2-spinors, Twistors and Supersymmetry in the Spacetime Algebra

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Abstract

We present a new treament of 2-spinors and twistors, using the spacetime algebra. The key rôle of bilinear covariants is emphasized. As a by-product, an explicit representation is found, composed entirely of real spacetime vectors, for the Grassmann entities of supersymmetric field theory.

1 Introduction

The aim of this presentation is to give a new translation of 2-spinors and twistors into the language of Clifford algebra. This has certainly been considered before [1, 2], but we differ from previous approaches by using the language of a particular form of Clifford algebra, the spacetime algebra (henceforth STA), in which the stress is on working in real 4-dimensional spacetime, with no use of a commutative scalar imaginary i. Moreover, the quantities which are Clifford multiplied together are always taken to be real geometric entities (vectors, bivectors, *etc.*), living in spacetime, rather than complex entities living in an abstract or internal space. Thus the real space geometry involved in any equation is always directly evident.

That such a translation can be achieved may seem surprising. It is generally believed that complex space notions and a unit imaginary i are fundamental in areas such as quantum mechanics, complex spin space, and 2-spinor and twistor theory. However using the spacetime algebra, it has already shown [3] how the i appearing in the Dirac, Pauli and Schrödinger equations has a geometrical explanation in terms of rotations in real spacetime. Here we extend this approach to 2-spinors and twistors, and thereby achieve a reworking that we believe is mathematically the simplest yet found, and which lays bare very clearly the real (rather than complex) geometry involved.

As another motivation for what follows, we should point out that the scheme we present has great computational power, both for hand working, and on computers. Every time two entities are written side by side algebraically a Clifford product is implied, thus all our expressions can be programmed into a computer in a completely definite and explicit fashion. There is no need either for an abstract spin space, containing objects which have to be operated on by operators, or for an abstract index convention. The requirement for an explicit matrix representation is also avoided, and all equations are automatically Lorentz invariant since they are written in terms of geometric objects.

Due to the restriction on space, we will only consider the most basic levels of 2-spinor and twistor theory. There are many more results in our translation programme for 2-spinors and twistors that have already been obtained, in particular for higher valence twistors, the conformal group on spacetime, twistor geometry and curved space differentiation, and these will be presented with proper technical details in a forthcoming paper. However, by spending some time being precise about the nature of our translation, we hope that even the basic level results presented here will still be of use and interest. A short introduction is also given of the equivalent process for field supersymmetry, and we end by discussing some implications for the rôle of 2-spinors and twistors in physics.

2 The Spacetime Algebra

The spacetime algebra is the geometric (Clifford) algebra of real 4-dimensional spacetime. Geometric algebra and the geometric product are described in detail in [4]. Our own conventions follow those of this reference, and are also described in [5]. Briefly we define a *multivector* as a sum of Clifford objects of arbitrary grade (grade 0 = scalar, grade 1 = vector, grade 2 = bivector, *etc.*). These are equipped with an associative (geometric) product. We will also need the operation of *reversion* which reverses the order of multivectors,

$$(AB)\tilde{} = \tilde{B}\tilde{A}, \tag{1}$$

but leaves vectors (and scalars) unchanged, so it simply reverses the order of the vectors in any product.

The Clifford algebra for 3-dimensional Euclidean space is generatated by three orthonormal vectors $\{\sigma_k\}$, and is spanned by

$$1, \qquad \{\sigma_k\}, \qquad \{i\sigma_k\}, \qquad i \tag{2}$$

where $i = \sigma_1\sigma_2\sigma_3$ is the *pseudoscalar* (highest grade multivector) for the space. The pseudoscalar i squares to -1, and commutes with all elements of the algebra in this 3-dimensional case, so is given the same symbol as the unit imaginary. Note, however, that it has a definite geometrical rôle as on oriented volume element, rather than just being an imaginary scalar. For future clarity, we will reserve the symbol j for the uninterpreted commutative imaginary i, as used for example in conventional quantum mechanics and electrical engineering. The algebra (2) is the Pauli algebra, but in geometric algebra the three Pauli σ_k are no longer viewed as three matrix-valued components of a single isospace vector, but as three independent basis vectors for real space.

A quantum spin state contains a pair of complex numbers, ψ_1 and ψ_2

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{3}$$

and has a one to one correspondence with an even multivector ψ. A general even element can be written as $\psi = a^0 + a^k i\sigma_k$, where a^0 and the a^k are scalars (summation convention assumed), and the correspondence works via the basic identification

$$|\psi\rangle = \begin{pmatrix} a^0 + ja^3 \\ -a^2 + ja^1 \end{pmatrix} \leftrightarrow \psi = a^0 + a^k i\sigma_k. \tag{4}$$

We will call ψ a *spinor*, as one of its key properties is that it has a single-sided transformation law under rotations (section 3).

To show that this identification works, we also need the translation of the angular momentum operators on spin space. We will denote these operators $\hat{\sigma}_k$, where as usual

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

The translation scheme is then

$$|\phi\rangle = \hat{\sigma}_k\, |\psi\rangle \leftrightarrow \phi = \sigma_k \psi \sigma_3 \qquad (k = 1, 2, 3). \tag{6}$$

Verifying that this works is a matter of computation, e.g.

$$\hat{\sigma}_x\, |\psi\rangle = \begin{pmatrix} -a^2 + ja^1 \\ a^0 + ja^3 \end{pmatrix} \leftrightarrow -a^2 + a^3 i\sigma_1 - a^0 i\sigma_2 + a^1 i\sigma_3 = \sigma_1 \left(a^0 + a^k i\sigma_k\right) \sigma_3, \tag{7}$$

demonstrates the correspondence for $\hat{\sigma}_x$. Finally we need the translation for the action of j upon a state $|\psi\rangle$. This can be seen to be

$$|\phi\rangle = j\, |\psi\rangle \leftrightarrow \phi = \psi\, i\sigma_3. \tag{8}$$

We note this operation acts solely to the *right* of ψ. The significance of this will be discussed later.

An implicit notational convention should be apparent above. Conventional quantum states will always appear as bras or kets, while their STA equivalents will be written using the same letter but without the brackets. Operators (e.g. upon spin space) will be denoted by carets. We do not at this stage need a special notation for operators in STA, because the rôle of operators is taken over by right or left multiplication by elements from the same Clifford algebra as the spinors themselves are taken from. This is the first example of a conceptual unification afforded by STA — 'spin space' and 'operators upon spin space' become united, with both being just multivectors in real space. Similarly the unit imaginary j is disposed of to become another element of the same kind, which in the next section we show has a clear geometrical meaning.

In order to extend these results to 4-dimensional spacetime, we need the full 16-component STA, which is generated by four vectors γ_μ. This has basis elements 1 (scalar), γ_μ (vectors), $i\sigma_k$ and σ_k (bivectors), $i\gamma_\mu$ (pseudovectors) and i (pseudoscalar) ($\mu = 0, \ldots, 3;\ k = 1, 2, 3$). The even elements of this space, 1, σ_k, $i\sigma_k$ and i, coincide with the full Pauli algebra. Thus vectors in the Pauli algebra become bivectors as viewed from the Dirac algebra. The precise definitions are

$$\sigma_k \equiv \gamma_k\gamma_0 \quad \text{and} \quad i \equiv \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3. \tag{9}$$

Note that though these algebras share the same pseudoscalar i, this *anti*-commutes with the spacetime vectors γ_μ. Note also that reversion in this algebra (also denoted by a tilde — $\tilde{R}$), reverses the sign of all bivectors, so does not coincide with Pauli reversion. In matrix terms this is the difference between the Hermitian and Dirac adjoints. It should be clear from the context which is implied.

A 4-component Dirac column spinor $|\psi\rangle$ is put into a one to one correspondence with an even element of the Dirac algebra ψ [6] via

$$|\psi\rangle = \begin{pmatrix} a^0 + ja^3 \\ -a^2 + ja^1 \\ -b^3 + jb^0 \\ -b^1 - jb^2 \end{pmatrix} \leftrightarrow \psi = a^0 + a^k i\sigma_k + i(b^0 + b^k i\sigma_k). \tag{10}$$

The resulting translation for the action of the operators $\hat{\gamma}_\mu$ is

$$\hat{\gamma}_\mu\,|\psi\rangle \leftrightarrow \gamma_\mu\psi\gamma_0 \quad (\mu = 0, \ldots, 3), \tag{11}$$

which follows if the $\hat{\gamma}$ matrices are defined in the standard Dirac-Pauli representation [7]. Verification is again a matter of computation, and further details are given in [5]. The action of j is the same as in the Pauli case,

$$j\,|\psi\rangle \leftrightarrow \psi\, i\sigma_3. \tag{12}$$

3 Rotations and Bilinear Covariants

In STA, the vectors σ_k are simply the basis vectors for 3-dimensional space, which means that the translation (6) for the action of the $\hat{\sigma}_k$ can be recast in a particularly suggestive form. Let n be a unit vector, then the eigenvalue equation for the measurement of spin in a direction n is conventionally

$$n\cdot\hat{S}\,|\psi\rangle = \pm\frac{\hbar}{2}\,|\psi\rangle\,, \tag{13}$$

where in this scheme $\hat{S}$ is a 'vector', with 'components' $\hat{S}_k = (\hbar/2)\hat{\sigma}_k$. Now $n\cdot\hat{S} = \frac{\hbar}{2}n^k\hat{\sigma}_k$, so the STA translation for this equation is just

$$n\psi\sigma_3 = \pm\psi, \tag{14}$$

where n is a (true) vector in ordinary 3-dimensional space. Multiplying on the right by $\sigma_3\tilde{\psi}$ ($\tilde{\psi} = a^0 - a^k i\sigma_k$), yields

$$n\psi\tilde{\psi} = \pm\psi\sigma_3\tilde{\psi}. \tag{15}$$

Now $\psi\tilde{\psi}$ is a scalar in the Pauli case

$$|\psi|^2 \equiv \psi\tilde{\psi} = \tilde{\psi}\psi \tag{16}$$

$$= (a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2, \tag{17}$$

so we can write

$$n = \pm\frac{\psi\sigma_3\tilde{\psi}}{|\psi|^2}. \tag{18}$$

This shows that the wavefunction ψ is in fact an instruction on how to rotate the fixed reference direction σ_3 and align it parallel or anti-parallel with the desired direction n. The amplitude just gives a change of scale. This idea, of taking a fixed or 'fiducial' direction, and transforming it to give the particle spin axis, is a central one for the development of our physical interpretation of quantum mechanics.

In the relativistic case, $\psi\tilde{\psi}$ is not necessarily a pure scalar, and we have $\psi\tilde{\psi} = \tilde{\psi}\psi = \rho e^{i\beta}$. The relativistic wavefunction ψ now specifies a spin axis s via $s = \rho^{-1}\psi\gamma_3\tilde{\psi}$, and a complete set of body axes e_μ via

$$e_\mu = \rho^{-1}\psi\gamma_\mu\tilde{\psi}. \tag{19}$$

$e_0 = v$ is interpreted as the particle 4-velocity, while ρv is the standard Dirac probabilty current — see [5] for further details. The main change in viewpoint on going to the STA should now be apparent — instead of the discrete and discontinous language of operators, eigenstates and eigenvalues we now have the idea of continuous families of transformations. This enables us to give a realistic physical description of particle tracks and spin directions in interaction with external apparatus [8].

One of the great advantages of geometric algebra is the way that rotation of a general multivector is achieved in exactly the same fashion as for a single vector. Thus to discuss Lorentz rotations for example, let us write $\psi = \rho^{1/2} e^{i\beta/2} R$. Then R is an even multivector satisfying $R\tilde{R} = \tilde{R}R = 1$ and therefore corresponds to a Lorentz rotation (combination of pure boost and spatial rotation). To rotate an arbitrary multivector M we just form the analogue of (19) and write

$$M' = RM\tilde{R}. \tag{20}$$

This is a very quick way of obtaining the transformation formulae for electric and magnetic fields for example. If we use the whole wavefunction, which incorporates information about the particle density, ρ, and also the β factor, and use it to rotate a given fixed Clifford entity such as the γ_0 and γ_3 considered above, then we get a physical density for some quantity. For example, the spin angular momentum density for a Dirac particle is the bivector $\frac{1}{2}\hbar\psi i\sigma_3\tilde{\psi}$. (Note the combination $\psi \ldots \tilde{\psi}$ preserves grade for objects of grade 1, 2 and 3.) Such expressions can generally be written equivalently as bilinear covariants in conventional Dirac theory notation — for example, $\rho v = \psi\gamma_0\tilde{\psi}$, the Dirac current, would be written conventionally as $j^\mu = \langle\overline{\psi}|\hat{\gamma}^\mu|\psi\rangle$ — but in the STA version the meaning of the expression is usually much clearer. We mention this point, since it will transpire that many of the quantities of importance for 2-spinors and twistors turn out to be bilinear covariants of the above kind, which could therefore in principle also be translated into the Dirac notation, but again, look more straightforward in our version.

As a final comment, we should discuss the way in which specific Clifford elements such as γ_0 and $i\sigma_3$ enter expressions such as $\rho v = \psi\gamma_0\tilde{\psi}$, and why general Lorentz covariance is not compromised by this. What is happening is that the wavefunction ψ is an instruction to rotate *from* some fixed set of multivectors *to* the configuration required (by the Dirac equation for example) at some given spacetime point. If we desire the final configurations (at all positions) to be rotated an extra amount R, then we must use a new wavefunction $\psi' = R\psi$. This of course explains the usual spinor transformation law under a global rotation of space, but also shows us why we do *not* want to rotate the elements we started from as well. Thus general covariance and invariance under global Lorentz rotations is assured if all quantities appearing to the *left* of the wavefunction make no mention of specific axes, directions *etc.*, while those to the *right* are allowed to do so, but must remain fixed under such a rotation.

As a complementary exercise, one might decide to rotate the elements (such as γ_0, $i\sigma_3$, *etc.*) we start from, by R say, leaving the *final* configuration fixed. In this case we have $\psi' = \psi\tilde{R}$. This is what happens under a change of 'phase' for example, where $|\psi\rangle \mapsto e^{j\theta}|\psi\rangle$. Here the STA equivalent undergoes $\psi \mapsto \psi e^{\theta i\sigma_3}$, which thus corresponds to a rotation of starting orientation through 2θ radians about the fiducial σ_3 direction. The action of j itself is thus a rotation through π about the σ_3 axis. Note particularly that only one copy of real spacetime is necessary to represent what is going on in this process.

4 2-spinors

Having been explicit about our translation of quantum Dirac and Pauli spinors, we are now in a position to begin the translation of 2-spinor theory. For the latter we adopt the notation and conventions of the standard exposition, [9, 10].

The basic translation is as follows. In 2-spinor theory, a spinor can be written either as an abstract index entity κ^A, or as a complex spin vector in spin-space (just like a quantum Pauli spinor) $\underline{\boldsymbol{\kappa}}$. We put a 2-spinor κ^A in 1-1 correspondence with a Clifford spinor κ via

$$\kappa^A \leftrightarrow \kappa(1+\sigma_3), \tag{21}$$

where κ is the Clifford Pauli spinor in one to one correspondence with the column spinor $\underline{\boldsymbol{\kappa}}$ (via 4). The function of the 'fiducial projector' $(1+\sigma_3)$ (actually half this must be taken to get a projection operator) relates to what happens under a 'spin transformation' represented by an arbitrary complex spin matrix $\underline{\boldsymbol{R}}$. The new spin vector is $\underline{\boldsymbol{R\kappa}}$ and has only 4 real degrees of freedom, whereas an arbitrary Lorentz rotation specified by a Clifford R applied to a Clifford κ gives the quantity $R\kappa$, which contains 8 degrees of freedom. However, applying R to $\kappa(1+\sigma_3)$ limits the degrees of freedom back to 4 again, in conformity with what happens in the 2-spinor formulation.

The complex conjugate spinor $\overline{\kappa}^{A'}$ belongs to the opposite ideal under the action of the projector $(1+\sigma_3)$,

$$\overline{\kappa}^{A'} \leftrightarrow -\kappa i\sigma_2(1-\sigma_3). \tag{22}$$

This explains why κ^A and its complex conjugate have to be treated as belonging to different 'modules' in the Penrose and Rindler theory. Note that in more conventional quantum notation our projectors $(1\pm\sigma_3)$ would correspond to the chirality operators $(1 \pm j\hat{\gamma}_5)$, or in the notation of the appendix of [10], to (multiples of) $\underline{\mathbf{\Pi}}$ and $\underline{\tilde{\mathbf{\Pi}}}$. We do not use these alternative notations since it is a vital part of what we are doing that the projection operators should be constructed from ordinary spacetime entities.

The most important quantities associated with a single 2-spinor κ^A are its *flagpole* $K^a = \kappa^A \overline{\kappa}^{A'}$, and the *flagplane* determined by the bivector $P^{ab} = \kappa^A \kappa^B \epsilon^{A'B'} + \epsilon^{AB} \overline{\kappa}^{A'} \overline{\kappa}^{B'}$. Here we use the Penrose notation in which a is a 'lumped index' representing the spinor indices AA' *etc.* Now in order to get a precise translation for quantities like $\kappa^A \overline{\kappa}^{A'}$, or $\kappa^A \kappa^B \epsilon^{A'B'}$, it is necessary to develop 'multiparticle STA' [11]. This still involves real spacetime, but with a separate copy for each particle. We have carried this out and thereby found the STA equivalents of 2-spinor outer product expressions. However, we have also discovered a mapping from the spin-$\frac{1}{2}$ space of a single spinor to the spin-1 space of general complex world vectors (as Penrose & Rindler call them), which applied in reverse enables us to find 'spin-$\frac{1}{2}$' (*i.e.* just one copy of spacetime) equivalents for the lumped index expressions. It is these equivalents we give now, and proper proofs are contained in [5].

Firstly, if we write $\psi = \kappa(1 + \sigma_3)$, the flagpole of the 2-spinor κ^A is just (up to a factor 2) the Dirac current associated with the wavefunction ψ,

$$K = \tfrac{1}{2}\psi\gamma_0\tilde{\psi} = \kappa(\gamma_0 + \gamma_3)\tilde{\kappa}. \tag{23}$$

We see that the projector $(1 + \sigma_3)$ has produced a massless (null) current.

Secondly, the flagplane bivector is a rotated version of the fiducial bivector σ_1:

$$P = \tfrac{1}{2}\psi\sigma_1\tilde{\psi} = \kappa(\gamma_1 \wedge (\gamma_0 + \gamma_3))\tilde{\kappa}. \tag{24}$$

Since σ_1 anticommutes with $i\sigma_3$, while γ_0 commutes, P responds at double rate to phase rotations $\kappa \mapsto \kappa e^{i\sigma_3\theta}$, whilst the flagpole is unaffected. A convenient spacelike vector L, perpendicular to the flagpole and satisfying $P = L \wedge K$, is $L = (\kappa\tilde{\kappa})^{-1/2}\kappa\gamma_1\tilde{\kappa}$, that is, just the 'body' 1-direction.

In 2-spinor theory, a 'spin-frame' is usually written o^A, ι^A, but for notational reasons, and to draw out the parallel with twistors, we prefer to write these as ω^A, π^A. In our translation, a spin-frame ω^A, π^A is packaged together to form a Clifford Dirac spinor ϕ via

$$\phi = \omega\tfrac{1}{2}(1 + \sigma_3) - \pi i\sigma_2\tfrac{1}{2}(1 - \sigma_3). \tag{25}$$

Now

$$\phi\tilde{\phi} = \tfrac{1}{2}\kappa(1 + \sigma_3)i\sigma_2\tilde{\omega} + \text{ reverse} = \lambda + i\mu \qquad \text{say.} \tag{26}$$

If one now calculates the 2-spinor inner product for the same spin-frame one finds

$$\{\underline{\boldsymbol{\omega}}, \underline{\boldsymbol{\pi}}\} = \omega_A \pi^A = -(\lambda + j\mu). \tag{27}$$

Thus the complex 2-spinor inner product is in fact a disguised version of the quantity $\phi\tilde{\phi}$. The 'disguise' consists of representing something that is in fact a pseudoscalar (the i in $\lambda + i\mu$) as an uninterpreted scalar j. The condition for a spin frame to be normalized, $\omega_A \pi^A = 1$, is in our approach the condition for ϕ to be a Lorentz transformation, that is $\phi\tilde{\phi} = 1$ (except for a change of sign which in twistor terms corresponds to negative helicity). We can thus say "*a normalized spin frame is equivalent to a Lorentz transformation*".

The orthonormal real tetrad, t^a, x^a, y^a, z^a, determined by such a spin-frame [9, p120], is in fact the same (up to signs) as the frame of 'body axes' $e_\mu = \phi\gamma_\mu\tilde{\phi}$ which we drew attention to in standard Dirac theory, whilst the null tetrad is just a rotated version of a certain 'fiducial' null tetrad as follows:

$$l^a = \frac{1}{\sqrt{2}}(t^a + z^a) = \omega^A \overline{\omega}^{A'} \quad \leftrightarrow \quad \phi(\gamma_0 + \gamma_3)\tilde{\phi}, \tag{28}$$

$$n^a = \frac{1}{\sqrt{2}}(t^a - z^a) = \pi^A \overline{\pi}^{A'} \quad \leftrightarrow \quad \phi(\gamma_0 - \gamma_3)\tilde{\phi}, \tag{29}$$

$$m^a = \frac{1}{\sqrt{2}}(x^a - jy^a) = \omega^A \overline{\pi}^{A'} \quad \leftrightarrow \quad -\phi(\gamma_1 + i\gamma_2)\tilde{\phi}, \tag{30}$$

$$\overline{m}^a = \frac{1}{\sqrt{2}}(x^a + jy^a) = \pi^A \overline{\omega}^{A'} \quad \leftrightarrow \quad -\phi(\gamma_1 - i\gamma_2)\tilde{\phi}. \tag{31}$$

Note that the x or y axis is inverted with respect to the world vector equivalents, which is a feature that occurs throughout our translation of 2-spinor theory. Note also that $\gamma_1 - i\gamma_2$ and $\gamma_1 + i\gamma_2$ involve *trivector* components. This is how complex world vectors in the Penrose & Rindler formalism appear when translated down to equivalent objects in a single-particle STA space. We shall find a use for these shortly as supersymmetry generators.

5 Valence-1 Twistors

On page 47 of [10] the authors state '*Any temptation to identify a twistor with a Dirac spinor should be resisted. Though there is a certain formal resemblance* at one point*, the vital twistor dependence on position has no place in the Dirac formalism.*' We argue on the contrary that a twistor *is* a Dirac spinor, with a particular dependence on position imposed. Our fundamental translation is

$$Z = \phi - r\phi\,\gamma_0\, i\sigma_3\, \tfrac{1}{2}(1+\sigma_3), \tag{32}$$

where ϕ is an arbitrary constant relativistic STA spinor, and $r = x^\mu \gamma_\mu$ is the position vector in 4-dimensions. To start making contact with the Penrose notation, we decompose the Dirac spinor Z, quite generally, as

$$Z = \omega\, \tfrac{1}{2}(1+\sigma_3) - \pi\, i\sigma_2\, \tfrac{1}{2}(1-\sigma_3). \tag{33}$$

Then the pair of Pauli spinors ω and π are the translations of the 2-spinors ω^A and $\pi_{A'}$ appearing in the usual Penrose representation

$$Z^\alpha = (\omega^A, \pi_{A'}). \tag{34}$$

In (34) $\pi_{A'}$ is constant and ω^A is meant to have the fundamental twistor dependence on position

$$\omega^A = \omega_0^A - j x^{AA'} \pi_{A'}, \tag{35}$$

where ω_0^A is constant. We thus see that the arbitary constant spinor ϕ in (32) is

$$\phi = \omega_0\, \tfrac{1}{2}(1+\sigma_3) - \pi\, i\sigma_2\, \tfrac{1}{2}(1-\sigma_3). \tag{36}$$

We note this is identical to the STA representation of a spin-frame.

This ability, in the STA, to package the two parts of a twistor together, and to represent the position dependence in a straightforward fashion, leads to some remarkable simplifications in twistor analysis. This applies both with regard to connecting the twistor formalism with physical properties of particles (spin, momentum, helicity, etc.), and to the sort of computations required for establishing the geometry associated with a given twistor.

For present purposes, we confine ourselves to establishing the link with massless particles, and define a set of quantities to represent various properties of such particles (most of which are useful in the formulation of twistor geometry as well). These are basically just the bilinear covariants of Dirac theory, adapted to the massless case. Firstly, the null momentum associated with the particle is

$$p = Z\,(\gamma_0 - \gamma_3)\,\tilde{Z}. \tag{37}$$

This is constant (independent of spacetime position), since

$$Z\,(\gamma_0-\gamma_3)\,\tilde{Z}=\phi\,(\gamma_0-\gamma_3)\,\tilde{\phi}=\pi\,(1+\sigma_3)\tilde{\pi}\,\gamma_0. \tag{38}$$

p thus points in the flagpole direction of π. Secondly, the flagpole of the twistor itself, defined as the flagpole of its principal part ω^A, is the null vector

$$w=Z\,(\gamma_0+\gamma_3)\,\tilde{Z}. \tag{39}$$

Evaluated at the origin, this becomes

$$w_0=\phi\,(\gamma_0+\gamma_3)\,\tilde{\phi}=\omega_0\,(1+\sigma_3)\,\tilde{\omega}_0\,\gamma_0. \tag{40}$$

Thirdly, we define an angular momentum bivector in the usual way for Dirac theory (see above)

$$M=Z\,i\sigma_3\,\tilde{Z}. \tag{41}$$

Substituting from (32) for Z yields (in two lines)

$$M=M_0+r\wedge p, \tag{42}$$

where the constant part M_0 is given by

$$M_0=\phi\,i\sigma_3\,\tilde{\phi}. \tag{43}$$

This angular momentum coincides with the real skew tensor field

$$M^{ab}=i\omega^{(A}\overline{\pi}^{B)}\epsilon^{A'B'}-i\overline{\omega}^{(A'}\pi^{B')}\epsilon^{AB}, \tag{44}$$

on page 68 of [10], who have

$$M^{ab}=M_0^{ab}-x^ap^b+x^bp^a. \tag{45}$$

The key calculation showing that (41) is the correct angular momentum, is to demonstrate that the Pauli-Lubanski vector for this massless case is proportional to the momentum. In the STA, the Pauli-Lubanski vector (the non-orbital part of the angular momentum, expressed as a vector) is given generally by

$$S=p\cdot(iM). \tag{46}$$

Now $p\cdot(iM) = p\cdot(iM_0 + ir\wedge p)$ and $p\cdot(ir\wedge p) = -i(p\wedge r\wedge p) = 0$. Also

$$p\, iM_0 = \phi\,(\gamma_0 - \gamma_3)\,\tilde{\phi}\, i\phi\, i\sigma_3\,\tilde{\phi}, \tag{47}$$

so that writing $\phi\tilde{\phi} = \tilde{\phi}\phi = \rho e^{i\beta}$, we have

$$p\, iM_0 = -\rho e^{-i\beta}\,\phi(-\gamma_3 + \gamma_0)\,\tilde{\phi} \tag{48}$$

and therefore

$$S = -\rho\cos\beta\, p. \tag{49}$$

The helicity s is thus just minus the scalar part of the product $\phi\tilde{\phi}$.

6 Field Supersymmetry Generators

A common version of the field supersymmetry generators required for the Poincaré super-Lie algebra uses 2-spinors Q_α with Grassmann entries:

$$Q_\alpha = -i\left(\frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\alpha'}\overline{\theta}^{\alpha'}\partial_\mu\right), \tag{50}$$

where the θ^α and $\overline{\theta}^\alpha$ are Grassmann variables, and μ is a spatial index [12, 13, 14]. A translation of Q_α into STA basically amounts to finding real spacetime representations for the θ^α variables. Using 2-particle STA we have found such representations, and they turn out to be two distinct copies of the complex null tetrad discussed above. The two copies arise in a natural fashion in our version of 2-spinor theory, but are harder to spot in a conventional approach.

This has an interesting 'single particle' equivalent, using the 4 quantities $\gamma_0 \pm \gamma_3$ and $\gamma_1 \pm i\gamma_2$ as effective Grassmann variables, with the anticommutator $\{A, B\}$ replaced by the symmetric product $\langle\tilde{A}B\rangle$. With

$$\begin{array}{ll} \theta_1 = \gamma_0 + \gamma_3 & \overline{\theta}_1 = \gamma_0 - \gamma_3 \\ \theta_2 = \gamma_1 + i\gamma_2 & \overline{\theta}_2 = -\gamma_1 + i\gamma_2 \end{array}$$

it is a simple exercise to verify that the θ_α satisfy the required supersymmetry algebra (with $\{A, B\} \equiv \langle\tilde{A}B\rangle$)

$$\{\theta_\alpha, \theta_\beta\} = \{\overline{\theta}_\alpha, \overline{\theta}_\beta\} = 0, \qquad \{\theta_\alpha, \overline{\theta}_\beta\} = 2\delta_{\alpha\beta}. \tag{51}$$

This raises interesting new possibilites, similar to those outlined in [15], of being able to reduce the arena of 'superspace' to ordinary spacetime, without in any way diminishing its richness or interest.

7 Conclusions

When 2-spinors and twistors are absorbed into the framework of spacetime algebra, they become both easier to manipulate and interpret, and many parallels are revealed with ordinary Dirac theory. In particular the bilinear covariants of Dirac theory (expressed in STA), turn out to be precisely those needed to understand the rôle of higher valence spinors and twistors. As a byproduct of the translation we have shown that a commutative scalar imaginary is unnecessary in the formulation of 2-spinor and twistor theory. Furthermore, had space permitted, we would have presented a discussion of the mapping we have constructed between lumped vector index expressions, and spin-$\frac{1}{2}$ equivalents. This would have made it evident that the notion that 2-spinor or twistor space is more fundamental than the space of ordinary vectors or tensors, is misplaced. In our version the spinor space itself is imbued with all the metrical properties of spacetime, and the construction of vectors and tensors using outer products of spinors (as given in Penrose & Rindler for example) can be shown via our translation to use precisely the metrical properties already present at the so-called spinor level (which is in fact just ordinary spacetime).

Normalized spin-frames have been shown to be identical to Lorentz transforms, with spin frames in general identical to constant Dirac spinors (even multivectors in the STA approach). Twistors themselves have been shown to be Dirac spinors, with a particular position dependence imposed, and the physical quantities constructed from them to be just the standard Dirac bilinear covariants. It is therefore clear that some of the claims of the 'strong twistor' programme, as described in e.g. [16], must appear in a new light, though the full implications remain to be worked out.

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