\tau=0}. \tag{2.8}$$

An arbitrary vector basis $\{e_k\}$, with reciprocal basis $\{e^k\}$, can be extended via exterior multiplication to define a basis for the entire algebra $\{e_J\}$, where J is a general (antisymmetric) index. With the reciprocal basis $\{e^k\}$ defined by $e^k{\cdot}e_j = \delta^k_j$, the multivector derivative is now defined as

$$\partial_X = \sum_J e^J e_J{*}\partial_X, \tag{2.9}$$

so that ∂_X inherits the multivector properties of its argument X, as well as a calculus from equation (2.8).

The most useful result for the multivector derivative is

$$\partial_X \langle XA\rangle = P_X(A), \tag{2.10}$$

where $P_X(A)$ is the projection of A on to the grades contained in X. From (2.10) it follows that

$$\begin{aligned} \partial_X \langle \tilde{X}A\rangle &= P_X(\tilde{A}) \\ \partial_{\tilde{X}} \langle \tilde{X}A\rangle &= P_X(A). \end{aligned} \tag{2.11}$$

Leibniz' rule can now be used in conjunction with (2.10) to build up results for the action of ∂_X on more complicated functions; for example

$$\partial_X \langle X\tilde{X}\rangle^{k/2} = k\langle X\tilde{X}\rangle^{(k-2)/2}\tilde{X}. \tag{2.12}$$

The multivector derivative acts on objects to its immediate right unless brackets are present, as in $\partial_X(AB)$ where ∂_X acts on both A and B. If ∂_X is only intended to act on B we write this as $\dot{\partial}_X A\dot{B}$, where the overdot denotes the multivector on which the derivative acts. Leibniz' rule can now be given in the form

$$\partial_X(AB) = \dot{\partial}_X \dot{A}B + \dot{\partial}_X A\dot{B}. \tag{2.13}$$

These conventions apply equally if the derivative is taken with respect to a scalar, where the overdot notation remains a useful way of encoding partial derivatives. In situations where the overdots could be confused with time derivatives, we replace the former with overstars.

The derivative with respect to spacetime position x is called the *vector derivative*, and is given the symbol

$$\partial_x = \nabla = \nabla_x. \tag{2.14}$$

Two useful results are

$$\begin{array}{rcl} \dot{\nabla}(\dot{x}\cdot A_r) & = & rA_r \\ \dot{\nabla}(\dot{x}\wedge A_r) & = & (n-r)A_r, \end{array} \tag{2.15}$$

where n is the dimension of the space. The left equivalent of ∇ is written as $\overleftarrow{\nabla}$ and acts on multivectors to its immediate left, although it is not always necessary to use $\overleftarrow{\nabla}$ since we can use the overdot notation to write $A\overleftarrow{\nabla}$ as $\dot{A}\dot{\nabla}$. The operator $\overleftrightarrow{\nabla}$ acts both to its left and right, and is usually taken as acting on everything within a given expression, for example

$$A\overleftrightarrow{\nabla}B = \dot{A}\dot{\nabla}B + A\dot{\nabla}\dot{B}. \tag{2.16}$$

Finally, we need a notation for dealing with functions of multivectors. If $F(X)$ is a multivector-valued function of X (not necessarily linear) we write

$$A{*}\partial_X F(X) = \underline{F}_X(A) = \underline{F}(A), \tag{2.17}$$

which is a linear function of A (the X-dependence is usually suppressed). The adjoint to $\underline{F}$ is defined via the multivector derivative as

$$\overline{F}(B) = \partial_A \langle \underline{F}(A) B \rangle. \tag{2.18}$$

It follows that

$$\langle A \overline{F}(B) \rangle = \langle \underline{F}(A) B \rangle. \tag{2.19}$$

A symmetric function is one for which $\underline{F} = \overline{F}$.

If $f(x)$ is a function which maps between spacetime points, we define the differential

$$\underline{f}(a) = a \cdot \nabla f(x), \tag{2.20}$$

which is a linear function mapping vectors to vectors. This is extended to act on all multivectors through the definition

$$\underline{f}(a \wedge b \wedge \ldots \wedge c) = \underline{f}(a) \wedge \underline{f}(b) \ldots \wedge \underline{f}(c), \tag{2.21}$$

from which the determinant is defined as

$$\underline{f}(I) = \det(\underline{f}) I, \tag{2.22}$$

where I is the highest-grade element (pseudoscalar) for the algebra. For linear vector functions, equation (2.19) has the useful extensions

$$\begin{aligned} A_r \cdot \overline{f}(B_s) &= \overline{f}[\underline{f}(A_r) \cdot B_s] \qquad r \leq s \\ \underline{f}(A_r) \cdot B_s &= \underline{f}[A_r \cdot \overline{f}(B_s)] \qquad r \geq s, \end{aligned} \tag{2.23}$$

from which the inverse functions can be constructed:

$$\begin{aligned} \underline{f}^{-1}(A) &= \det(f)^{-1} \overline{f}(AI) I^{-1} \\ \overline{f}^{-1}(A) &= \det(f)^{-1} I^{-1} \underline{f}(IA). \end{aligned} \tag{2.24}$$

These are all the definitions and conventions we require; further details and proofs can be found in Hestenes & Sobczyk [1].

3 Point-Particle Lagrangians

Before turning to field theory, it is instructive to see how the formalism of Section 2 applies to the simpler case of classical mechanics. This analysis introduces some of the concepts needed in later sections as well as demonstrating how the multivector derivative can extend classical mechanics through the use of multivector-parameterised symmetries.

To illustrate these techniques, we shall treat a classical model for a spin-$\frac{1}{2}$ fermion. This is a useful preliminary to the full Dirac theory and also demonstrates that internal spin-$\frac{1}{2}$ has a satisfactory classical formulation without the introduction of Grassmann variables [5, 6].

3.1 Euler-Lagrange Equations and Noether's Theorem

Consider a scalar-valued function

$$L = L(\psi_i, \dot{\psi}_i), \tag{3.1}$$

where ψ_i are a set of general multivectors, and $\dot{\psi}_i$ denotes differentiation with respect to some scalar parameter, which we will usually take to be time. We shall assume here that L is not a function of time explicitly, and depends on time only through ψ_i and $\dot{\psi}_i$.

We wish to extremise the action

$$S = \int_{t_1}^{t_2} dt L(\psi_i, \dot{\psi}_i) \tag{3.2}$$

with respect to ψ_i. We write the variables ψ_i in the form [7]

$$\psi_i(t) = \psi_i^0(t) + \epsilon\phi_i(t), \tag{3.3}$$

where ϕ_i is a multivector of the same grade(s) as ψ_i, ϵ is a scalar, and ψ_i^0 represents the extremal path. We now take the derivative with respect to the parameter ϵ and find that

$$\begin{aligned} \partial_\epsilon L &= \int_{t_1}^{t_2} dt \left(\phi_i * \partial_{\psi_i} L + \dot{\phi}_i * \partial_{\dot{\psi}_i} L\right) \\ &= \int_{t_1}^{t_2} dt \phi_i * \left(\partial_{\psi_i} L - \partial_t(\partial_{\dot{\psi}_i} L)\right), \end{aligned} \tag{3.4}$$

where ∂_{ψ_i} is the multivector derivative with respect to ψ_i. For the action to be stationary, (3.4) must vanish for all ϕ_i, and we can read off the Euler-Lagrange equations

$$\partial_{\psi_i} L - \partial_t(\partial_{\dot{\psi}_i} L) = 0. \tag{3.5}$$

These extend naturally if higher-order derivatives are present. Equation (3.5) could alternatively have been derived by decomposing ψ_i in an explicit basis, varying each component separately, and recombining the separate equations. The multivector derivative approach is manifestly quicker, more elegant and leads to a clearer understanding of the role of the variables in any Lagrangian.

The multivector derivative facilitates a more general version of Noether's theorem, by allowing for transformations parameterised by multivectors. This makes it possible to derive conserved quantities conjugate to discrete symmetries, something which cannot be done if only infinitesimal transformations are considered. The standard results for scalar-parameterised transformations are a special case of this more general result. Any transformation written in geometric algebra is necessarily *active*, because the freedom from coordinates in geometric algebra prevents us from writing down passive transformations. Passive transformations can accordingly be eliminated from physics altogether and we contend that they should be.

The most general transformation parameterised by a single multivector M is

$$\psi_i' = f(\psi_i, M), \tag{3.6}$$

where f and M are, respectively, time-independent functions and multivectors. The transformation f need not be grade-preserving, and can therefore provide an analogue to supersymmetric transformations [5]. The symmetries we consider here will preserve grade, however, as their associated geometry is much clearer. The differential notation of Section 2 is helpful at this point and we define

$$\underline{f}_A(\psi_i, M) = A{*}\partial_M f(\psi_i, M). \tag{3.7}$$

Defining now

$$L' = L(\psi_i', \dot{\psi_i}'), \tag{3.8}$$

we have

$$\begin{aligned} A{*}\partial_M L' &= \underline{f}_A(\psi_i, M){*}\partial_{\psi_i'} L' + \underline{f}_A(\dot{\psi}_i, M){*}\partial_{\dot{\psi}_i'} L' \\ &= \underline{f}_A(\psi_i, M){*}\Big(\partial_{\psi_i'} L' - \partial_t(\partial_{\dot{\psi}_i'} L')\Big) + \partial_t\left(\underline{f}_A(\psi_i, M){*}\partial_{\dot{\psi}_i'} L'\right). \end{aligned} \tag{3.9}$$

If we now assume that the equations of motion are satisfied for ψ_i' (an assumption which must be confirmed in any given instance) it follows that

$$A{*}\partial_M L' = \partial_t\left(\underline{f}_A(\psi_i, M){*}\partial_{\dot{\psi}_i'} L'\right). \tag{3.10}$$

We now differentiate out the A-dependence, yielding

$$\begin{aligned} \partial_M L' &= \partial_t \left(\partial_A \underline{f}_A(\psi_i, M) * \partial_{\dot{\psi}_i'} L' \right) \\ &= \partial_t \left(\overset{*}{\partial}_M \overset{*}{\psi}_i{}' * \partial_{\dot{\psi}_i'} L' \right), \end{aligned} \tag{3.11}$$

where, as mentioned in Section 2, we have employed overstars rather than overdots to avoid confusion with time derivatives. If L' is independent of M the quantity $\partial_A \underline{f}_A(\psi_i, M) * \partial_{\dot{\psi}_i'} L'$ is conserved, although both forms in (3.11) are useful in practice. Equation (3.11) is the general result, appropriate to any transformation parameterised by a multivector. These can include discrete symmetries, such as reflections, and our result therefore extends the conventional theory based on infinitesimal transformations [7].

If M is a scalar parameter, α, say, (3.11) reduces to the more familiar form

$$\partial_\alpha L' = \partial_t \left((\partial_\alpha \psi_i') * \partial_{\dot{\psi}_i'} L' \right), \tag{3.12}$$

and if L' is α-dependent, useful results are still obtained by setting $\alpha = 0$:

$$\partial_\alpha L'|_{\alpha=0} = \partial_t \left((\partial_\alpha \psi_i') * \partial_{\dot{\psi}_i'} L' \right)\Big|_{\alpha=0}. \tag{3.13}$$

As an application of this we consider time-translation, for which

$$\psi_i'(t, \alpha) = \psi_i(t + \alpha) \tag{3.14}$$

$$\Rightarrow \partial_\alpha \psi_i'|_{\alpha=0} = \dot{\psi}_i. \tag{3.15}$$

If all t-dependence enters L through the dynamical variables only, equation (3.13) gives

$$\partial_t L = \partial_t (\dot{\psi}_i * \partial_{\dot{\psi}_i} L), \tag{3.16}$$

and we define the conserved Hamiltonian as

$$H = \dot{\psi}_i * \partial_{\dot{\psi}_i} L - L. \tag{3.17}$$

Many of the results in this section generalise to to the case where the Lagrangian is multivector-valued [5, 8]. 'Multivector Lagrangians' allow for large numbers of coupled scalar Lagrangians to be combined into a single entity and both the Euler-Lagrange equations and Noether's theorem have satisfactory formulations in this case.

3.2 Point-Particle Lagrangians with Spin

As an interesting application of these results, we consider the classical model for spin-$\frac{1}{2}$ particles [5] introduced by Barut & Zanghi [9] (this has also been analysed previously by one of us [10]). The Lagrangian contains spinor variables and can be written in the STA as [3]

$$L = \langle \dot{\psi} i\sigma_3 \tilde{\psi} + p(\dot{x} - \psi\gamma_0\tilde{\psi}) + eA(x)\psi\gamma_0\tilde{\psi}\rangle. \tag{3.18}$$

Our dynamical variables are x, p and ψ, where ψ is an even multivector, and the dot denotes differentiation with respect to some arbitrary parameter τ. In order to derive the equations of motion we first consider the ψ equation,

$$\begin{aligned} \partial_\tau(i\sigma_3\tilde{\psi}) &= -i\sigma_3\dot{\tilde{\psi}} - 2\gamma_0\tilde{\psi}p + 2\gamma_0\tilde{\psi}A \\ \Rightarrow \dot{\psi} i\sigma_3 &= P\psi\gamma_0, \end{aligned} \tag{3.19}$$

where $P = p - eA$ and we have used (2.10). In deriving (3.19) there is no pretence that ψ and $\tilde{\psi}$ are independent variables — we have just one variable and everything else is taken care of by the multivector derivative.

The p equation is simple:

$$\dot{x} = \psi\gamma_0\tilde{\psi}, \tag{3.20}$$

although since $\dot{x}^2 = \rho^2$ is not, in general, equal to 1, τ cannot necessarily be viewed as the proper time for the particle.

The x equation is

$$\begin{aligned} \dot{p} &= e\nabla A(x)\cdot(\psi\gamma_0\tilde{\psi}) \\ &= e(\nabla\wedge A)\cdot\dot{x} + e\dot{x}\cdot\nabla A \\ \Rightarrow \dot{P} &= eF\cdot\dot{x}. \end{aligned} \tag{3.21}$$

We can now use (3.13) to derive some consequences for this model. The Hamiltonian is given by

$$\begin{aligned} H &= \dot{x}*\partial_{\dot{x}}L + \dot{\psi}*\partial_{\dot{\psi}}L - L \\ &= P\cdot\dot{x}, \end{aligned} \tag{3.22}$$

and is conserved absolutely. The 4-momentum and angular momentum are conserved only if $A=0$, when (3.18) reduces to the free-particle Lagrangian

$$L_0 = \langle \dot{\psi} i\sigma_3 \tilde{\psi} + p(\dot{x} - \psi\gamma_0\tilde{\psi})\rangle. \tag{3.23}$$

The 4-momentum is found by considering translations,

$$x' = x + \alpha a, \tag{3.24}$$

and is simply p. The component of p in the $\dot{x}$ direction gives the energy (3.22). The angular momentum is found by considering rotational invariance, so we set

$$\begin{aligned} x' &= e^{\alpha B/2} x e^{-\alpha B/2} \\ p' &= e^{\alpha B/2} p e^{-\alpha B/2} \\ \psi' &= e^{\alpha B/2} \psi \end{aligned} \tag{3.25}$$

(spinors have a single-sided transformation law under rotations) in which case L_0' is independent of α. It follows that the quantity

$$(B\cdot x)*\partial_{\dot{x}} L_0 + \tfrac{1}{2}(B\psi)*\partial_{\dot{\psi}} L_0 = B\cdot(x\wedge p + \tfrac{1}{2}\psi i\sigma_3\tilde{\psi}) \tag{3.26}$$

is conserved for arbitrary B. The angular momentum is therefore $p\wedge x - \frac{1}{2}\psi i\sigma_3\tilde{\psi}$, which exhibits the required spin-$\frac{1}{2}$ behaviour. The factor of $\frac{1}{2}$ originates from the transformation law (3.25).

We can also consider transformations in which the spinor is acted on to the right. These correspond to gauge transformations, though a wider class is now available than for the standard column-spinor formulation. These transformations quickly yield interesting results when used in conjunction with (3.13). For example,

$$\psi' = \psi e^{\alpha i\sigma_3} \tag{3.27}$$

can be used to show that $\langle\psi\tilde{\psi}\rangle$ is constant, and

$$\psi' = \psi e^{\alpha\sigma_3} \tag{3.28}$$

leads to the equation

$$\partial_\tau \langle i\psi\tilde{\psi}\rangle = -2P\cdot(\psi\gamma_3\tilde{\psi}). \tag{3.29}$$

These may be combined to give

$$\partial_\tau(\psi\tilde{\psi}) = 2iP\cdot(\psi\gamma_3\tilde{\psi}). \tag{3.30}$$

Finally, the duality transformation

$$\psi' = \psi e^{\alpha i} \tag{3.31}$$

yields

$$2\langle\dot{\psi}\sigma_3\tilde{\psi}\rangle = \partial_\tau\langle\psi\sigma_3\tilde{\psi}\rangle = 0. \tag{3.32}$$

In fact, the Lagrangian (3.18) is unsatisfactory for a number of reasons. It is not reparameterisation-invariant, so that it is not possible to define a proper time; it is not gauge-invariant; and it predicts a zero gyromagnetic moment [10]. Indeed, it is clear from (3.21) that something already has gone wrong, since we expect to see $\dot{p}$ rather than $\dot{P}$ coupling to $F\cdot x$. However, the derivation of a suitable angular momentum is sufficient reason to continue constructing Lagrangians of this type, in an effort to find one having all the required properties. This subject will be taken further in a later paper.

4 Field Theory

The potential of the multivector derivative is more fully realised when the formalism of Section 3.1 is extended to encompass field theory. It provides great formal clarity by allowing spinors and tensors to be treated in a unified way (*c.f.* the approach of Belinfante [11]) and it inherits the computational advantages of geometric algebra. This is manifest in derivations of the stress-energy and angular-momentum tensors for Maxwell and coupled Maxwell-Dirac theory. The formalism provides a clearer understanding of the role of antisymmetric terms in the stress-energy tensor, and their relation to spin.

Noether's theorem is also formulated in terms of the multivector derivative, and this is used to derive new conjugate currents in Dirac theory, using the spacetime algebra approach to spinors described in Paper II [3]. This greatly simplifies the derivations of many results for local observables in the Dirac theory.

4.1 Euler-Lagrange Equations and Noether's Theorem

In this section we will restrict our attention to relativistic field theory (though the results are easily reproduced for the non-relativistic case). Consider a scalar-valued Lagrangian density

$$\mathcal{L} = \mathcal{L}(\psi_i, \nabla\psi_i), \tag{4.1}$$

where ψ_i is a multivector. Here we have assumed that $\mathcal{L}$ can be written as a function of ψ_i and $\nabla\psi_i$ only, which must be confirmed; an example is provided by electromagnetism in Section 4.3. The action is defined as

$$S = \int |d^4x| \mathcal{L}, \tag{4.2}$$

where $|d^4x|$ is the invariant measure. Proceeding as in Section 3.1, we write

$$\psi_i(x) = \psi_i^0(x) + \epsilon\phi_i(x), \tag{4.3}$$

where ϕ_i contains the same grades as ψ_i. We now find that

$$\partial_\epsilon S = \int |d^4x| \left(\phi_i * \partial_{\psi_i}\mathcal{L} + (\nabla\phi_i) * \partial_{\nabla\psi_i}\mathcal{L}\right). \tag{4.4}$$

The last term here can be written, employing the overdot notation of Section 2, as

$$\langle \dot{\nabla}\dot{\phi}_i \partial_{\nabla\psi_i}\mathcal{L}\rangle = \nabla\cdot\langle \phi_i \partial_{\nabla\psi_i}\mathcal{L}\rangle_1 - \phi_i * \Big((\partial_{\nabla\psi_i}\mathcal{L})\overleftarrow{\nabla}\Big) \tag{4.5}$$

and, assuming the boundary term vanishes, we find that

$$\partial_\epsilon S = \int |d^4x| \phi_i * \Big(\partial_{\psi_i}\mathcal{L} - (\partial_{\nabla\psi_i}\mathcal{L})\overleftarrow{\nabla}\Big). \tag{4.6}$$

From (4.6) we can read off versions of the Euler-Lagrange equations appropriate to the multivector character of ψ_i. If ψ_i only contains grade-r terms, for example, we deduce that the grade-r part of the quantity enclosed in brackets vanishes:

$$\langle \partial_{\psi_i}\mathcal{L} - (\partial_{\nabla\psi_i}\mathcal{L})\overleftarrow{\nabla}\rangle_r = 0. \tag{4.7}$$

If, on the other hand, ψ is a general even multivector (as is the case for the Dirac equation) our Euler-Lagrange equation is

$$\partial_\psi \mathcal{L} = (\partial_{\nabla\psi}\mathcal{L})\overleftarrow{\nabla}, \tag{4.8}$$

or

$$\partial_{\tilde{\psi}}\mathcal{L} = \nabla(\partial_{(\nabla\psi)\tilde{}}\,\mathcal{L}). \tag{4.9}$$

Equation (4.7) allows for vectors, tensors and spinor variables to be handled in a single equation: a considerable unification!

Noether's theorem for field Lagrangians can also be derived in the same way as in Section 3.1. We begin by considering a general multivector-parameterised transformation,

$$\psi_i' = f(\psi_i, M). \tag{4.10}$$

With $\mathcal{L}' = \mathcal{L}(\psi_i', \nabla\psi_i')$, we have

$$\begin{aligned} A*\partial_M \mathcal{L}' &= \underline{f}_A(\psi_i, M)*\partial_{\psi_i'}\mathcal{L}' + \langle \nabla \underline{f}_A(\psi_i, M)\partial_{\nabla\psi_i'}\mathcal{L}'\rangle \\ &= \nabla\cdot\langle \underline{f}_A(\psi_i, M)\partial_{\nabla\psi_i'}\mathcal{L}'\rangle_1 + \underline{f}_A(\psi_i, M)*\Big(\partial_{\psi_i'}\mathcal{L}' - (\partial_{\nabla\psi_i'}\mathcal{L}')\overleftarrow{\nabla}\Big) \end{aligned} \tag{4.11}$$

If we now assume that the ψ_i' satisfy their equations of motion (which must again be verified) we find that

$$\partial_M \mathcal{L}' = \partial_A \nabla\cdot\langle \underline{f}_A(\psi_i, M)\partial_{\nabla\psi_i'}\mathcal{L}'\rangle_1. \tag{4.12}$$

This is the most general result. It applies even if ψ_i' is evaluated at a different spacetime point from ψ_i, when

$$\psi_i'(x) = f\left(\psi_i(h(x)), M\right). \tag{4.13}$$

If we now take M to be a scalar, α, we find that

$$\partial_\alpha \mathcal{L}' = \nabla\cdot\langle \partial_\alpha \psi_i' \partial_{\nabla\psi_i'}\mathcal{L}'\rangle_1 \tag{4.14}$$

so that, if $\mathcal{L}'$ is independent of α, the current

$$j = \langle \partial_\alpha \psi_i' \partial_{\nabla\psi_i'}\mathcal{L}'\rangle_1\Big|_{\alpha=0} \tag{4.15}$$

satisfies the conservation equation

$$\nabla\cdot j = 0. \tag{4.16}$$

An inertial frame relative to the constant time-like velocity γ_0 sees charge

$$Q = \int |d^3x| j\cdot\gamma_0 \tag{4.17}$$

as conserved with respect to its local time.

If $\mathcal{L}'$ is dependent on α, useful consequences can be derived from the important formula

$$\partial_\alpha \mathcal{L}'\big|_{\alpha=0} = \nabla\cdot\langle \partial_\alpha \psi_i' \partial_{\nabla\psi_i'}\mathcal{L}'\rangle_1\Big|_{\alpha=0}. \tag{4.18}$$

4.2 Spacetime Transformations and their Conjugate Tensors

In this section we use (4.18) to analyse the consequences of Poincaré and conformal invariance. This enables us to identify conserved stress-energy and angular-momentum tensors, while further demonstrating the effectiveness of the multivector derivative.

We first consider translations:

$$\begin{aligned} x' &= x + \alpha n \\ \psi_i'(x) &= \psi_i(x') \end{aligned} \tag{4.19}$$

and, assuming $\mathcal{L}'$ is only x-dependent through the fields, (4.18) gives

$$n\cdot\nabla\mathcal{L} = \nabla\cdot\langle n\cdot\nabla\psi_i\partial_{\nabla\psi_i}\mathcal{L}\rangle_1. \tag{4.20}$$

From this we define the adjoint to the canonical stress-energy tensor as

$$\overline{T}(n) = \langle n\cdot\nabla\psi_i\partial_{\nabla\psi_i}\mathcal{L} - n\mathcal{L}\rangle_1 \tag{4.21}$$

which satisfies

$$\nabla\cdot\overline{T}(n) = 0. \tag{4.22}$$

The canonical stress-energy tensor is the adjoint function, which, from (2.18), is

$$T(n) = \dot{\nabla}\langle\dot{\psi}_i\partial_{\nabla\psi_i}\mathcal{L}n\rangle - n\mathcal{L}. \tag{4.23}$$

It follows from (4.22) that

$$\begin{aligned} \dot{T}(\dot{\nabla})\cdot n &= 0 \qquad \text{for all } n, \\ \Rightarrow \dot{T}(\dot{\nabla}) &= 0, \end{aligned} \tag{4.24}$$

so that $T(n)$ is a conserved tensor. In the γ_0 frame there is now a conserved 4-vector

$$p = \int |d^3x| T(\gamma_0), \tag{4.25}$$

which is identified as the total momentum. The total energy is

$$E = \int |d^3x| \gamma_0\cdot T(\gamma_0). \tag{4.26}$$

We next consider rotations, assuming initially that all fields ψ_i transform as vectors. We define

$$\begin{aligned} x' &= e^{-\alpha B/2} x e^{\alpha B/2} \\ \psi_i'(x) &= e^{\alpha B/2}\psi_i(x')e^{-\alpha B/2}, \end{aligned} \tag{4.27}$$

which again we regard as an active rotation of fields from one spacetime point to another. This differs from (3.25) in the relative direction of the rotation for the position vector x and the fields ψ_i, resulting in a sign difference in the contribution of the spin. In order to apply (4.18) we use

$$\partial_\alpha \psi_i'|_{\alpha=0} = B\times\psi_i - (B\cdot x)\cdot\nabla\psi_i \tag{4.28}$$

and

$$\partial_\alpha \mathcal{L}'|_{\alpha=0} = \nabla\cdot(x\cdot B\mathcal{L}). \tag{4.29}$$

Together, these yield the conserved vector

$$\overline{J}(B) = \langle (B\times\psi_i - (B\cdot x)\cdot\nabla\psi_i)\partial_{\nabla\psi_i}\mathcal{L}\rangle_1 + B\cdot x\mathcal{L}, \tag{4.30}$$

which satisfies

$$\begin{aligned} \dot{\nabla}\cdot\dot{\overline{J}}(B) &= 0 \\ \Rightarrow \underline{\dot{J}}(\dot{\nabla})\cdot B &= 0 \quad \text{for all } B \\ \Rightarrow \underline{\dot{J}}(\dot{\nabla}) &= 0. \end{aligned} \tag{4.31}$$

The adjoint function $\underline{J}(n)$ is, therefore, a conserved bivector-valued function of position, which we identify as the canonical angular-momentum tensor. The calculation of $\underline{J}(n)$ is a simple application of (2.18):

$$\begin{aligned} \underline{J}(n) &= \partial_B\langle (B\times\psi_i - (B\cdot x)\cdot\nabla\psi_i)\partial_{\nabla\psi_i}\mathcal{L}n + B\cdot x\mathcal{L}n\rangle \\ &= \langle \psi_i\times(\partial_{\nabla\psi_i}\mathcal{L}n)\rangle_2 - x\wedge\dot{\nabla}\langle\dot{\psi}_i\partial_{\nabla\psi_i}\mathcal{L}n\rangle + x\wedge(n\mathcal{L}) \\ &= T(n)\wedge x + \langle\psi_i\times(\partial_{\nabla\psi_i}\mathcal{L}n)\rangle_2. \end{aligned} \tag{4.32}$$

If one of the fields ψ, say, transforms single-sidedly (as a spinor), then (4.32) contains a term $\langle \frac{1}{2}\psi\partial_{\nabla\psi}\mathcal{L}n\rangle_2$.

The first term in (4.32) is the routine $p\wedge x$ component, and the second term is due to the spin of the field. The general form of $\underline{J}(n)$ is therefore

$$\underline{J}(n) = T(n)\wedge x + S(n). \tag{4.33}$$

By applying (4.31) to (4.33) and using (4.24), we find that

$$T(\dot{\nabla})\wedge\dot{x} + \dot{S}(\dot{\nabla}) = 0. \tag{4.34}$$

The first term in (4.34) can be written as $-\partial_a\wedge T(a)$, which returns the *characteristic bivector* [1] B of $T(n)$. The antisymmetric part of $T(n)$ can always be written in terms of this bivector as

$$T_A(a) = \tfrac{1}{2}B\cdot a, \tag{4.35}$$

so that

$$T(a)\wedge\partial_a = \tfrac{1}{2}(B\cdot a)\wedge\partial_a = B. \tag{4.36}$$

Equation (4.34) now gives

$$B = -\dot{S}(\dot{\nabla}), \tag{4.37}$$

so that, in any Poincaré-invariant theory, the antisymmetric part of the stress-energy tensor is a total divergence. In order for (4.32) to hold, however, the antisymmetric part of $T(n)$ must be retained, since it cancels the divergence of the spin term: although $T_A(n)$ is a total divergence, $x\wedge T_A(n)$ certainly is not.

By inserting (4.23) into (4.32) and setting $\underline{\dot{J}}(\dot{\nabla}) = 0$, we find the interesting equation

$$\langle \psi_i\times(\partial_{\psi_i}\mathcal{L}) + (\nabla\psi_i)\times(\partial_{\nabla\psi_i}\mathcal{L})\rangle_2 = 0, \tag{4.38}$$

which is satisfied by any Poincaré-invariant theory. If spinor terms are present, the left-hand side includes terms of the type

$$\tfrac{1}{2}\langle \psi_i(\partial_{\psi_i}\mathcal{L}) + (\nabla\psi_i)(\partial_{\nabla\psi_i}\mathcal{L})\rangle_2. \tag{4.39}$$

Equation (4.38) is a generalised Euler homogeneity condition, and is a consequence of the assumed isotropy of space.

While all fundamental theories should be Poincaré-invariant, an interesting class go beyond this and are invariant under conformal transformations. The conformal group contains two further symmetries, of which the first is scale invariance. (In fact dilation symmetry does not imply full conformal invariance, and the results below are appropriate to any scale-invariant theory.) We define

$$\begin{aligned} x' &= e^{\alpha}x \\ \psi_i'(x) &= e^{d_i\alpha}\psi_i(x'), \end{aligned} \tag{4.40}$$

so that

$$\nabla\psi_i'(x) = e^{(d_i+1)\alpha}\nabla_{x'}\psi_i(x'). \tag{4.41}$$

If the theory is scale-invariant, it is possible to assign the conformal weights d_i such that the left-hand side of (4.18) reduces to

$$\partial_\alpha \mathcal{L}'|_{\alpha=0} = \nabla\cdot(x\mathcal{L}). \tag{4.42}$$

Equation (4.18) now takes the form

$$\nabla\cdot(x\mathcal{L}) = \nabla\cdot\langle(d_i\psi_i + x\cdot\nabla\psi_i)\partial_{\nabla\psi_i}\mathcal{L}\rangle_1, \tag{4.43}$$

so that

$$\begin{aligned} \Rightarrow \nabla\cdot\langle d_i\psi_i\partial_{\nabla\psi_i}\mathcal{L}\rangle_1 &= -\nabla\cdot(\overline{T}(x)) \\ &= -\mathrm{tr}(T). \end{aligned} \tag{4.44}$$

Thus, in a scale-invariant theory, the trace of the canonical stress-energy tensor is a total divergence. The current conjugate to dilations is

$$j = \langle d_i\psi_i\partial_{\nabla\psi_i}\mathcal{L} + \overline{T}(x)\rangle_1. \tag{4.45}$$

By using the equations of motion, equation (4.44) can be written, in four dimensions, as

$$d_i\langle\psi_i\partial_{\psi_i}\mathcal{L}\rangle + (d_i+1)\langle\nabla\psi_i\partial_{\nabla\psi_i}\mathcal{L}\rangle = 4\mathcal{L}, \tag{4.46}$$

which is an Euler homogeneity requirement and can be taken as an alternative definition of a scale-invariant theory.

The further generator of the conformal group is inversion:

$$x' = x^{-1}. \tag{4.47}$$

As it stands this is not parameterised by anything, and cannot be applied to (4.18). In order to derive a conserved tensor [12], (4.47) is combined with a translation to define a special conformal transformation [1]:

$$\begin{aligned} x' &= h(x) \\ &= (x^{-1} + \alpha n)^{-1} \\ &= x(1 + \alpha n x)^{-1}, \end{aligned} \tag{4.48}$$

from which it follows that

$$\Rightarrow \underline{h}(a) = (1+\alpha xn)^{-1} a (1+\alpha nx)^{-1}, \tag{4.49}$$

and $\underline{h}$ is therefore a spacetime-dependent rotation/dilation. This can be used to postulate transformation laws for all fields (including spinors, which transform single-sidedly) such that

$$\mathcal{L}' = \mathcal{L}(\psi_i', \nabla\psi_i') = \det \underline{h} \mathcal{L}(\psi_i(x'), \nabla_{x'}\psi_i(x')), \tag{4.50}$$

and hence

$$\partial_\alpha \mathcal{L}'|_{\alpha=0} = \partial_\alpha \det \underline{h}\mathcal{L}|_{\alpha=0} + \det \underline{h}(\partial_\alpha x')\cdot\nabla_{x'}\mathcal{L}(\psi_i(x'), \nabla_{x'}\psi_i(x'))|_{\alpha=0}\,. \tag{4.51}$$

It can be shown that

$$\partial_\alpha \det \underline{h}|_{\alpha=0} = -8x\cdot n \tag{4.52}$$

and

$$(\partial_\alpha x')\cdot\nabla_{x'} = -(xnx)\cdot\nabla, \tag{4.53}$$

whence

$$\partial_\alpha \mathcal{L}'|_{\alpha=0} = -\nabla\cdot(xnx\mathcal{L}). \tag{4.54}$$

Special conformal transformations therefore lead to the tensor

$$\overline{T}(xnx) - \langle(\partial_\alpha\psi_i')_{\alpha=0}\,\partial_{\nabla\psi_i}\mathcal{L}\rangle_1, \tag{4.55}$$

whose adjoint is a tensor of the form

$$xT(n)x - K(n), \tag{4.56}$$

which is conserved in a conformally-invariant theory.

By adding a total divergence, $T(n)$ can be redefined to give a $T'(n)$ which is symmetric and traceless. In this case (4.45) can be written as $T'(x)$ and (4.56) becomes $xT'(n)x$. We now have a set of four tensors, $T'(x)$, $T'(n)$, $xT'(n)x$ and $J(n)$, which are all conserved in conformally-invariant theories. This yields a set of $1+4+4+6 = 15$ conserved quantities — the dimension of the conformal group. All this is well known, of course, but we believe this is the first time that geometric algebra has been systematically applied to this problem. In doing so we have simplified many of the derivations, and generated a clearer understanding of the results.

4.3 Electromagnetism

As an application of the results of Section 4.1 and Section 4.2 we consider the electromagnetic Lagrangian [13]

$$\mathcal{L} = -A\cdot J + \tfrac{1}{2}F\cdot F, \tag{4.57}$$

where A is the vector potential, $F = \nabla\wedge A$, and A couples to an external current J which is not varied. To find the equations of motion we must first write $F\cdot F$ as a function of ∇A:

$$\begin{aligned} F\cdot F &= \tfrac{1}{4}\langle(\nabla A - (\nabla A)\tilde{\ })^2\rangle \\ &= \tfrac{1}{2}\langle\nabla A\nabla A - \nabla A(\nabla A)\tilde{\ }\rangle. \end{aligned} \tag{4.58}$$

Since A is a pure vector, the appropriate form of the Euler-Lagrange equations is (4.7)

$$\begin{aligned} \langle\partial_{\tilde{A}}L - \nabla\partial_{(\nabla A)\tilde{\ }}\mathcal{L}\rangle_1 &= 0 \\ \Rightarrow \nabla\cdot F &= J. \end{aligned} \tag{4.59}$$

With the identity $\nabla\wedge F = \nabla\wedge(\nabla\wedge A) = 0$, this yields the full Maxwell equations $\nabla F = J$.

To calculate the free-field stress-energy tensor, we set $J = 0$ in (4.57) and work with

$$\mathcal{L} = -\tfrac{1}{2}\langle F\tilde{F}\rangle, \tag{4.60}$$

so that (4.23) gives

$$T(n) = \dot{\nabla}\langle(n\wedge\dot{A})\cdot F\rangle - \tfrac{1}{2}n\langle F^2\rangle. \tag{4.61}$$

This expression is physically unsatisfactory, because it is not gauge-invariant. In order to find a gauge-invariant form of (4.61), we write [14]

$$\begin{aligned} \dot{\nabla}\langle\dot{A}F\cdot n\rangle &= (\nabla\wedge A)\cdot(F\cdot n) + (F\cdot n)\cdot\nabla A \\ &= F\cdot(F\cdot n) - (F\cdot\dot{\nabla})\cdot n\dot{A} \end{aligned} \tag{4.62}$$

and observe that, since $\nabla\cdot F = 0$, the second term is a total divergence and can therefore be ignored. We are left with [15]

$$\begin{aligned} T_{\text{em}}(n) &= F\cdot(F\cdot n) - \tfrac{1}{2}nF\cdot F \\ &= \tfrac{1}{2}Fn\tilde{F}, \end{aligned} \tag{4.63}$$

which is both gauge-invariant and symmetric. As a check, the energy in the γ_0 frame is given by

$$\begin{aligned} \int |d^3x|\langle\gamma_0 F\gamma_0\tilde{F}\rangle &= \int |d^3x|\tfrac{1}{2}\langle(\boldsymbol{E}+i\boldsymbol{B})(\boldsymbol{E}-i\boldsymbol{B})\rangle \\ &= \int |d^3x|\tfrac{1}{2}(\boldsymbol{E}^2+\boldsymbol{B}^2), \end{aligned} \tag{4.64}$$

in agreement with standard formulations [7].

The angular momentum is found from (4.32):

$$\underline{J}(n) = (\dot{\nabla}\langle\dot{A}Fn\rangle - \tfrac{1}{2}n\langle F^2\rangle)\wedge x + A\wedge(F\cdot n), \tag{4.65}$$

where we have used the stress-energy tensor in the form (4.61). This expression therefore suffers from the same lack of gauge invariance, and is fixed up in the same way, using (4.62) and

$$(F\cdot n)\wedge A - x\wedge\Big((F\cdot\dot{\nabla})\cdot n\dot{A}\Big) = -x\wedge((F\cdot\overset{\leftrightarrow}{\nabla})\cdot nA), \tag{4.66}$$

which is a total divergence. This leaves simply

$$\underline{J}(n) = T_{\text{em}}(n)\wedge x, \tag{4.67}$$

and conservation is ensured by the result $\dot{T}_{em}(\dot{\nabla}) = 0$ and the symmetry of T_{em}. The angular momentum in the γ_0 frame is now

$$\int |d^3x| T_{em}(\gamma_0)\wedge x = \int |d^3x| \left(\boldsymbol{P}t - \tfrac{1}{2}(\boldsymbol{E}^2+\boldsymbol{B}^2)\boldsymbol{x} + \boldsymbol{x}\times\boldsymbol{P}\right), \tag{4.68}$$

where $\boldsymbol{P}$ is the Poynting vector $-i(\boldsymbol{E}\times\boldsymbol{B})$. The relative 3-space vector terms in (4.68) give the centre of energy, and the relative bivector term is the angular momentum (recall that the $\times$ signifies the commutator product (2.2) and not the vector cross product).

By redefining the stress-energy tensor to be symmetric, the spin term in the angular momentum has been absorbed into (4.63). For the case of electromagnetism this has the advantage that gauge invariance is manifest, but it also suppresses the spin-1 nature of the field. Suppressing the spin term in this manner is not always desirable, as we shall see with the Dirac equation.

The Lagrangian (4.60) is not only Poincaré-invariant; it is invariant under the full conformal group of spacetime. To see this, consider an arbitrary transformation h, so that

$$\begin{aligned} x' &= h(x) \\ A'(x) &= \overline{h}(A(x')). \end{aligned} \tag{4.69}$$

It follows that

$$\begin{aligned} \nabla\wedge A'(x) &= \overline{h}(\nabla_{x'})\wedge\overline{h}(A(x')) \\ &= \overline{h}(\nabla_{x'}\wedge A(x')) \\ &= \overline{h}(F(x')), \end{aligned} \tag{4.70}$$

since $\overline{h}$ commutes with the exterior derivative. The transformed action is therefore

$$\int |d^4x|\tfrac{1}{2}\langle\overline{h}(F(x'))^2\rangle = \int |d^4x'|(\det \underline{h})^{-1}\langle F(x')\cdot\underline{h}\overline{h}(F(x'))\rangle, \tag{4.71}$$

and h generates a symmetry if

$$\underline{h}\overline{h}(B) = \lambda\det(\underline{h})B \tag{4.72}$$

for any bivector B, where λ is an arbitrary scalar constant. The set of transformations h satisfying (4.72) generates the conformal group, as we observe by writing (4.72) as

$$\overline{h}(A)\cdot\overline{h}(B) = \lambda\det(\underline{h})A\cdot B \tag{4.73}$$

where A and B are bivectors; this relation holds for any h which satisfies

$$\overline{h}(a)\cdot\overline{h}(b) = e^{\alpha(x)}a\cdot b, \tag{4.74}$$

with a, b vectors. Equation (4.74) provides a standard definition of the conformal group [1]. All translations satisfy (4.72) trivially, since $\underline{h} = 1$. Reflections and rotations also satisfy (4.72) immediately, since for both of these $\underline{h}\overline{h} = 1$ and $\det\underline{h} = \pm 1$.

The remaining conformal transformations are dilations and inversions, as we studied in Section 4.2. Dilations clearly satisfy (4.72) and, as a check, the trace of the canonical stress-energy tensor is a total divergence:

$$\partial_n T(n) = -F\cdot F = \langle A \overset{\leftrightarrow}{\nabla} F\rangle. \tag{4.75}$$

The conserved current conjugate to dilations can be written in the form

$$\tfrac{1}{2}Fx\tilde{F} - F\cdot\nabla(x\cdot A) \tag{4.76}$$

but, since the second term is already a total divergence, we can write the conserved vector as $T_{\rm em}(x)$. This is conserved since $T_{\rm em}(n)$ is traceless.

The final conformal transformation is inversion,

$$\begin{aligned} h(x) &= x^{-1} = \frac{x}{x^2} \\ \underline{h}(a) &= -\frac{xax}{x^4} \\ \det \underline{h} &= -\frac{1}{x^8}, \end{aligned} \tag{4.77}$$

which again satisfies (4.72). The current conjugate to this is given by (4.56), and is

$$xT_{\rm em}(n)x. \tag{4.78}$$

The complete list of conserved tensors in free-field electromagnetism is therefore $T_{\rm em}(x)$, $T_{\rm em}(n)$, $xT_{\rm em}(n)x$, and $T_{\rm em}(n)\wedge x$, and it is a simple matter to calculate the modified conservation equations when a current is present.

4.4 Dirac Theory

The multivector derivative is particularly powerful when applied to the Dirac equation. To proceed, we must first eliminate column spinors and matrix operators from Dirac theory, and work instead with multivectors in the STA. This reformulation is carried out in Paper II [3], where it is shown that the Lagrangian for the Dirac equation becomes

$$\mathcal{L} = \langle \nabla\psi i\gamma_3\tilde{\psi} - eA\psi\gamma_0\tilde{\psi} - m\psi\tilde{\psi}\rangle, \tag{4.79}$$

where ψ is an even multivector and A is an external field (which is not varied). The appropriate form of the Euler-Lagrange equations is (4.9), giving

$$\begin{aligned} \nabla\psi i\gamma_3 - 2eA\psi\gamma_0 - 2m\psi &= -\nabla(\psi i\gamma_3) \\ \Rightarrow \nabla\psi i\sigma_3 - eA\psi &= m\psi\gamma_0, \end{aligned} \tag{4.80}$$

which is the familiar STA form of the Dirac equation [3, 16].

We now analyse the Dirac equation from the Lagrangian (4.79), employing the Noether theorem described in Section 4.1. There are two classes of symmetry, according to whether or not the position vector x is transformed. For the rest of this section we will consider position-independent transformations of the spinor ψ. Spacetime transformations are dealt with in Section 4.5.

The transformations we study at this point are of the type

$$\psi' = \psi e^{\alpha M}, \tag{4.81}$$

where M is a general multivector and α and M are independent of position. Operations on the right of ψ arise naturally in the STA formulation of Dirac theory, and should be thought of as generalised gauge transformations. In the standard Dirac theory with column spinors, however, transformations like (4.81) cannot be written down simply, and many of the results presented here are much harder to derive.

Applying (4.18) to (4.81), we find that

$$\nabla\cdot\langle\psi M i\gamma_3\tilde{\psi}\rangle_1 = \partial_\alpha \mathcal{L}'\big|_{\alpha=0}, \tag{4.82}$$

which is a result we shall exploit by substituting various quantities for M. If M is odd, equation (4.82) yields no information, since both sides vanish identically. The first even M we consider is a scalar, λ, so that $\langle\psi M i\gamma_3\tilde{\psi}\rangle_1$ is zero. It follows that

$$\begin{aligned} \partial_\alpha \left(e^{2\alpha\lambda}\mathcal{L}\right)\Big|_{\alpha=0} &= 0 \\ \Rightarrow \mathcal{L} &= 0, \end{aligned} \tag{4.83}$$

so that, when the equations of motion are satisfied, the Dirac Lagrangian vanishes.

We next consider a duality transformation. Setting $M = i$, equation (4.82) gives

$$\begin{aligned} -\nabla\cdot(\rho s) &= -m\partial_\alpha\langle e^{2i\alpha}\rho e^{i\beta}\rangle\Big|_{\alpha=0}, \\ \Rightarrow \nabla\cdot(\rho s) &= -2m\rho\sin\beta, \end{aligned} \tag{4.84}$$

where $\psi\tilde{\psi} = \rho e^{i\beta}$ and the spin current ρs is defined as $\psi\gamma_3\tilde{\psi}$. The role of the β-parameter in the Dirac equation remains unclear [13, 16], although (4.84) relates it to non-conservation of the spin current. Equation (4.84) is already known [13], but it does not seem to have been pointed out before that the spin current is the conjugate current to duality rotations. In conventional versions, these would be called 'axial rotations', with the role of i is taken by γ_5. However, in our approach, these rotations are identical to duality transformations for the electromagnetic field — another unification provided by geometric algebra. The duality transformation $e^{i\alpha}$ is also the continuous analogue of discrete mass conjugation symmetry, since $\psi \mapsto \psi i$ changes the sign of the mass term in $\mathcal{L}$. Hence we expect that the conjugate current, ρs, is conserved for massless particles.

Finally, taking M to be an arbitrary bivector B yields

$$\begin{aligned} \nabla\cdot(\psi B\cdot(i\gamma_3)\tilde{\psi}) &= 2\langle \nabla\psi iB\cdot\gamma_3\tilde{\psi} - eA\psi B\cdot\gamma_0\tilde{\psi}\rangle \\ &= 2\left\langle eA\psi(\sigma_3 B\sigma_3 - B)\gamma_0\tilde{\psi}\right\rangle, \end{aligned} \tag{4.85}$$

where we have used the equations of motion (4.80). Both sides of (4.85) vanish for $B = i\sigma_1, i\sigma_2$ and σ_3, with useful equations arising on taking $B = \sigma_1, \sigma_2$ and $i\sigma_3$. The last of these, $B = i\sigma_3$, corresponds to the usual $U(1)$ gauge transformation of the spinor field, and gives

$$\nabla\cdot(\rho v) = 0, \tag{4.86}$$

where $\rho v = \psi\gamma_0\tilde{\psi}$ is the current conjugate to phase transformations, and is strictly conserved. The remaining transformations, $e^{\alpha\sigma_1}$ and $e^{\alpha\sigma_2}$, give

$$\begin{aligned} \nabla\cdot(\rho e_1) &= 2e\rho A\cdot e_2 \\ \nabla\cdot(\rho e_2) &= -2e\rho A\cdot e_1, \end{aligned} \tag{4.87}$$

where $\rho e_\mu = \psi\gamma_\mu\tilde{\psi}$. Although these equations have been found before [13], the role of ρe_1 and ρe_2, as currents conjugate to right-sided $e^{\alpha\sigma_2}$ and $e^{\alpha\sigma_1}$ transformations, has not been noted. Right multiplication by σ_1 and σ_2 provide continuous versions of charge conjugation, since the transformation $\psi \mapsto \psi\sigma_1$ takes (4.80) into

$$\nabla\psi i\sigma_3 + eA\psi = m\psi\gamma_0. \tag{4.88}$$

It follows that the conjugate currents are conserved exactly if the external potential vanishes, or the particle has zero charge.

Many of the results in this section have been derived by David Hestenes [13, 16], through an analysis of the local observables of the Dirac theory. The Lagrangian approach simplifies many of these derivations and, more importantly, reveals that many of the observables in the Dirac theory are conjugate to symmetries of the Lagrangian, and that these symmetries have natural geometric interpretations.

4.5 Spacetime Transformations in Maxwell-Dirac Theory

We now consider spacetime symmetries in the Dirac theory, and derive the canonical stress-energy tensor and angular-momentum tensors. In doing so, we include the free-field term for the electromagnetic field, and work with the coupled Lagrangian

$$\mathcal{L} = \langle (\nabla\psi) i\gamma_3 \tilde{\psi} - eA\psi\gamma_0\tilde{\psi} - m\psi\tilde{\psi} + \tfrac{1}{2}F^2 \rangle, \tag{4.89}$$

in which ψ and A are both dynamical variables. Including both fields ensures that the Lagrangian is Poincaré-invariant.

From (4.23) and (4.83), the stress-energy tensor is

$$T(n) = \dot{\nabla}\langle \dot{\psi} i\gamma_3 \tilde{\psi} n \rangle + \dot{\nabla}\langle \dot{A} F n \rangle - \tfrac{1}{2} n F\cdot F, \tag{4.90}$$

which once again is not gauge-invariant. We can manipulate the last two terms as in Section 4.3, the only difference being that we now pick up a term from $\nabla\cdot F = J$ ($J \equiv e\psi\gamma_0\tilde{\psi}$), giving

$$T_{\mathrm{md}}(n) = \dot{\nabla}\langle \dot{\psi} i\gamma_3 \tilde{\psi} n \rangle - n\cdot J A + \tfrac{1}{2}\tilde{F} n F, \tag{4.91}$$

which *is* now gauge-invariant. Conservation of (4.91) can be confirmed using the equations of motion (4.80) and (4.59). The first and last terms are the free-field stress-energy tensors, and the middle term, $-n\cdot JA$, arises from the coupling. The stress-energy tensor for the Dirac theory in the presence of an external field A is conventionally defined by the first two terms of (4.91), since the combination of these is gauge-invariant.

Only the free-field electromagnetic contribution in (4.91) is symmetric; the other terms each contain antisymmetric parts. The overall antisymmetric contribution is

$$\begin{aligned} T_A &= \tfrac{1}{2}(T(n) - \overline{T}(n)) \\ &= n\cdot(\nabla\cdot(\tfrac{1}{4} i\rho s)) \\ &= n\cdot(-i\nabla\wedge(\tfrac{1}{4}\rho s)), \end{aligned} \tag{4.92}$$

and is therefore completely determined by the exterior derivative of the spin current [17].

The angular momentum is found from (4.32), using the rearrangement carried out in Section 4.3, and is

$$\underline{J}(n) = T_{\mathrm{md}}(n)\wedge x + \tfrac{1}{2} i\rho s\wedge n, \tag{4.93}$$

in full agreement with (4.92). The ease of derivation of (4.93) compares favourably with the traditional operator approach [14]. It was crucial to the derivation that the antisymmetric component of $T_{\mathrm{md}}(n)$ was retained, in order to identify the spin-$\frac{1}{2}$ contribution to $\underline{J}(n)$. In (4.93) the spin term is determined by the trivector is, and the fact that this trivector can be dualised to the vector s is a unique property of four-dimensional spacetime.

The sole term breaking conformal invariance in (4.89) is the mass term $\langle m\psi\tilde{\psi}\rangle$, and it is useful to consider the currents conjugate to dilations and special conformal transformations, and show how their non-conservation results from this term. For dilations, since the conformal weight of a spinor field is $\frac{3}{2}$, (4.45) yields the current

$$j_d = \overline{T}_{\mathrm{md}}(x) \tag{4.94}$$

(after subtracting out a total divergence). The conservation equation is

$$\begin{aligned} \nabla\cdot j_d &= \partial_n\cdot T_{\mathrm{md}}(n) \\ &= \langle m\psi\tilde{\psi}\rangle. \end{aligned} \tag{4.95}$$

For special conformal transformations, we know from (4.49) and (4.69) that the A-field transforms as

$$A'(x) = (1+\alpha n x)^{-1} A(x')(1+\alpha x n)^{-1}, \tag{4.96}$$

and, since this is a rotation/dilation, we postulate for ψ the single-sided transformation

$$\psi'(x) = (1+\alpha n x)^{-2}(1+\alpha x n)^{-1}\psi(x'). \tag{4.97}$$

In order to verify that (4.50) is satisfied, we need the result

$$\nabla\left((1+\alpha n x)^{-2}(1+\alpha x n)^{-1}\right) = 0. \tag{4.98}$$

From (4.55) we find that the conserved tensor is

$$T_c(n) = x T_{\mathrm{md}}(n) x + n\cdot(ix\wedge(\rho s)), \tag{4.99}$$

and the conservation equation is

$$\dot{T}_c(\dot{\nabla}) = 2\langle m\psi\tilde{\psi}\rangle x. \tag{4.100}$$

In both (4.95) and (4.100) the conjugate tensors are conserved as the mass goes to zero, as expected.

5 Multivector Techniques For Functional Differentiation

In the previous section we dealt with Lagrangians which are functions of multivector-valued fields. In this section we will outline how to extend the formalism to include multivector-valued *functions* as the dynamical variables in the Lagrangian. This is, in fact, crucial to a complete formulation of gauge theory within geometric algebra, which will be presented elsewhere.

In order to generalise the results of Section 4.1, it is necessary to extend the multivector derivative so that it becomes possible to differentiate with respect to a multivector-valued function. The resulting operator defines a 'functional calculus' for linear functions, and provides a clear understanding of the meaning of 'functional differentiation with respect to the metric'.

The advantage of the present approach is that only a slight elaboration of the techniques outlined in Section 2 is required; no new notation or conventions are needed. The quantities we wish to calculate are of the type

$$\partial_{\underline{f}(b)}\underline{f}(A_r), \tag{5.1}$$

where b is a vector, A_r is an grade-r multivector, and $\underline{f}$ is a linear vector function. Recall that $\underline{f}(b)$ is a shorthand notation for $\underline{f}(b,x) = b\cdot\nabla f(x)$, so that in writing (5.1) we must assume that $\underline{f}(b)$ and $\underline{f}(A_r)$ are evaluated at the same spacetime point x. This can be enforced by the inclusion of a Dirac delta-function, but that is not necessary for the manipulations carried out here.

To obtain an explicit formula for the derivative (5.1), we first project out from A_r those terms which include the vector b. The appropriate projection operator is

$$P_b(A_r) = (A_r\cdot b^{-1})b = b\wedge(b^{-1}\cdot A_r), \tag{5.2}$$

so that

$$\begin{aligned}\partial_{\underline{f}(b)}\underline{f}(A_r) &= \partial_{\underline{f}(b)}\underline{f}(b\wedge(b^{-1}\cdot A_r)) \\ &= \partial_{\underline{f}(b)}\left(\underline{f}(b)\wedge\underline{f}(b^{-1}\cdot A_r)\right). \end{aligned} \tag{5.3}$$

It is easily shown (for example, by expanding in a basis) that the factor $\underline{f}(A_r\cdot b^{-1})$ does not depend on $\underline{f}(b)$, so, recalling (2.15), we obtain

$$\partial_{\underline{f}(b)}\underline{f}(A_r) = (n-r+1)\underline{f}(b^{-1}\cdot A_r), \tag{5.4}$$

where n is the dimension of the space. We can extend (5.4) in a number of ways, and will from now on take b to be a unit (time-like) vector ($b^{-1} = b$). Employing the obvious result

$$\partial_{\underline{f}(b)}\underline{f}(b)\cdot a = a, \tag{5.5}$$

we find that

$$\begin{aligned}\partial_{\underline{f}(b)}\underline{f}(a)\cdot c &= b\cdot a\partial_{\underline{f}(b)}\underline{f}(b)\cdot c \\ &= b\cdot ac. \end{aligned} \tag{5.6}$$

This may be compared with the result of standard functional calculus, in which the derivative of a scalar by a 2-tensor gives a 2-tensor (in this case $t(b) = b\cdot ac$). Equation (5.6) extends to

$$\begin{aligned}\partial_{\underline{f}(b)}\langle\underline{f}(c\wedge d)\cdot B_2\rangle &= \dot{\partial}_{\underline{f}(b)}\langle\dot{\underline{f}}(c)\cdot(\underline{f}(d)\cdot B_2)\rangle - \dot{\partial}_{\underline{f}(b)}\langle\dot{\underline{f}}(d)\cdot(\underline{f}(c)\cdot B_2)\rangle \\ &= \underline{f}(b\cdot(c\wedge d))\cdot B_2, \end{aligned} \tag{5.7}$$

so that

$$\partial_{\underline{f}(b)}\langle\underline{f}(A)B_2\rangle = \underline{f}(b\cdot\langle A\rangle_2)\cdot B_2, \tag{5.8}$$

where B_2 is an arbitrary bivector. Proceeding in this manner, we find the general formula

$$\partial_{\underline{f}(b)}\langle\underline{f}(A)B\rangle = \sum_r\langle\underline{f}(b\cdot A_r)B_r\rangle_1. \tag{5.9}$$

Equation (5.9) can be used to derive formulae for the functional derivative of the adjoint. The general result can be expressed as

$$\partial_{\underline{f}(b)}\overline{f}(A_r) = \langle\underline{f}(b\cdot\dot{X}_r)A_r\rangle_1\dot{\partial}_{X_r} \tag{5.10}$$

and when A is a vector, this admits the simpler form

$$\partial_{\underline{f}(b)}\overline{f}(a) = ab. \tag{5.11}$$

If $\underline{f}$ is a symmetric function then $\underline{f} = \overline{f}$, but we cannot exploit this for functional differentiation, since $\underline{f}$ and $\overline{f}$ are independent for the purposes of calculus.

Our final results concern the functional derivative of the inverse function, given by (2.24). We first need the result for the derivative of the determinant, as defined by (2.22):

$$\begin{aligned} \partial_{\underline{f}(b)}\underline{f}(I) &= \underline{f}(b\cdot I) \\ \Rightarrow \partial_{\underline{f}(b)}\det(f) &= \underline{f}(b\cdot I)I^{-1} \\ &= \det(f)\overline{f}^{-1}(b). \end{aligned} \tag{5.12}$$

This again coincides with the standard formula for functional differentiation of the determinant by its corresponding tensor. The present proof, which follows directly from the definitions (2.22) and (2.24) and the formula (5.4), is considerably more concise than by conventional matrix/tensor methods. The result for the inverse is now found to be:

$$\begin{aligned} \partial_{\underline{f}(b)}\overline{f}^{-1}(A_r) &= \partial_{\underline{f}(b)}\Big((\det f)^{-1}\underline{f}(A_r I)I^{-1}\Big) \\ &= r\overline{f}^{-1}(b\wedge A_r) - \overline{f}^{-1}(b)\cdot\overline{f}^{-1}(A), \end{aligned} \tag{5.13}$$

and the analogue of (5.9) is

$$\begin{aligned} \partial_{\underline{f}(b)}\langle\overline{f}^{-1}(A)M\rangle &= \sum_r \partial_{\underline{f}(b)}\langle(\det f)^{-1}\underline{f}(A_r I)I^{-1}M_r\rangle \\ &= \sum_r -\overline{f}^{-1}(A_r)\cdot(M_r\cdot\overline{f}^{-1}(b)). \end{aligned} \tag{5.14}$$

In both cases we have made repeated use of (2.23).

An extension of these results can be expected to provide a rich elaboration of multivector calculus. Some further developments will be given in a forthcoming paper, but here we concentrate on a single application — the stress-energy tensor.

5.1 The Stress-Energy Tensor Revisited

Functional differentiation with respect to the metric has become the standard method for deriving the stress-energy tensor in the context of general relativity [18, 19], so we need to examine how this is incorporated into our framework. As an example we take free-field electromagnetism, for which the standard approach involves writing the Lagrangian as

$$\tfrac{1}{2}F\cdot F = \tfrac{1}{2}F^{\mu\nu}F^{\rho\sigma}g_{\mu\sigma}g_{\nu\rho}, \tag{5.15}$$

where $F^{\mu\nu}$ are the components of F with respect to an arbitrary frame. In order to imagine varying $g_{\mu\nu}$ we do not need to introduce any concept of curved space, since all that is required is an understanding of how coordinate frames are defined in flat space. Following the approach of chapter 6 of Hestenes & Sobczyk [1], we introduce a set of scalar coordinates $\{x^\mu\}$, so that $x = x(x^0, \ldots x^3)$. From this we define a coordinate frame as

$$e_\mu = \partial_{x^\mu} x, \tag{5.16}$$

which induces the metric

$$g_{\mu\nu} = e_\mu \cdot e_\nu. \tag{5.17}$$

The reciprocal frame is defined as

$$e^\mu = \nabla x^\mu, \tag{5.18}$$

and it is easily verified that

$$e^\mu \cdot e_\nu = \delta^\mu_\nu. \tag{5.19}$$

The metric is now understood as the tensor mapping the reciprocal frame to the coordinate frame:

$$g(e^\mu) = e_\mu. \tag{5.20}$$

The metric is therefore tied to a given spacetime frame and so, in order to vary the metric, we must vary this frame. This is achieved by a linear redefinition of the spacetime point at which the coordinate fields x^μ are evaluated, so that

$$\begin{aligned} x^\mu(x) &\mapsto x^\mu(h(x)) \\ e_\mu &\mapsto \underline{h}^{-1}(e_\mu). \end{aligned} \tag{5.21}$$

Variation of the metric therefore gives us information about the variation of the action under reparameterisation, so we define

$$\begin{aligned} x' &= h(x), \\ \psi'(x') &= \psi(x), \end{aligned} \tag{5.22}$$

where $h = \underline{h}$ is a general linear function which need not be symmetric. The action is now

$$\begin{aligned} S &= \int |d^4x| \mathcal{L}(\psi_i(x), \nabla\psi_i(x)) \\ &= \int |d^4x'| (\det \underline{h})^{-1} \mathcal{L}(\psi_i'(x'), \overline{h}(\nabla_{x'})\psi_i'(x')). \end{aligned} \tag{5.23}$$

Since we have chosen h to be linear, $\underline{h}$ is not a function of position. We can therefore unambiguously relabel the parameter in (5.23) so as to give

$$\begin{aligned} S &= \int |d^4x| (\det \underline{h})^{-1} \mathcal{L}(\psi_i'(x), \overline{h}(\nabla)\psi_i'(x)) \\ &= \int |d^4x| (\det \underline{h})^{-1} \mathcal{L}'. \end{aligned} \tag{5.24}$$

The form of ψ_i' depends on the variable, with

$$\begin{aligned} \psi_i' &= \overline{h}(\psi_i) && \text{– vector field,} \\ \psi_i' &= \psi_i && \text{– spinor field.} \end{aligned} \tag{5.25}$$

The cost of making the action integral invariant under active transformations of spacetime is the introduction of a new tensor variable h, which is similar to the vierbein of general relativity [19]. The h-tensor has no kinetic term and variation of S with respect to $\underline{h}$ yields

$$\underline{h}(e^\mu)*\partial_{\underline{h}(e_\mu)} S\Big|_{h=1} = \int |d^4x| \underline{h}(\partial_n)*\partial_{\underline{h}(n)} \left((\det \underline{h})^{-1}\mathcal{L}'\right)\Big|_{h=1}. \tag{5.26}$$

Upon defining

$$\partial_{\underline{h}(n)} \left((\det \underline{h})^{-1}\mathcal{L}'\right)\Big|_{h=1} = T(n), \tag{5.27}$$

equation (5.26) becomes

$$\begin{aligned}\delta S &= \int |d^4x| \underline{h}(\partial_n) * T(n) \\ &= \int |d^4x| \dot{h}(x) \cdot T(\dot{\nabla}) \\ &= -\int |d^4x| h(x) \cdot \dot{T}(\dot{\nabla}).\end{aligned} \tag{5.28}$$

However, if the equations of motion are satisfied, δS must vanish for arbitrary h, so that variation with respect to $\underline{h}$ leads to the conservation equation

$$\dot{T}(\dot{\nabla}) = 0, \tag{5.29}$$

and the tensor $T(n)$ is identified as the functional stress-energy tensor. To see that this is equivalent to the canonical tensor, we use (5.12) in (5.27) to give

$$T(n) = \partial_{\underline{h}(n)} \mathcal{L}' \Big|_{h=1} - n\mathcal{L}. \tag{5.30}$$

The first term in (5.30) can be written as

$$\begin{aligned}&\partial_{\underline{h}(n)} \left(\psi_i' * \partial_{\psi_i} \mathcal{L} + (\overline{h}(\nabla)\psi_i') * \partial_{\nabla\psi_i} \mathcal{L} \right) \Big|_{h=1} \\ &= \dot{\nabla} \langle \dot{\psi}_i \partial_{\nabla\psi_i} \mathcal{L} n \rangle + \partial_{\underline{h}(n)} \dot{\psi}_i' * (\partial_{\psi_i} \mathcal{L} + \dot{\nabla} \partial_{\nabla\psi_i} \mathcal{L}) \Big|_{h=1}\end{aligned} \tag{5.31}$$

and, assuming the equations of motion are satisfied, the second term in (5.31) becomes $\langle \partial_{\underline{h}(n)} \psi_i' \overset{\leftrightarrow}{\nabla} (\partial_{\nabla\psi_i} \mathcal{L}) \rangle$, which is a total divergence. On comparing (5.31) with (4.23) we see that the tensors now agree, up to a total divergence.

We illustrate this with free-field electromagnetism and Dirac theory. For electromagnetism, we have

$$\begin{aligned}T(n) &= \tfrac{1}{2} \partial_{\underline{h}(n)} \overline{h}(F) \cdot \overline{h}(F) \Big|_{h=1} - \tfrac{1}{2} n F \cdot F \\ &= (n \cdot F) \cdot F - \tfrac{1}{2} n F \cdot F \\ &= \tfrac{1}{2} F n \tilde{F},\end{aligned} \tag{5.32}$$

which agrees with (4.63). The functional derivative approach automatically preserves gauge invariance, since $\mathcal{L}'$ is gauge-invariant. The derivation of the symmetric tensor (5.32) has nothing to do with any imposed symmetry on $\underline{h}$; it follows purely from the form of $\mathcal{L}$. We can see that this approach does not necessarily yield a symmetric stress-energy by considering the free-field Dirac Lagrangian, for which we find

$$\begin{aligned} T(n) &= \partial_{\underline{h}(n)}\langle\overline{h}(\nabla)\psi i\gamma_3\tilde{\psi}\rangle \\ &= \dot{\nabla}\langle\dot{\psi}i\gamma_3\tilde{\psi}n\rangle, \end{aligned} \tag{5.33}$$

in agreement with (4.91).

Traditional derivations of the stress-energy tensor by differentiating with respect to the metric always finish up by imposing symmetry as a constraint on $T(n)$ [19, 20]. In our approach, however, we have differentiated with respect h, which is a 'square root' of the metric tensor g and is in general asymmetric. It is therefore natural that our approach can give rise to asymmetric terms, which is very gratifying since we have already seen that these are central to the correct treatment of spin [21, 22, 23].

6 Summary and Conclusions

Geometric calculus is the natural language for the study of Lagrangian field theory. Geometric algebra clarifies the physics, and the multivector derivative simplifies the algebra. Passive transformations are eliminated, and only active transformations, in which the particles (or experiments) are transferred from one spacetime point to the other, are discussed. The geometric algebra approach allows for spinor and vector variables to be treated in the same way and, applied to the Dirac theory, leads to the identification of new conjugate currents.

Functional differentiation with respect to linear functions can also be handled within multivector calculus, leading to powerful ways of manipulating the 'vierbein' fields of general relativity. The functional stress-energy tensor is not necessarily symmetric, and the symmetry of the electromagnetic stress-energy tensor is a consequence solely of gauge invariance.

In future work, a more complete formulation of gauge theory will be presented utilising the techniques introduced in this paper. This will include treatments of both electroweak symmetries and gravity.

References

[1] D. Hestenes and G. Sobczyk. *Clifford Algebra to Geometric Calculus.* Reidel, Dordrecht, 1984.

[2] S.F. Gull, A.N. Lasenby, and C.J.L. Doran. Imaginary numbers are not real — the geometric algebra of spacetime. *Found. Phys.*, 23(9):1175, 1993.

[3] C.J.L. Doran, A.N. Lasenby, and S.F. Gull. States and operators in the spacetime algebra. *Found. Phys.*, 23(9):1239, 1993.

[4] S.F. Gull, A.N. Lasenby, and C.J.L. Doran. Electron paths, tunnelling and diffraction in the spacetime algebra. *Found. Phys.*, 23(10):1329, 1993.

[5] A.N. Lasenby, C.J.L. Doran, and S.F. Gull. Grassmann calculus, pseudoclassical mechanics and geometric algebra. *J. Math. Phys.*, 34(8):3683, 1993.

[6] F.A. Berezin and M.S. Marinov. Particle spin dynamics as the Grassmann variant of classical mechanics. *Ann. Phys.*, 104:336, 1977.

[7] H. Goldstein. *Classical Mechanics*. Addison–Wesley, Reading MA, 1950.

[8] C.J.L. Doran, A.N. Lasenby, and S.F. Gull. Grassmann mechanics, multivector derivatives and geometric algebra. In Z. Oziewicz, B. Jancewicz, and A. Borowiec, editors, *Spinors, Twistors, Clifford Algebras and Quantum Deformations*, page 215. Kluwer Academic, Dordrecht, 1993.

[9] A.O. Barut and N. Zanghi. Classical models of the Dirac electron. *Phys. Rev. Lett.*, 52(23):2009, 1984.

[10] S.F. Gull. Charged particles at potential steps. In A. Weingartshofer and D. Hestenes, editors, *The Electron*, page 37. Kluwer Academic, Dordrecht, 1991.

[11] F.J. Belinfante. On the current and density of charge, energy, momentum and angular momentum of fields. *Physica*, 8(5):449, 1940.

[12] S. Coleman. *Aspects of Symmetry*. Cambridge University Press, 1985.

[13] D. Hestenes. Real Dirac theory. In Preparation, 1994.

[14] C. Itzykson and J-B. Zuber. *Quantum Field Theory*. McGraw–Hill, New York, 1980.

[15] D. Hestenes. *Space–Time Algebra*. Gordon and Breach, New York, 1966.

[16] D. Hestenes. Observables, operators, and complex numbers in the Dirac theory. *J. Math. Phys.*, 16(3):556, 1975.

[17] H. Tetrode. Der impuls-energiesatz in der Diracschen quantentheorie des elektrons. *Z. Physik*, 49:858, 1928.

[18] S.W. Hawking and G.F.R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge University Press, 1973.

[19] S. Weinberg. *Gravitation and Cosmology*. Wiley, New York, 1972.

[20] C.W. Misner, K.S. Thorne, and J.A. Wheeler. *Gravitation*. W.H. Freeman and Company, San Francisco, 1973.

[21] A. Papapetrou. Non-symmetric stress-energy-momentum tensor and spin-density. *Phil. Mag.*, 40:937, 1949.

[22] F.W. Hehl, P. von der Heyde, G.D. Kerlick, and J.M. Nester. General relativity with spin and torsion: Foundations and prospects. *Rev. Mod. Phys.*, 48:393, 1976.

[23] F.W. Hehl. On the energy tensor of spinning massive matter in classical field theory and general relativity. *Reports on Math. Phys.*, 9:55, 1976.