Grassmann Calculus, Pseudoclassical Mechanics and Geometric Algebra

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Abstract

We present a reformulation of Grassmann calculus in terms of geometric algebra — a unified language for physics based on Clifford algebra. In this reformulation, Grassmann generators are replaced by vectors, so that every product of generators has a natural geometric interpretation. The calculus introduced by Berezin [1] is shown to be unnecessary, amounting to no more than an algebraic contraction. Our approach is not only conceptually clearer, but it is computationally more efficient, which we demonstrate by treatments of the 'Grauss' integral and the Grassmann Fourier Transform. Our reformulation is applied to pseudoclassical mechanics [2], where it is shown to lead to a new concept, the multivector Lagrangian. To illustrate this idea, the 3-dimensional Fermi oscillator is reformulated and solved, and its symmetry properties discussed. As a result, a new and highly compact formula for generating super-Lie algebras is revealed. We finish with a discussion of quantization, outlining a new approach to fermionic path integrals.

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1 Introduction

Grassmann variables have become of great importance in modern theoretical physics, playing a central rôle in areas such as second quantization, non-Abelian gauge theory and supersymmetry [1, 3]. They are generally thought to require extra 'Grassmann' degrees of freedom, quite separate from the degrees of freedom associated with ordinary vectors in spacetime. Our aim in this paper is to show that the introduction of extra Grassmann dimensions is unnecessary, and that the ordinary geometric properties of vectors in real Euclidean space are sufficient to account for all the properties of both Grassmann algebra and Berezin calculus. In order to achieve this, we make use of the associative 'geometric product' between vectors, $ab = a{\cdot}b + a{\wedge}b$ (this was defined independently by Clifford [4] and Grassmann [5]). In this expression, $a{\cdot}b$ is the usual inner product, and $a{\wedge}b$ is Grassmann's exterior product. The latter results in a 'bivector', which can be thought of as a section of an oriented plane containing a and b. By utilising the exterior part of this product, it is a simple matter to represent a Grassmann algebra within a Clifford algebra. The great advantage of this approach is that the remaining, interior, part of the product is precisely what is needed to carry out the calculations which are conventionally done with Berezin calculus.

If $\{\sigma_i\}$ are a set of orthonormal frame vectors, then, under the geometric product, these satisfy the relations

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}, \tag{1.1}$$

and thus generate a Clifford algebra. Clifford algebras have, of course, been used in physics for many years, in the guise of the Pauli and Dirac matrices. Our point of departure, prompted by the work of Hestenes [6], is to drop any connection with matrix representations, and treat the $\{\sigma_i\}$, and all quantities formed from these, as geometric entities in real space. The reward for this shift in view is that it becomes possible to 'geometrize' many of the concepts of modern theoretical physics, by locating them in the real physical geometry of space or spacetime. Hestenes [7] has already demonstrated that the Dirac, Pauli and Schrödinger equations can all be expressed geometrically in real space or spacetime. Over the course of a series of papers [8, 9, 10, 11] we shall demonstrate that Grassmann dimensions, point-particle and field supersymmetry, 2-spinors and twistors can similarly be expressed geometrically. Furthermore, this is achieved without the use of a commutative scalar imaginary, hitherto thought to be essential in modern physics (see for example [12]).

For the general theory of Grassmann calculus (the algebra of fermion creation and annihilation operators) dealt with in this paper, the geometrization is carried out by identifying the Grassmann variables as ordinary vectors in n-dimensional Euclidean space. This enables quantities to be manipulated in ways that have no counterpart in the prototype Grassmann system, by utilising the associative geometric product. We illustrate this with treatments of the 'Grauss' integral, and the Grassmann Fourier transform. The latter can be formulated in geometric algebra as a rotation through $\pi/2$, so that the 'Grassmann Fourier inversion theorem' reduces to the simple fact that a rotation, followed by its inverse, gives the identity. Similarly, once a Grassmann system has been formulated in geometric algebra, it can be extended in ways previously unavailable, producing new mathematics and the possibility of new physics. An example of this is the concept of a multivector Lagrangian, which arises from the translation of Grassmann-valued Lagrangians, but requires geometric algebra to be developed to its full potential [8].

Having introduced geometric algebra in Section 2, and dealt with the translation of Grassmann calculus into geometric algebra in Section 3, much of the rest of this paper is devoted to the illustrative example of 'pseudoclassical' mechanics. A pseudoclassical system is one in which the dynamical variables are Grassmann-valued, and such systems are often introduced as models for the classical mechanics of spin-$\frac{1}{2}$ particles. After reformulating a particular example, the 3-dimensional Fermi oscillator, we are able to solve it explicitly, and study its symmetry properties with a generalization of Noether's theorem. In doing so, we find that a key rôle is played by the fiducial tensor, which is the symmetric square root of the metric tensor. The ubiquity of this tensor in our approach suggests that it has a fundamental importance, and the techniques we introduce for handling it are likely to prove useful in other fields. A by-product of this work worth emphasising is a new, matrix-free, way of generating super-Lie algebras. This generalizes the approach to Lie algebras developed in [13], and should have significant applications beyond the Grassmann Poisson bracket context treated here.

Our treament of pseudoclassical mechanics ends with a discussion of quantization, from both the canonical and path-integral viewpoints. Canonical quantization is shown to amount to a restriction to a *classical* $g = 2$ spinning particle (though in the non-relativistic case the g-factor is put in by hand). Hamilton's equations also have a natural classical interpretation after quantization, in which time derivatives are given by the commutator with a bivector. The meaning of path-integral quantization is less clear, and we outline an alternative possibility, in which Berezin integration (contraction) is replaced by genuine Riemann integration over the dynamical variables of the system, as expressed in geometric algebra. An appendix contains the details of the translation of the Grassmann Fourier Transform.

2 An Introduction to Geometric Algebra

In this section we give an outline of geometric algebra, concentrating on the definitions and results needed for this paper. We have endeavoured to keep this self-contained, whilst being as succinct as possible. Those familiar with geometric algebra will only need to read this section to discover our conventions, but others may like to study one or two of the following references. The most detailed and comprehensive text on geometric algebra is [14], and most of the results of this section can be found in greater detail there. More pedagogical introductions are provided in [6] and [15], and some aspects are covered in detail in [16]. A useful list of recommended additional texts is contained in [17].

2.1 Axioms and Definitions

It should be stressed from the outset that there is more to geometric algebra than just Clifford algebra. To paraphrase from the introduction to [14], Clifford algebra provides the *grammar* out of which geometric algebra is constructed, but it is only when this grammar is augmented with a number of secondary definitions and concepts that one arrives at a true geometric algebra. It is therefore preferable to introduce geometric algebra through the axioms outlined in [14], rather than through the more abstract definitions conventionally used to introduce Clifford algebras (see [18] for example).

A geometric algebra consists of a graded linear space, the elements of which are called multivectors. These are defined to have an addition, and an associative product which is distributive. The space is assumed to be closed under these operations. Multivectors are given geometric significance by identifying the grade-1 elements as vectors. The final axiom that distinguishes a geometric algebra is that the square of any vector is a *real* scalar.

From these rules it follows that the geometric product of 2 vectors a, b can be decomposed as

$$ab = a\cdot b + a\wedge b, \tag{2.1}$$

where

$$a\cdot b = \tfrac{1}{2}(ab + ba) \tag{2.2}$$

is the standard scalar, or inner, product (a real scalar), and

$$a\wedge b = \tfrac{1}{2}(ab - ba) \tag{2.3}$$

is the wedge, or outer, product originally introduced by Grassmann. This gives rise to a new quantity, a *bivector*, which represents a directed plane segment containing the vectors a and b, and is a grade-2 multivector.

This decomposition extends to the geometric product of a vector with a grade-r multivector A_r as,

$$aA_r = a\cdot A_r + a\wedge A_r, \tag{2.4}$$

where

$$a\cdot A_r = \langle aA_r\rangle_{r-1} = \tfrac{1}{2}(aA_r - (-1)^r A_r a) \tag{2.5}$$

is known as the inner product, and lowers the grade of A_r by one. Similarly,

$$a\wedge A_r = \langle aA_r\rangle_{r+1} = \tfrac{1}{2}(aA_r + (-1)^r A_r a) \tag{2.6}$$

raises the grade by one. This is usually referred to as the exterior product with a vector, and defines the grading for the entire algebra inductively. We have used the notation $\langle A\rangle_r$ to denote the result of the operation of taking the grade-r part of A (this is a projection operation). As a further abbreviation we write the scalar (grade 0) part of A simply as $\langle A\rangle$.

The entire multivector algebra can be built up by repeated multiplication of vectors. Multivectors which contain elements of only one grade are termed *homogeneous*, and will usually be written as A_r to show that A contains only a grade-r component. Homogeneous multivectors which can be expressed purely as the outer product of a set of (independent) vectors are termed *blades*.

The geometric product of two multivectors is (by definition) associative, and for two homogeneous multivectors of grade r and s this product can be decomposed as follows:

$$A_r B_s = \langle AB\rangle_{r+s} + \langle AB\rangle_{r+s-2} \ldots + \langle AB\rangle_{|r-s|}. \tag{2.7}$$

'$\cdot$' and '$\wedge$' will continue to be used for the lowest-grade and highest-grade terms of this series, i.e.

$$A_r \cdot B_s = \langle AB \rangle_{|s-r|} \tag{2.8}$$
$$A_r \wedge B_s = \langle AB \rangle_{s+r}, \tag{2.9}$$

which we call the interior and exterior products respectively. The exterior product is associative, and satisfies the symmetry property

$$A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r. \tag{2.10}$$

Two final pieces of notation are as follows. Reversion, $\tilde{A}$, reverses the order of vectors in any multivector, so that

$$(AB)\tilde{} = \tilde{B}\tilde{A}, \tag{2.11}$$

and $\tilde{a} = a$ for any vector a. It is simple to check that this implies

$$\tilde{A}_r = (-1)^{r(r-1)/2} A_r. \tag{2.12}$$

The modulus $|A|$ is defined for positive definite spaces by

$$|A|^2 = \langle A\tilde{A} \rangle \geq 0, \tag{2.13}$$

and $|A| = 0$ if and only if A=0.

Finally, we use the convention throughout that, in the absence of brackets, an inner or outer product always takes precedence over a geometric product.

2.2 Orthonormal Bases and Clifford Algebras

The definitions of Section 2.1 are general to all geometric algebras, regardless of metric signature, however in this paper we shall be concerned almost entirely with finite-dimensional Euclidean algebras. A finite algebra is generated by the introduction of a set of n independent frame vectors $\{e_i\}$, which leads to a geometric algebra with the basis

$$1, \quad \{e_i\}, \quad \{e_i \wedge e_j\}, \quad \{e_i \wedge e_j \wedge e_k\}, \quad \ldots, \quad e_1 \wedge e_2 \ldots \wedge e_n. \tag{2.14}$$

Any multivector can now be expanded in this basis, but it should be emphasised that one of the strengths of geometric algebra is that it possible to carry out many calculations in a *basis-free* way. The above basis need not be orthonormal, and for much of this paper we will be concerned with frames where no restrictions are placed on the inner product.

The highest-grade blade in this algebra is given the name 'pseudoscalar' (or directed volume element), and is of special significance in geometric algebra. Its unit is given the special symbol I (or i in three or four dimensions). It is a pure blade, and a knowledge of I is sufficient to specify the vector space over which the algebra is defined (see [14]). This pseudoscalar also defines the duality operation for the algebra, since multiplication of a grade-r multivector by I results in a grade-$(n-r)$ multivector.

If we choose an orthonormal set of basis vectors $\{\sigma_k\}$, these satisfy

$$\sigma_j \cdot \sigma_k = \delta_{jk} \tag{2.15}$$

or

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk}, \tag{2.16}$$

which is the conventional starting point for the matrix representation theory of finite Clifford algebras [18, 19] (this has an obvious extension for indefinite metrics). Orthogonality of the basis vectors implies

$$\sigma_i \wedge \sigma_j = \sigma_i \sigma_j \qquad (i \neq j). \tag{2.17}$$

Note that in order to distinguish types of frame, we will use Greek letters for orthonormal vectors, and reserve Roman letters for arbitrary, i.e. not necessarily orthonormal, frames.

In Sections 5.2 and 4 we will be interested in geometric algebra in two and three dimensions respectively. The Clifford algebra of the Euclidean plane is generated by a pair of vectors $\{\sigma_1, \sigma_2\}$ satisfying (2.15), and is spanned by

$$1, \qquad \sigma_1, \sigma_2 \qquad I, \tag{2.18}$$

where $I \equiv \sigma_1 \sigma_2$. The unit pseudoscalar here satisfies $I^2 = -1$, and anticommutes with vectors. The even-grade part of this algebra forms a subalgebra, and can be put in a one-to-one correspondence with the complex field. Furthermore, there is a natural map between even elements z (complex numbers) and vectors x, via

$$x = \sigma_1 z, \tag{2.19}$$

where the vector σ_1 has been singled out as a unit vector along the real axis. In this way the complex number z can be viewed as a rotation/dilation acting on σ_1 to generate x.

The Clifford algebra for 3-dimensional space is generated by three orthonormal vectors $\{\sigma_1, \sigma_2, \sigma_3\}$, and is spanned by

$$1, \qquad \{\sigma_k\}, \qquad \{i\sigma_k\}, \qquad i \tag{2.20}$$

where $i \equiv \sigma_1\sigma_2\sigma_3$. Here the pseudoscalar squares to -1 and commutes with all elements of the algebra, and so is naturally given the symbol of the unit imaginary (in a matrix representation it will be i times the unit matrix). The algebra (2.20) is the Pauli algebra, but in geometric algebra the three Pauli matrices are no longer viewed as three components of a single isospace vector, but as three independent basis vectors for space.

3-dimensional space has the distinguishing feature that the dual of any bivector is a vector, and this is used to define the standard vector cross product as

$$a \times b = \frac{1}{i} a \wedge b. \tag{2.21}$$

A detailed development of 3-dimensional geometric algebra is contained in [20].

2.3 Linear Functions and the Outermorphism

Geometric algebra has many advantages when used for developing the theory of linear functions, as is shown in [14, Chapter 3] and [16]. Below we will summarize the conventions and notation of [16], and state a number of results without proof.

If $f(a)$ is a linear function mapping vectors to vectors (in the same space), then it can be extended via 'outermorphism' to act linearly on multivectors as follows,

$$\underline{f}(a \wedge b \wedge \ldots \wedge c) = f(a) \wedge f(b) \ldots \wedge f(c), \tag{2.22}$$

so that $\underline{f}$ is grade-preserving. An example of this, which will be useful later, is a rotation, the action of which on a vector a can be written as

$$R(a) = e^{B/2} a e^{-B/2}, \tag{2.23}$$

where B is the plane(s) of rotation. The outermorphism extension of this is

$$\underline{R}(A) = e^{B/2} A e^{-B/2}, \tag{2.24}$$

which provides a very compact way to handle rotations.

Since the pseudoscalar is unique up to a scale factor, we can define the determinant via

$$\underline{f}(I) = \det(f) I, \tag{2.25}$$

which demonstrates its rôle as the volume scale factor.

The adjoint $\overline{f}$ to f, is defined to satisfy

$$\langle \underline{f}(A) B \rangle = \langle A \overline{f}(B) \rangle, \tag{2.26}$$

which turns out to be a special case of the more general formulae,

$$\begin{array}{rclc} A_r \cdot \overline{f}(B_s) & = & \overline{f}[\underline{f}(A_r) \cdot B_s] & \quad r \leq s \\ \underline{f}(A_r) \cdot B_s & = & \underline{f}[A_r \cdot \overline{f}(B_s)] & \quad r \geq s. \end{array} \tag{2.27}$$

A symmetric function is one for which $\underline{f} = \overline{f}$. Equations (2.27) can be used to derive the inverse functions,

$$\begin{array}{rcl} \underline{f}^{-1}(A) & = & \det(f)^{-1} \overline{f}(AI) I^{-1} \\ \overline{f}^{-1}(A) & = & \det(f)^{-1} I^{-1} \underline{f}(IA). \end{array} \tag{2.28}$$

The concept of an eigenvector is generalized to that of an eigenblade A, which is a blade satisfying

$$\underline{f}(A) = \alpha A, \tag{2.29}$$

where α is a *real* eigenvalue. Eigenvectors with complex eigenvalues are replaced by eigenbivectors with real eigenvalues. These bivector blades each specify a plane for which they are the pseudoscalar, and thus define a complex structure containing more geometrical information than the scalar imaginary i.

2.4 Non-Orthonormal Frames

We shall make frequent use of non-orthonormal frames, which we usually designate $\{e_i\}$ or $\{f_i\}$. We now summarise a few results concerning these.

From the non-orthonormal set of n vectors, $\{e_i\}$, we can define the (non-zero) pseudoscalar for this frame as

$$E_n = e_1 \wedge e_2 \wedge \ldots \wedge e_n. \tag{2.30}$$

The reciprocal frame $\{e^i\}$ satisfies

$$e^i \cdot e_j = \delta^i_j, \tag{2.31}$$

and is constructed via [14, Chapter 1]

$$e^i = (-1)^{i-1} e_1 \wedge \ldots \check{e}_i \ldots \wedge e_n E^n, \tag{2.32}$$

where the check symbol on $\check{e}_i$ signifies that this vector is missing from the product, and E^n is the pseudoscalar for the reciprocal frame, defined as

$$E^n = e^n \wedge e^{n-1} \wedge \ldots \wedge e^1. \tag{2.33}$$

This satisfies

$$E_n E^n = 1 \tag{2.34}$$

$$\Rightarrow E^n = E_n/(E_n)^2. \tag{2.35}$$

The components of the vector a in the e^i frame are given by $a \cdot e_i$, so that

$$a = (a \cdot e_i) e^i, \tag{2.36}$$

where the summation convention is implied. Since $e_i e^i = n$, it follows from (2.2) that

$$e_i a e^i = (2-n)a. \tag{2.37}$$

For a multivector of grade r, this can be extended to give

$$e_i A_r e^i = (-1)^r (n-2r) A_r. \tag{2.38}$$

Thus,

$$e_i (e^i \cdot A_r) = e_i \wedge (e^i \cdot A_r) = r A_r, \tag{2.39}$$

so that the operator $\sum_i e^i \wedge (e_i \cdot$ counts the grade of its multivector argument.

The metric tensor g is an example of a symmetric linear operator, and is defined by

$$g(e^i) = e_i. \tag{2.40}$$

As a matrix, it has components $g_{ij} = e_i \cdot e_j$, and it follows from (2.25), that

$$\underline{g}(E^n) = \tilde{E}_n \tag{2.41}$$

$$\Rightarrow \det(g) = E_n \tilde{E}_n = |E_n|^2. \tag{2.42}$$

It turns out to be very convenient to work with the 'fiducial frame' $\{\sigma_k\}$, which is the orthonormal frame determined by the $\{e_i\}$ via

$$e_k = h(\sigma_k) = h_k^j \sigma_j, \tag{2.43}$$

where h is the *unique, symmetric* fiducial tensor. The requirement that h be symmetric means that the $\{\sigma_k\}$ frame must satisfy

$$\sigma_k \cdot e_j = \sigma_j \cdot e_k, \tag{2.44}$$

which, together with orthonormality, defines a set of n^2 equations that determine the σ_k (and hence h) uniquely, up to permutation. These permutations only alter the labels for the frame vectors, and do not re-define the frame itself. From (2.43) it is simple to prove that

$$\overline{h}(e^j) = h(e^j) = \sigma^j = \sigma_j, \tag{2.45}$$

from which it can be seen that h is the 'square-root' of g,

$$g(e^i) = e_i = h^2(e^i). \tag{2.46}$$

It follows that

$$\det(h) = |E_n|. \tag{2.47}$$

The fiducial tensor, together with other non-symmetric square-roots of the metric tensor, correspond to what are usually called *vierbeins* in 4-dimensional spacetime. These find many applications in the geometric calculus approach to differential geometry [21].

3 Grassmann Variables and Berezin Calculus

In this section we will outline the basis of our translation between Grassmann calculus and geometric algebra. It will be shown that the geometric algebra defined in Section 2 is sufficient to formulate all of the required concepts, thus integrating them into a single unifying framework. This is illustrated with a simple example, the 'Grauss' integral, with the more interesting example of the Grassmann Fourier transform, which demonstrates the full potential of the geometric algebra approach, contained in Appendix A. We finish the section with a discussion of further developments and some potential applications.

3.1 The Translation to Geometric Algebra

The basis of Grassmann calculus is described in many sources. Reference [1] is one of the earliest, and now classic, texts, a useful summary of which is contained in the Appendices to [2]. More recently, Grassmann calculus has been extended to the field of superanalysis [22, 23], as well as in other directions [24, 25].

The basis of our approach is to utilise the natural embedding of Grassmann algebra within geometric algebra, thus reversing the usual progression from Grassmann to Clifford algebra via quantization. Throughout this paper we will retain the symbol ζ_i for Grassmann variables, and use the symbol $\leftrightarrow$ to show that we are translating from one language to the other. We start with a set of Grassmann variables ζ_i, satisfying the anticommutation relations

$$\{\zeta_i, \zeta_j\} = 0. \tag{3.1}$$

In this paper we are only concerned with Grassmann variables which carry vector indices; spinors with Grassmann entries will be treated in a later paper. In geometric algebra we will represent each Grassmann variable ζ_i by a vector e_i, and the product of Grassmann variables by an *exterior product*, so

$$\zeta_i\zeta_j \quad \leftrightarrow \quad e_i \wedge e_j, \tag{3.2}$$

where $\{e_i\}$ are a set of arbitrary vectors spanning an n-dimensional space. Equation (3.1) is now satisfied by virtue of the antisymmetry of the exterior product,

$$e_i \wedge e_j + e_j \wedge e_i = 0. \tag{3.3}$$

The $\{e_i\}$ are not necessarily orthonormal because as these vectors represent Grassmann variables, nothing can be assumed about their inner product. Any arbitrary Grassmann element built out of a string of the $\{\zeta_i\}$ can now be translated into a multivector.

Next, we need a translation for the calculus introduced by Berezin [1]. In this calculus, differentiation is defined by the rules

$$\begin{aligned} \frac{\partial \zeta_j}{\partial \zeta_i} &= \delta_{ij}, && (3.4)\\ \zeta_j \frac{\overleftarrow{\partial}}{\partial \zeta_i} &= \delta_{ij}, && (3.5) \end{aligned}$$

together with the 'graded Leibnitz rule',

$$\frac{\partial}{\partial \zeta_i}(f_1 f_2) = \frac{\partial f_1}{\partial \zeta_i} f_2 + (-1)^{[f_1]} f_1 \frac{\partial f_2}{\partial \zeta_i}, \tag{3.6}$$

where $[f_1]$ is the parity (even/odd) of f_1. Our translation of this is achieved by introducing the reciprocal frame $\{e^i\}$, and replacing

$$\frac{\partial}{\partial \zeta_i}(\quad \leftrightarrow \quad e^i\cdot(\tag{3.7}$$

so that

$$\frac{\partial \zeta_j}{\partial \zeta_i} \quad \leftrightarrow \quad e^i\cdot e_j = \delta^i_j. \tag{3.8}$$

We stress that we are using upper and lower indices to distinguish a frame from its reciprocal frame, whereas Grassmann algebra only uses these indices to distinguish metric signature.

The graded Leibnitz rule follows simply from the axioms of geometric algebra. For example, if f_1 and f_2 are grade-1 and so, upon translation, are replaced by vectors a and b, then the rule (3.6) becomes

$$e^i\cdot(a\wedge b) = e^i\cdot ab - ae^i\cdot b. \tag{3.9}$$

This expresses one of the most useful identities of geometric algebra,

$$a\cdot(b\wedge c) = a\cdot bc - a\cdot cb, \tag{3.10}$$

for any three vectors a, b, c.

Right differentiation translates in the same way,

$$)\frac{\overleftarrow{\partial}}{\partial \zeta_i} \quad \leftrightarrow \quad)\cdot e^i, \tag{3.11}$$

and the standard results for Berezin second derivatives [1] can also be verified simply. For example, given that F is the multivector equivalent of the Grassmann variable $f(\zeta)$,

$$\begin{aligned}
\frac{\partial}{\partial \zeta_i}\frac{\partial}{\partial \zeta_j} f(\zeta) \quad \leftrightarrow & \quad e^i\cdot(e^j\cdot F) \\
= & \quad (e^i\wedge e^j)\cdot F \\
= & \quad -e^j\cdot(e^i\cdot F)
\end{aligned} \tag{3.12}$$

shows that second derivatives anticommute, and

$$\begin{aligned}
\left(\frac{\partial f}{\partial \zeta_i}\right)\frac{\overleftarrow{\partial}}{\partial \zeta_j} \quad \leftrightarrow & \quad (e^i\cdot F)\cdot e^j \\
= & \quad e^i\cdot(F\cdot e^j)
\end{aligned} \tag{3.13}$$

shows that left and right derivatives commute.

The final concept we need is that of integration over a Grassmann algebra. In Berezin calculus, this is defined to be the same as right differentiation (apart perhaps from some unimportant extra factors of i and 2π [23]), so that

$$\int f(\zeta) d\zeta_n d\zeta_{n-1}\ldots d\zeta_1 \equiv f(\zeta)\frac{\overleftarrow{\partial}}{\partial \zeta_n}\frac{\overleftarrow{\partial}}{\partial \zeta_{n-1}}\cdots\frac{\overleftarrow{\partial}}{\partial \zeta_1}. \tag{3.14}$$

We can translate these in exactly the same way as the right derivative (3.7). The only important formula is that for the total integral

$$\begin{aligned}
\int f(\zeta) d\zeta_n d\zeta_{n-1}\ldots d\zeta_1 \quad \leftrightarrow & \quad (\ldots((F\cdot e^n)\cdot e^{n-1})\ldots)\cdot e^1 \\
= & \quad \langle F E^n\rangle,
\end{aligned} \tag{3.15}$$

where again F is the multivector equivalent of $f(\zeta)$, as defined by (3.1). Equation (3.15) picks out the coefficient of the pseudoscalar part of F via

$$\langle F\rangle_n = \alpha E_n \tag{3.16}$$

$$\Rightarrow \langle F E^n\rangle = \alpha, \tag{3.17}$$

so that the Grassman integral simply returns the coefficient α.

A change of variables is performed by a linear transformation f, say (see

Section 2.3)

$$e_i{}' = f(e_i) \tag{3.18}$$

$$\Rightarrow \quad E_n' = \underline{f}(E_n) = \det(f) E_n. \tag{3.19}$$

However e^i transforms under $\overline{f}^{-1}$ to preserve orthonormality, so

$$e^{i\prime} = \overline{f}^{-1}(e^i) \tag{3.20}$$

$$\Rightarrow \quad E^{n\prime} = \det(f)^{-1} E^n, \tag{3.21}$$

which is the usual result for a change of variables in a Grassmann multiple integral. That $E_n' E^{n\prime} = 1$ follows from the definitions.

In this way all the basic formulae of Grassmann calculus can be derived in geometric algebra, and often the derivations are simpler. Moreover, they allow for the results of Grassmann algebra to be incorporated into a wider scheme, where they may find applications in other fields. Finally, this translation makes it clear why there can be no measure associated with Grassmann integrals: nothing is being added up!

3.2 Example: The 'Grauss' Integral

The Grassmann analogue of the Gaussian integral [1],

$$\int e^{\frac{1}{2} a^{jk} \zeta_j \zeta_k} d\zeta_n \ldots d\zeta_1 = \det(a)^{\frac{1}{2}}, \tag{3.22}$$

where a^{jk} is an antisymmetric matrix, is one of the most important results in applications of Grassmann algebra, finding use for example in fermionic path integration. It is instructive to see how this is formulated and proved in geometric algebra. First, we translate

$$\tfrac{1}{2} a^{jk} \zeta_j \zeta_k \quad \leftrightarrow \quad \tfrac{1}{2} a^{jk} e_j \wedge e_k = A, \text{ say}, \tag{3.23}$$

where A is a general bivector. The integral now becomes

$$\int e^{\frac{1}{2} a^{jk} \zeta_j \zeta_k} d\zeta_n \ldots d\zeta_1 \quad \leftrightarrow \langle (1 + A + \frac{A \wedge A}{2!} \ldots) E^n \rangle. \tag{3.24}$$

We see immediately that this is only non-zero for even n ($= 2m$ say), in which case (3.24) becomes

$$\frac{1}{m!}\langle (A)^m E^n\rangle. \tag{3.25}$$

Precisely this type of expression was considered in a different context in [14, Chapter 3], which provides a good illustration of how the systematic use of a unified language leads to new analogies and insights. In order to prove that (3.25) equals $\det(a)^{\frac{1}{2}}$, we need the result (proved in [14]) that any bivector can be written, not necessarily uniquely, as a sum of orthogonal commuting blades[1],

$$A = \alpha_1 A_1 + \alpha_2 A_2 + \ldots \alpha_m A_m, \tag{3.26}$$

where

$$\begin{aligned} A_i \cdot A_j &= -\delta_{ij} && (3.27)\\ [A_i, A_j] &= 0 && (3.28)\\ A_1 A_2 \ldots A_m &= I. && (3.29)\end{aligned}$$

Equation (3.25) now becomes, using (2.42),

$$\langle (\alpha_1\alpha_2 \ldots \alpha_m) I E^n\rangle = \det(g)^{-\frac{1}{2}}\alpha_1\alpha_2 \ldots \alpha_m. \tag{3.30}$$

If we now introduce the function

$$f(a) = A\cdot a, \tag{3.31}$$

it can be seen that the A_i blades are the eigenblades of f, with

$$\underline{f}(A_i) = \alpha_i^2 A_i, \tag{3.32}$$

so that

$$\underline{f}(I) = \underline{f}(A_1 \wedge A_2 \wedge \ldots A_m) = (\alpha_1\alpha_2 \ldots \alpha_m)^2 \underline{f}(I) \tag{3.33}$$

$$\Rightarrow \det(f) = (\alpha_1\alpha_2 \ldots \alpha_m)^2. \tag{3.34}$$

[1]This result only holds in spaces with Euclidean or Lorentzian signature [26]. Because of the way we use the inner product to represent Berezin differentiation, we are implicitly assuming a Euclidean space.

In terms of components, however,

$$\begin{aligned} f_j^k &= e_j \cdot f(e^k) \\ &= g_{jl} a^{lk}, \end{aligned} \tag{3.35}$$

$$\Rightarrow \det(f) = \det(g)\det(a). \tag{3.36}$$

Inserting (3.36) into (3.30), we have

$$\frac{1}{m!}\langle (A)^m E^n \rangle = \det(a)^{\frac{1}{2}}, \tag{3.37}$$

as required.

This result can be derived more succinctly using the fiducial frame $\sigma_i = h^{-1}(e_i)$ to write (3.24) as

$$\frac{1}{m!}\langle (A')^m I \rangle, \tag{3.38}$$

where $A' = \frac{1}{2}a^{jk}\sigma_j\sigma_k$. This automatically takes care of the factors of $\det(g)^{\frac{1}{2}}$, though it is instructive to note how these appear naturally otherwise.

Although this translation has not added much new algebraically, it has demonstrated that notions of Grassmann calculus are completely unnecessary to the problem. In many other applications, however, the geometric algebra formulation does provide for important algebraic simplifications, as we demonstrate in Appendix A. There, the Grassmann Fourier transform is expressed in geometric algebra as a rotation followed by a duality transformation. This reduces the Grassmann Fourier inversion theorem to a simple identity, the proof of which requires much more work if carried out solely within Grassmann calculus.

3.3 Further Development and Comments

Before dealing with pseudoclassical mechanics, we make some further observations. It is well known [1] that the operators

$$\hat{Q}_k = \zeta_k + \frac{\partial}{\partial \zeta_k}, \tag{3.39}$$

satisfy the Clifford algebra generating relation[2]

$$\{\hat{Q}_j, \hat{Q}_k\} = 2\delta_{jk}. \tag{3.40}$$

This can be seen from an interesting perspective in geometric algebra by utilising the fiducial tensor, as follows:

$$\begin{aligned} \hat{Q}_k a(\zeta) \quad \leftrightarrow \quad & e_k \wedge A + e^k \cdot A \\ = \quad & h(\sigma_k) \wedge A + h^{-1}(\sigma_k) \cdot A \\ = \quad & \underline{h}[\sigma_k \underline{h}^{-1}(A)], \end{aligned} \tag{3.41}$$

where A is the multivector equivalent of $a(\zeta)$ and we have used (2.27). The operator $\hat{Q}_k$ thus becomes an orthogonal Clifford vector (now Clifford multiplied), sandwiched between a symmetric distortion and its inverse. (For details on the how this h can be viewed as generating an induced geometry on the flat space of the σ_k see [14].) It is now simple to see that

$$\begin{aligned} \{\hat{Q}_j, \hat{Q}_k\} a(\zeta) \quad \leftrightarrow \quad & \underline{h}(2\sigma_j \cdot \sigma_k \underline{h}^{-1}(A)) \\ = \quad & 2\delta_{jk} A. \end{aligned} \tag{3.42}$$

The above is an example of the ubiquity of the fiducial tensor in applications involving non-orthonormal frames (we will see many more in Section 4), which makes it all the more surprising that this object is not more prominent in standard expositions of linear algebra.

Berezin [1] defines dual operators to the $\hat{Q}_k$ by

$$\hat{P}_k = \frac{1}{i}(\zeta_k - \frac{\partial}{\partial \zeta_k}), \tag{3.43}$$

though a more useful structure is derived by dropping the i, and defining

$$\hat{P}_k = \zeta_k - \frac{\partial}{\partial \zeta_k}. \tag{3.44}$$

[2] This is used by Sherry in [27, 28] as an alternative approach to quantizing a Grassmann system.

These satisfy

$$\begin{aligned} \{\hat{P}_j, \hat{P}_k\} &= -2\delta_{jk} && (3.45) \\ \{\hat{P}_j, \hat{Q}_k\} &= 0, && (3.46) \end{aligned}$$

so that the $\hat{P}_k, \hat{Q}_k$ span a $2n$-dimensional balanced algebra (signature n, n). The $\hat{P}_k$ can be translated in the same manner, this time giving (for a homogeneous multivector)

$$\begin{aligned} \hat{P}_k a(\zeta) \quad \leftrightarrow \quad & e_k \wedge A_r - e^k \cdot A_r \\ =\ & (-1)^r \underline{h}[\underline{h}^{-1}(A_r)\sigma_k], \end{aligned} \qquad (3.47)$$

so that the $\{\sigma_k\}$ frame now sits to the right of the multivector on which it operates. The factor of $(-1)^r$ accounts for the minus sign in (3.45) and for the fact that the left and right multiples anticommute in (3.46). $\hat{Q}_k$ and $\hat{P}_k$ can both be given right analogues if desired. The geometric analogues of the $\hat{P}_k$ and $\hat{Q}_k$ operators, and their relationship to the balanced (n, n) algebra, turn out to be very useful for studying linear functions. This is demonstrated in [13], where they are used to provide a new approach to linear function theory, in which all linear functions are represented as (Clifford) polynomials of vectors.

The idea of using two frames, one on either side of a multivector, is a very powerful one in many applications of geometric algebra. For example, in rigid body dynamics [20] the two frames can be used to represent the laboratory and body axes, and in the geometric algebra versions of the Pauli and Dirac equations [7, 10, 29], the second frame is connected with the spin of the electron.

As a final comment in this section, we outline our philosophy on the use of complex numbers. It was noted in Section 2.2 that within the 2-dimensional and 3-dimensional *real* Clifford algebras there exist multivectors that naturally play the rôle of a unit imaginary, and in general there can exist many of these objects. All of the results of complex analysis therefore follow, and in many cases are enhanced. Similarly, functions of several complex variables can be studied in a real $2n$-dimensional algebra. Elsewhere [9, 10] we will show that many other concepts of modern theoretical physics can also be given real formulations, including (as has been shown by Hestenes [7]) the Dirac, Pauli and Schrödinger equations. This leads us to speculate that, though often mathematically convenient, a scalar unit imaginary may be unnecessary for fundamental physics. For most occurrences of the unit imaginary, it can be replaced by something geometrically meaningful (usually a bivector), however the literature on supersymmetry and superanalysis contains many instances where a unit imaginary is introduced for purely formal reasons, and where it plays no rôle in calculations. When dealing with such ocurrences, our policy will be to drop all reference to the imaginary, and keep everything real.

4 Pseudoclassical Mechanics

Pseudoclassical mechanics [2, 3, 30] was originally introduced as the classical analogue of quantum spin one-half (i.e. for particles obeying Fermi statistics). Recent work based on classical Lagrangians with spinor variables [31] has now provided alternative models for classical spin one-half, so it is interesting to return to the original models, and analyse them from the perspective of geometric algebra. We will concentrate on the simplest non-trivial 3-dimensional model, and analyse its equations of motion. It can be seen that this system is ultimately straightforward and, after quantization, is very similar to those derived from classical Lagrangians with spinor variables. Some interesting new concepts will be presented, however, including a new method of generating super-Lie algebras, which could form the basis for an alternative approach to their representation theory.

4.1 A Model Lagrangian and its Equations of Motion

The Lagrangian studied here was introduced by Berezin and Marinov [2], and has become the standard example of non-relativistic pseudoclassical mechanics [3, Chapter 11]. With a slight change of notation, and dropping the irrelevant factors of i, the Lagrangian can be written as

$$L = \tfrac{1}{2}\zeta_i\dot{\zeta}_i - \tfrac{1}{2}\epsilon_{ijk}\omega_i\zeta_j\zeta_k, \tag{4.1}$$

where ω_i are a set of three scalar constants. We immediately translate this to

$$L = \tfrac{1}{2}e_i{\wedge}\dot{e}_i - \omega, \tag{4.2}$$

where

$$\omega = \omega_1(e_2{\wedge}e_3) + \omega_2(e_3{\wedge}e_1) + \omega_3(e_1{\wedge}e_2). \tag{4.3}$$

It is worth re-emphasising that our translation has taken what was originally thought of as a single vector with Grassmann entries, and replaced it by three ordinary vectors in Euclidean 3-space. Thus, as promised, we lose the need for additional Grassmann dimensions.

A possible surprise is that our Lagrangian is no longer a scalar, but a bivector-valued object. This raises interesting questions; in particular, which of the many techniques applied to scalar Lagrangians remain applicable when the Lagrangian becomes multivector-valued. Many of these questions are answered in [8], where a general theory for analysing multivector Lagrangians, and studying their symmetries, is outlined. In fact, multivector Lagrangians are straightforward generalisations of scalar Lagrangians, allowing large numbers of coupled variables to be handled simultaneously (there are nine independent parameters in (4.2)), and the variational principle, symmetry properties and Noether's theorem all extend naturally. However, there is an important restriction on the type of multivector Lagrangian which can be allowed. If we expand the resulting multivector action in a basis, then stationarity of each scalar coefficient alone determines a classical motion for the system, since we can derive Euler-Lagrange equations from that coefficient, via the usual scalar methods. The crucial requirement is that the equations derived from each basis element should be mutually consistent, so that the whole multivector action, $\int dt\, L$, can be genuinely stationary for the classical motion. How the present system meets this requirement is discussed at the end of this sub-section.

In [8] the variational principle is formalised using the 'multivector derivative' [14], and a general expression of the Euler-Lagrange equations for a multivector Lagrangian is derived. Here we adopt a less formal approach and simply vary the $\{e_i\}$ vectors independently, keeping their end-points fixed[3]. If we consider varying e_1, say, we find

$$\delta S = \int dt\ \delta e_1 \wedge (\dot{e}_1 + \omega_2 e_3 - \omega_3 e_2). \tag{4.4}$$

Setting this equal to zero, and repeating for e_2 and e_3, yields the equations of motion,

$$\begin{aligned} \dot{e}_1 &= -\omega_2 e_3 + \omega_3 e_2 \\ \dot{e}_2 &= -\omega_3 e_1 + \omega_1 e_3 \\ \dot{e}_3 &= -\omega_1 e_2 + \omega_2 e_1. \end{aligned} \tag{4.5}$$

We now have a set of three coupled first-order *vector* equations, which, in terms of components, is a set of nine equations for the nine unknowns. These equations can

[3]This provides another motivation for not fixing the inner product of our frame vectors. Had we enforced, say, the 'quantum' condition that the vectors were orthonormal, this would have been inconsistent with the variational principle. This situation is analogous to the problems encountered in quantising constrained systems.

be neatly combined into a single equation by introducing the reciprocal frame $\{e^i\}$ and writing

$$\dot{e}_i = e^i \cdot \omega, \tag{4.6}$$

which demonstrates some interesting geometry at work, relating the equations of motion of a frame to its reciprocal. Furthermore, feeding (4.5) into (4.3), we see that

$$\dot{\omega} = 0, \tag{4.7}$$

so that the ω plane is constant.

Geometric algebra now allows us to develop this system further than previously possible with pseudoclassical mechanics, by both solving the equations of motion and studying their symmetries. Although some of the equations we derive do have Grassmann analogues, it is clear that the system defined by (4.5) is richer when studied in geometric algebra.

We now give an alternative derivation of the equations of motion which shows that our bivector Lagrangian is admissable in the sense mentioned above. To derive an equation from a single scalar coefficient, we contract L with an arbitrary bivector B to form a new Lagrangian $L' = \langle LB \rangle$. The equations of motion formed from this (scalar) Lagrangian are found via simple variations of the e_i, or via the multivector derivative again, and are

$$\dot{e}_i \cdot B = (e^i \cdot \omega) \cdot B. \tag{4.8}$$

Because B was arbitrary, this equation directly implies the equations of motion (4.6). It also means that the three equations of motion obtained separately from (4.8) by letting B range over an independent set of basis bivectors are consistent, and we see that the bivector action based upon the Lagrangian (4.2) is indeed capable of simultaneous extremization in each coefficient. Although this equivalent derivation uses a set of scalar Lagrangians, the equations only make sense in the context of their derivation from a full multivector Lagrangian. This follows from the observation that restricting B in (4.8) to a *single* basis bivector only gives part of the full equations of motion. Thus, the bivector Lagrangian is a vital part of the formulation of this system, and is central to establishing and understanding the conserved quantities, as we show in Section 4.3.

4.2 General Solution and some Constants of Motion

The first step in solving equations (4.5) is finding the equivalent equations for the reciprocal frame, as defined by (2.32),

$$e^1 = e_2 \wedge e_3 E_n^{-1} \quad etc, \tag{4.9}$$

where here $n = 3$. We first observe that equations (4.5) imply that

$$\dot{E}_n = 0, \tag{4.10}$$

which is important, as it shows that if the $\{e_i\}$ frame initially spans 3-dimensional space, then it will do so for all time. Equation (4.9) now gives

$$\dot{e}^1 = -\omega_2 e^3 + \omega_3 e^2, \tag{4.11}$$

so that, defining the reciprocal bivector

$$\begin{aligned} \omega^* &= \underline{g}^{-1}(\omega) \\ &= \omega_1(e^2 \wedge e^3) + \omega_2(e^3 \wedge e^1) + \omega_3(e^1 \wedge e^2), \end{aligned} \tag{4.12}$$

we have

$$\begin{aligned} \dot{e}^i &= e_i \cdot \omega^* \\ &= e_i \cdot \underline{g}^{-1}(\omega) \\ &= g^{-1}(e^i \cdot \omega), \end{aligned} \tag{4.13}$$

where (2.27) has been used. Now, using (4.6), we have

$$g(\dot{e}^i) = \dot{e}_i = \frac{d}{dt} g(e^i) \tag{4.14}$$

$$\Rightarrow \dot{g} = 0, \tag{4.15}$$

so the metric tensor is constant, even though its matrix coefficients are varying. The variation of the coefficients of the metric tensor is, therefore, purely the result of the time variation of the frame, and is not a property of the frame-independent tensor. This implies that the fiducial tensor is also constant, and suggests that we should look at the equations of motion for the fiducial frame $\sigma_i = h^{-1}(e_i)$,

$$\begin{aligned} \dot{\sigma}_i &= h^{-1}(\dot{e}_i) \\ &= h^{-1}(h^{-1}(\sigma_i)\cdot\omega) \\ &= \sigma_i\cdot\underline{h}^{-1}(\omega). \end{aligned} \tag{4.16}$$

If we define the bivector

$$\Omega = \underline{h}^{-1}(\omega) = \omega_1\sigma_2\sigma_3 + \omega_2\sigma_3\sigma_1 + \omega_3\sigma_1\sigma_2 \tag{4.17}$$

(which must be constant, since both h and ω are), we have

$$\dot{\sigma}_i = \sigma_i\cdot\Omega, \tag{4.18}$$

so that the underlying fiducial frame simply rotates at a constant frequency in the Ω plane. If $\sigma_i(0)$ is the fiducial frame specified by the initial setup of the $\{e_i\}$ frame, then the solution to (4.18) is

$$\sigma_i(t) = e^{-\Omega t/2}\sigma_i(0)e^{\Omega t/2}, \tag{4.19}$$

and the general solution for the $\{e_i\}$ frame is

$$\begin{aligned} e_i(t) &= h(e^{-\Omega t/2}\sigma_i(0)e^{\Omega t/2}) \\ e^i(t) &= h^{-1}(e^{-\Omega t/2}\sigma_i(0)e^{\Omega t/2}). \end{aligned} \tag{4.20}$$

We recognise that, ultimately, we are simply looking at a rotating orthonormal frame viewed through a constant (symmetric) distortion. The $\{e_i\}$ frame and its reciprocal represent the same thing viewed through the distortion and its inverse. It follows that there is only one frequency in this system, ν, which is found via

$$\begin{aligned} \nu^2 &= -\Omega^2 \\ &= {\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2. \end{aligned} \tag{4.21}$$

It is now simple to derive some further conserved quantities in addition to ω, E_n and their reciprocals ω^* and E^n. Since

$$\Omega = i(\omega_1\sigma_1 + \omega_2\sigma_2 + \omega_3\sigma_3), \tag{4.22}$$

the vectors

$$u \equiv \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3, \tag{4.23}$$

and

$$u^* = g^{-1}(u), \tag{4.24}$$

are conserved. This follows from

$$\begin{aligned} u &= E^n \omega^* &(4.25)\\ u^* &= E_n \omega. &(4.26)\end{aligned}$$

Furthermore,

$$\begin{aligned} e_i e_i &= h(\sigma_i)h(\sigma_i) \\ &= \sigma_i g(\sigma_i) \\ &= \mathrm{Tr}(g) &(4.27)\end{aligned}$$

must also be time-independent (as can be verified from the equations of motion). The reciprocal quantity $e^i e^i = \mathrm{Tr}(g^{-1})$ is also conserved. We thus have the set of four standard rotational invariants, $\sigma_i \sigma_i$, the axis, the plane of rotation and the volume scale-factor, each viewed through the pair of distortions h, h^{-1}, giving a large set of related conserved quantities.

Despite the original intentions for pseudoclassical mechanics, it has not proved possible to identify the motion of (4.20) with any physical system, except in the simple case where $h = 1$ (see Section 5).

4.3 Lagrangian Symmetries and Conserved Currents

Although we have solved the equations of motion exactly, it is instructive to derive some of their consequences directly from the Lagrangian. A more complete formalism for constructing conserved quantities from multivector Lagrangians, utilising the multivector derivative, is described in [8], but for the present paper we just quote the necessary results. Before listing the symmetries contained in (4.2), we note one symmetry it does not contain — time reversal. This is a consequence of the first-order nature of the equations, which therefore sets this system apart from many others studied in physics. Of course, under time reversal the system simply rotates in the other direction, and the Lagrangian is invariant under the combined operations of time reversal and $\omega_i \to -\omega_i$.

Of the symmetries of interest, most are parameterized by a scalar, and can be written as

$$e_i' = f(e_i, \alpha), \tag{4.28}$$

where α is the controlling scalar. If we define $L' = L(e_i', \dot{e}_i')$, then, for the L of equation (4.2), the main result that we need from [8] is

$$\partial_\alpha L' = \frac{d}{dt}\left(\tfrac{1}{2} e_i' \wedge (\partial_\alpha e_i')\right). \tag{4.29}$$

Hence, if L' is independent of α, the quantity

$$\tfrac{1}{2} e_i' \wedge (\partial_\alpha e_i') \tag{4.30}$$

is conserved. In most cases it is convenient to set $\alpha = 0$ in (4.29), so that

$$\partial_\alpha L'|_{\alpha=0} = \frac{d}{dt}\left(\tfrac{1}{2} e_i' \wedge (\partial_\alpha e_i')\right)\bigg|_{\alpha=0}. \tag{4.31}$$

In writing this we are explicitly making use of the equations of motion, and so are finding 'on-shell' symmetries. The Lagrangian could be modified to extend these symmetries off-shell, but we will not consider this here.

The first example we consider is dilation symmetry:

$$e_i' = e^\alpha e_i. \tag{4.32}$$

Applying (4.31) gives

$$2L = \frac{d}{dt}\left(\tfrac{1}{2} e_i \wedge e_i\right) = 0, \tag{4.33}$$

so dilation symmetry implies that the Lagrangian vanishes along a classical path. This is quite common for first-order systems (*cf.* the Dirac equation), and is important in deriving other conserved quantities.

The next symmetry is rotation,

$$e_i' = e^{\alpha B/2} e_i e^{-\alpha B/2}. \tag{4.34}$$

Equation (4.31) now gives

$$B \times L = \frac{d}{dt}\left(\tfrac{1}{2} e_i \wedge (B \cdot e_i)\right), \tag{4.35}$$

where $\times$, known as the commutator product, is one-half of the actual commutator. Since $L = 0$ when the equations of motion are satisfied, the left hand side of (4.35) vanishes, and we find the conserved bivector

$$\tfrac{1}{2} e_i \wedge (B \cdot e_i). \tag{4.36}$$

If our Lagrangian were a scalar, we would derive a scalar-valued function of B at this point, from which we could read off a single conserved bivector — the angular momentum. Here our Lagrangian is a bivector, so we get a conserved bivector function of a bivector — a set of $3 \times 3 = 9$ conserved quantities. However, as (4.36) is a symmetric function of B, this reduces to 6 independently conserved quantities. To see what these are, re-write (4.36) as

$$\tfrac{1}{2}(e_i B e_i - B e_i e_i) = e_i e_i \wedge B - B e_i e_i, \tag{4.37}$$

and introduce the dual vector $b = iB$, leading to the conserved vector function

$$e_i \cdot b e_i - b e_i e_i = g(b) - b \mathrm{T}r(g). \tag{4.38}$$

Since this is conserved for all b, we can take the b derivative and observe that $-2\mathrm{T}r(g)$ is constant, as found in Section 4.2. It follows that $g(b)$ is constant for all b, so rotational symmetry implies conservation of the metric tensor — a total of 6 quantities, as expected.

The final 'classical' symmetry we consider is time translation,

$$e_i' = e_i(t + \alpha), \tag{4.39}$$

for which (4.31) gives

$$\frac{dL}{dt} = \frac{d}{dt}\left(\tfrac{1}{2} e_i \wedge \dot{e}_i\right). \tag{4.40}$$

From this we define the constant Hamiltonian as

$$H = \tfrac{1}{2} e_i \wedge \dot{e}_i - L = \omega. \tag{4.41}$$

Since the Lagrangian is a bivector, the Hamiltonian must be also. This has interesting implications for quantum mechanics, which are discussed in Section 5.

Now that we have derived conservation of g and ω, all the remaining conserved quantities follow. For example, $E_n = \det(g)^{\frac{1}{2}} i$ shows that E_n is constant. However, there is one interesting scalar-controlled symmetry which remains, namely

$$e_i' = e_i + \alpha \omega_i a, \tag{4.42}$$

where a is an arbitrary constant vector (in the same space). In this case (4.31) gives

$$\tfrac{1}{2} a \wedge \dot{u} = \frac{d}{dt}\Big(\tfrac{1}{2} e_i \wedge (\omega_i a)\Big) \tag{4.43}$$

$$\Rightarrow a \wedge \dot{u} = 0, \tag{4.44}$$

which gives us conservation of u directly. The symmetry (4.42) bears a striking resemblance to the transformation law for the fermionic sector of a fully supersymmetric theory [32], a fact which provides a promising start to the incorporation of supersymmetric Lagrangians into our scheme. The geometry behind (4.42) is not fully understood, though it is interesting to note that the pseudoscalar transforms as

$$E_n' = E_n + \alpha a \wedge \omega, \tag{4.45}$$

and is therefore not invariant.

Finally we consider a symmetry which cannot be parameterised by a scalar — reflection symmetry. In this case equation (4.31) must be modified so that it contains a multivector derivative, as described in [8]. If we define

$$e_i' = n e_i n^{-1} \tag{4.46}$$

where n is an arbitrary vector, so that $L' = nLn^{-1}$ vanishes, we obtain a conserved vector-valued function of a vector. Using the formulae given in [8], this function is

$$e_i e_i n^{-1} + e_i \cdot n n e_i n^{-1} = n(\mathrm{T}r(g) n^{-1} + g(n^{-1}))n^{-1}, \tag{4.47}$$

which shows that the symmetric function $\mathrm{T}r(g)a + g(a)$ is conserved. This can also be used to prove conservation of g. Since rotations are even products of reflections, we expect to derive the same conserved quantities when considering rotations and reflections separately. The fact that we can derive conserved currents from discrete symmetries illustrates the power of the multivector approach to the analysis of Lagrangians.

4.4 Poisson Brackets and the Hamiltonian Formalism

We can re-derive many of the preceding results from a Hamiltonian approach which, as a by-product, reveals a new, and remarkably compact formula for a super-Lie bracket.

We have already shown that the Hamiltonian for (4.2) is ω, so we next need a translation for the Poisson bracket, introduced in [2]. Dropping the i and adjusting some signs, this is

$$\{a(\zeta), b(\zeta)\}_{PB} = a\frac{\overleftarrow{\partial}}{\partial\zeta_k}\frac{\partial}{\partial\zeta_k}b, \tag{4.48}$$

which translates to

$$\{A, B\}_{PB} = (A\cdot e^k)\wedge(e^k\cdot B). \tag{4.49}$$

Utilising the fiducial tensor, and (2.27), this can be written as

$$\begin{aligned}(A\cdot h^{-1}(\sigma_k))\wedge(h^{-1}(\sigma_k)\cdot B) &= \underline{h}(\underline{h}^{-1}(A)\cdot\sigma_k)\wedge\underline{h}(\sigma_k\cdot\underline{h}^{-1}(B)) \\ &= \underline{h}\left((\underline{h}^{-1}(A)\cdot\sigma_k)\wedge(\sigma_k\cdot\underline{h}^{-1}(B))\right).\end{aligned} \tag{4.50}$$

If we assume that A and B are homogeneous, we can use (2.38) to get this into the form

$$\{A_r, B_s\}_{PB} = \underline{h}\langle\underline{h}^{-1}(A_r)\underline{h}^{-1}(B_s)\rangle_{r+s-2}, \tag{4.51}$$

which is a wonderfully compact representation of the super-Poisson bracket. The combination rule is simple, since the $\underline{h}$ always sits outside everything:

$$\{A_r, \{B_s, C_t\}_{PB}\}_{PB} = \underline{h}\left\langle\underline{h}^{-1}(A_r)\langle\underline{h}^{-1}(B_s)\underline{h}^{-1}(C_t)\rangle_{s+t-2}\right\rangle_{r+s+t-4}. \tag{4.52}$$

Since Clifford multiplication is associative, and

$$\langle A_rB_s\rangle_{r+s-2} = -(-1)^{rs}\langle B_sA_r\rangle_{r+s-2}, \tag{4.53}$$

it follows that (4.51) generates a super-Lie algebra, as it is well known that a graded associative algebra satisfying the graded commutator relation (4.53) satisfies the super-Jacobi identity [33, 34].

There has been considerable work on how various Lie algebras can be realised by multivectors within Clifford algebras [13, 14, 35, 36]. For example, all Lie algebras can be represented as bivector algebras under the commutator product [13]. We can see that the bivector commutator is a special case of (4.51), where all the elements are grade 2, and $h = 1$ (setting $h \neq 1$ enforces a type of deformation). The bracket (4.51) should now allow for this work to be extended to super-Lie algebras, where we can expect to find many improvements over the traditional matrix-based approach[4]. In particular, the abstract algebraic generators of a super-Lie algebra can be replaced by mixed-grade multivectors (directed lines, planes *etc.*), thus providing a concrete geometrical picture.

We can now derive the equations of motion from the Poisson bracket for our system as follows,

$$\begin{aligned} \dot{e}_i &= \{e_i, H\}_{PB} \\ &= h(\sigma_i \cdot \Omega) \\ &= e^i \cdot \omega. \end{aligned} \tag{4.54}$$

Similarly, some conservation laws can be checked, for example,

$$\{E_n, H\}_{PB} = \underline{h}\langle i\Omega\rangle_3 = 0, \tag{4.55}$$

and

$$\{\omega, H\}_{PB} = \underline{h}\langle \Omega\Omega\rangle_2 = 0. \tag{4.56}$$

However, this bracket gives zero for any scalar-valued functions, so is no help in deriving conservation of $e_i e_i$; furthermore, it only gives the correct equations of motion for the $\{e_i\}$ frame, since these are the genuine dynamical variables.

It is conventional to define the spin operators (again dropping an i)

$$S_i = \tfrac{1}{2}\epsilon_{ijk}\zeta_j\zeta_k \quad \leftrightarrow \quad \tfrac{1}{2}\epsilon_{ijk}e_j \wedge e_k = \underline{h}(i\sigma_i), \tag{4.57}$$

so that

$$\{S_i, S_j\}_{PB} = \underline{h}\langle i\sigma_i i\sigma_j\rangle_2. \tag{4.58}$$

This gives the commutation relations for orthogonal bivectors in the Pauli algebra, as viewed through the $\underline{h}$ tensor. These bivectors are well known to generate the su(2) Lie algebra, a fact that is usually interpreted as showing that (4.1) describes the pseudoclassical mechanics of spin. However, since the Pauli algebra is as applicable to classical mechanics as to quantum mechanics [20], the immediate identification of the su(2) algebra relations with quantum spin is unjustified. Indeed, the su(2) algebra expressed by (4.58) is nothing more than an expression of the behaviour of orthonormal vectors under the vector cross product.

[4]An attempt to study super-Lie algebras within Clifford algebras was carried out in [37], though their approach was very different.

Finally, we consider the density function, which is Grassmann-odd, and translates to the odd multivector

$$\rho = c + E_n, \tag{4.59}$$

(c is a vector). This is used to define the expectation of an operator by

$$\|f\| = \int f(\zeta)\rho d^3\zeta \quad \leftrightarrow \quad \langle F\rho E^n\rangle, \tag{4.60}$$

and ρ is normalised such that $\|1\| = 1$. ρ must satisfy the Liouville equation, which is

$$\frac{\partial\rho}{\partial t} + \{\rho, H\}_{PB} = 0. \tag{4.61}$$

The spin operators (4.57) now have the property

$$\begin{aligned} \|S_j\| &= \langle \underline{h}(i\sigma_j)\rho E^n\rangle \\ &= \langle h^{-1}(\sigma_j)cE_nE^n\rangle \\ &= e^j\cdot c, \end{aligned} \tag{4.62}$$

which is usually identified as finding the expectation of the S_j operator, but in geometric algebra is seen merely to pick out the components of the c vector in the $\{e_j\}$ frame (a similar point of view arises in the full quantum theory of spin [7, 10]). The components c^j are constrained to be constant, so for c to satisfy (4.61), it must have

$$c^j e^j\cdot\omega = \epsilon_{ijk}\omega_i c^j e_k = 0 \tag{4.63}$$

$$\Rightarrow \quad c^j = \lambda\omega_j \tag{4.64}$$

$$\Rightarrow \quad c = \lambda u, \tag{4.65}$$

so c is a constant multiple of u.

This is about as far as this simple model can be taken. We have demonstrated that analysing its properties in geometric algebra sheds new light on the geometry behind the model. Furthermore, geometric algebra has enabled us to develop a richer theory, in which the usual concepts introduced for scalar Lagrangians generalise naturally. It is to be hoped that further applications of this approach can be found, utilising the true power of geometric algebra.

5 Quantization

The quantization of the system arising from (4.2) is carried out in standard treatments in two ways, via the path-integral and canonical routes. The path integral will be discussed in Section 5.2, where a preliminary sketch of a new approach is presented, but first we consider the canonical approach.

5.1 Canonical Quantization

The Poisson bracket of Section 4.4 is defined such that

$$\{e_i, e_j\}_{PB} = \delta_{ij}. \tag{5.1}$$

The canonical quantization procedure therefore replaces the e_i by operators $\hat{\sigma}_i$ satisfying the Clifford-algebra generating relation

$$\hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i = 2\delta_{ij}. \tag{5.2}$$

These operators generate the 3-dimensional Euclidean Clifford algebra — the Pauli algebra (Section 2.2). The presence of the Pauli algebra is usually taken as evidence that we have arrived at a quantum system, but in fact this need not be the case. We have already shown in Section 2.2 that the generators of the Pauli algebra can be viewed as vectors, and (5.2) amounts to no more than the condition that these vectors are orthonormal. It is therefore natural to identify the $\hat{\sigma}_i$ with the fiducial frame σ_i, in which case the quantum condition amounts to

$$\underline{h} \to 1. \tag{5.3}$$

This equation cannot be enforced at the level of the Lagrangian (4.2), as this is inconsistent with the variational principle, which requires each of the vectors to be varied independently. Hence (5.3) can only be applied after the equations of motion, or in this case their solutions, have been found.

A further aspect of quantization is that the $\hat{\sigma}_i$ operators are now Clifford-multiplied everywhere, rather than exterior-multiplied. In terms of the σ_i vectors this makes little difference, for two reasons. The first is that orthonormality of the σ_i implies that

$$\sigma_i\wedge\sigma_j = \sigma_i\sigma_j \qquad (i \neq j). \tag{5.4}$$

The second is that, for a rotating orthonormal frame [38],

$$\sigma_i \dot{\sigma}_i = \sigma_i \wedge \dot{\sigma}_i. \tag{5.5}$$

Consequently, almost all of the calculations of Section 4 go through unchanged if the vectors are chosen orthonormal, and the wedges are dropped.

After (5.2) is applied, all that remains is a simple rotating orthonormal frame, with the rotation in the fixed plane orthogonal to the $\omega_i \sigma_i$ axis. This is an entirely classical system, though it is possible to make contact with one aspect of quantum electron behaviour. If Ω is chosen to be eB/m, where B a constant magnetic field bivector, and the 3-axis is identified with the spin axis s, then

$$\dot{s} = \frac{e}{m} s \cdot B. \tag{5.6}$$

This is the correct equation for a particle with gyromagnetic ratio two, though in this non-relativistic theory the g-factor has been put in by hand. In the relativistic pseudoclassical theory [2], however, the magnetic bivector B is replaced by the full electromagnetic field bivector F [38], and a g-factor of 2 is derived by demanding consistency with the Lorentz force law. This result was viewed as another success of the pseudoclassical program, but again this claim does not look so convincing when formulated in geometric algebra. The calculations are in fact very similar to those carried out by Rohrlich [39] and Hestenes [38], who showed that $g = 2$ is the natural value for a classical relativistic point particle. The equations used in [38] to demonstrate this are precisely those for a rotating orthonormal frame, with (5.6) obtaining in the non-relativistic limit. The point is not that the pseudoclassical mechanics is wrong, but that it is more classical than was previously realised.

Contact can now be made with a second approach to the classical mechanics of spin, in which particle Lagrangians are written down containing spinor variables [31]. Details of how to translate these into geometric algebra are given in [10, 11], but the essence is as follows. An arbitrary orthonormal frame can be written as

$$\sigma_i = R\sigma_i(0)\tilde{R}, \tag{5.7}$$

where R is a time-dependent 'rotor', satisfying $R\tilde{R} = 1$ (Section 2.3). Lagrangians with spinor variables then turn out to give equations for the rotor R, rather than the σ_i frame [8, 11, 10]. Typically, after translating into geometric algebra, we find an equation of motion of the type

$$\dot{R} = -\frac{e}{2m}BR. \tag{5.8}$$

With the spin defined by

$$s = R\sigma_3(0)\tilde{R}, \tag{5.9}$$

we find that $\dot{s}$ satisfies (5.6), and the two models lead to the same motion. This is a good illustration of how formulating apparently different systems in a single, unified language can reveal unexpected parallels.

Looking now at the Hamiltonian formalism, setting $\underline{h} = 1$ means that the Poisson bracket takes on the simple form

$$\{A_r, B_s\}_{PB} = \langle A_r B_s \rangle_{r+s-2}. \tag{5.10}$$

This is the form of the Poisson bracket most applicable to the study of super-Lie algebras within geometric algebra. Hamilton's equations now become

$$\dot{M} = \tfrac{1}{2}[M, H], \tag{5.11}$$

so time derivatives are determined by one-half the commutator with the (bivector) Hamiltonian. Furthermore, the rotor (spinor) equation (5.8) can be viewed as the 'Schrödinger representation' equivalent of (5.11), with the same bivector-valued Hamiltonian. This analogy with quantum mechanics is remarkable, and it is interesting to see how far the idea of a bivector Hamiltonian can be pushed. In particular, in the real geometric algebra formalism of the Dirac equation, in which the rôle of the unit imaginary is played by a bivector, the operator $i\hat{H}$ is also a bivector. Another reason for pursuing this idea is provided by the path integral, to which we now turn.

5.2 Path-Integral Quantization

The path integral over Grassmann variables plays an important rôle in many areas of field theory, for example for fermionic systems and Faddeev-Popov ghost fields in quantum field theory. A path-integral quantization of the system arising from (4.1) is carried out in [2], and similar calculations have been performed in greater detail in [23]. Elsewhere a Grassmann path integration of a supersymmetric model has been used to derive the Dirac propagator [40]. These calculations all involve Berezin integrals, which (as we showed in Section 3.1) can be replaced by algebraic contractions. These integrals can therefore be simply incorporated into our framework, though our formalism will suggest an entirely new approach to Grassmann-type path integrals, in which Berezin integrals are replaced by Riemann integrals.

Grassmann path integrals make repeated use of the 'Grauss' integral of Section 3.2. This calls for a space of even dimension, which is enforced in [2] by adopting a phase-space approach analogous to that of standard quantum-mechanical path integration [41][5]. This phase-space formalism translates via introduction of a second set of vectors $\{f_i\}$, so that the set $\{e_i, f_i\}$ span a 6-dimensional space. The phase-space action functional translates as

$$S = \int_0^T dt \, \Big(f_i \wedge \dot{e}_i - \tfrac{1}{2} e_i \wedge \dot{e}_i - H(f) \Big) \, , \tag{5.12}$$

where $H(f)$ is the Hamiltonian, expressed as a function of the f_i. For the Hamiltonian of (4.2), the equations of motion turn out to be

$$\begin{aligned} \dot{f}_i &= \dot{e}_i \qquad &(5.13)\\ \dot{e}_i &= -\tfrac{1}{2} \epsilon_{ijk} \omega_j f_k \qquad &(5.14)\\ \Rightarrow \dot{f}_i &= f^i \cdot \omega(f), \qquad &(5.15) \end{aligned}$$

where $\omega(f)$ is the bivector of (4.3) expressed in terms of the f_i. If equation (5.13) is integrated with the boundary conditions chosen so that $e_i = f_i$, we then recover the e_i equations of motion (4.5). To carry out the path integral, the action integral (5.12) is replaced by the sum

$$S \cong \sum_{k=1}^{N} f_i(k) \wedge (e_i(k+1) - e_i(k)) - \tfrac{1}{2} e_i(k) \wedge e_i(k+1) - \tfrac{1}{2} \epsilon_{ijk} \omega_i \Delta T f_j(k) \wedge f_k(k), \tag{5.16}$$

where $N\Delta t = T$, and $e_i(k)$ is an abbreviation for $e_i(k\Delta t)$. The final term $e_i(N+1) = e_i$ is the remaining frame of which the resultant propagator is a function. It is also required that each time-slice frame $\{f_i(k), e_i(k)\}$ be viewed as an independent (anticommuting) set of variables, so the path integral can be written as

$$G(e_i, T) = \lim_{N \to \infty} \langle \prod_{k=1}^{N} E^n(k) F^n(k) \frac{S^n}{n!} \rangle_{0,2}. \tag{5.17}$$

[5]de Witt [23] attempts to carry out path integrals in two dimensions without using a phase-space approach. This results in oscillatory behaviour, with the value of the integral depending on whether an even or odd number of steps are taken.

This limit is well-defined, but from our point of view the formalism looks unsatisfactory for a number of reasons. The introduction of a new frame for each time-slice is unattractive, and the propagator derived is only a function of one endpoint, e_i, rather than the start and end-points of a trajectory in phase space.

This leads us to propose an alternative approach to the path-integral quantization of (5.12), which again has no couterpart in Grassmann calculus. The idea is to utilise two properties of bivectors in Euclidean spaces: first, they naturally have negative square, thus precluding the need for a unit imaginary; second, they have a well-defined parameter space associated with them, so we can replace Berezin integrals by Riemann integrals over these parameters. This enables us to consider integrals of the type $\int \ldots \exp(S)$, where S is the bivector action, and this can then have the same oscillatory and classical path properties as the usual path integral of iS', where S' is some scalar action. We are now considering $\exp(S)$ as a Clifford bivector exponential, and so are relinquishing all ties with the original Grassmann algebra.

A further motivation for considering bivector path integrals is provided by the geometric algebra formalism of the Dirac equation [7, 10]. In this approach wavefunctions of pure states are the exponentials of bivectors, so that the superposition of wavefunctions corresponding to all paths linking initial and final states also results in integrals of the type $\int \ldots \exp(B)$. We hope that this new approach will eventually provide insights into the meaning of conventional path integrals in space and spacetime, but we resrict ourselves here to 2-dimensional systems, for which it is possible to exploit the correspondence between the unit bivector I and the scalar unit imaginary i (Section 2.2). In particular we shall make use of the result

$$\begin{aligned} \int d^2x e^{x\wedge a} &= \int dx_1 dx_2 e^{i(x_1a_2 - x_2a_1)} \\ &= 2\pi\delta(a_1)2\pi\delta(a_2) \\ &= (2\pi)^2\delta^2(a). \end{aligned} \tag{5.18}$$

(A similar result holds for Berezin integration of Grassmann variables [23].)

We now consider the simplest 2-dimensional 'free-frame' action,

$$S_0 = \int_0^T dt\, \left(f_i\wedge\dot{e}_i - \tfrac{1}{2}e_i\wedge\dot{e}_i \right), \tag{5.19}$$

where f_i and e_i ($i = 1, 2$) are vectors in the same 2-dimensional space. We approximate (5.19) by

$$S_0 \cong \sum_{k=0}^{N} f_i(k)\wedge(e_i(k+1) - e_i(k)) - \tfrac{1}{2}e_i(k)\wedge e_i(k+1), \tag{5.20}$$

with $e_i(0)$ and $e_i(N+1) = e_i(T)$ the boundary points. Our approach is now to integrate out the f_i, leaving an effective action for the e_i, and then perform the e_i integrals, so that just the boundary points remain. That is,

$$\begin{aligned}\int \mathcal{D}f_i\mathcal{D}e_i e^{S_0} &= \lim_{N\to\infty}\left[\prod_{k=0}^{N} d^2 f_i(k)\right]\left[\prod_{k=1}^{N}\frac{d^2 e_i(k)}{(2\pi)^2}\right]\exp(S_0)\\ &= \delta^2(e_1(0) - e_1(T))\delta^2(e_2(0) - e_2(T)).\end{aligned} \tag{5.21}$$

This could be interpreted as showing that the system is still constrained to follow the classical path.

An 'interaction' can now be included, so that the action becomes

$$S = \int_0^T dt\, \left(f_i\wedge\dot{e}_i - \tfrac{1}{2}e_i\wedge\dot{e}_i - \omega f_1\wedge f_2\right), \tag{5.22}$$

where ω is a scalar constant. This is the 2-dimensional reduction of (5.12). The path integral is defined in the same way as (5.21) and, on carrying out the $\{f_i\}$ integrals, we obtain the following effective action,

$$S_{\text{eff}} = \int_0^T dt\, \left(-\tfrac{1}{2}e_i\wedge\dot{e}_i + \frac{1}{\omega}\dot{e}_1\wedge\dot{e}_2\right). \tag{5.23}$$

As a check, the equations of motion derived from (5.23) are

$$\begin{aligned}\ddot{e}_1 &= \omega\dot{e}_2\\ \ddot{e}_2 &= -\omega\dot{e}_1,\end{aligned} \tag{5.24}$$

which are the same as would have been derived from (5.22) had the f_i been eliminated. Performing the remaining e_i integrals leads to the propagator,

$$\frac{e^{(\omega T/2)^2}}{(2\pi\omega T)^2}\exp\left\{-\tfrac{1}{2}e_i(0)\wedge e_i(T) + \frac{1}{\omega T}(e_1(0) - e_1(T))\wedge(e_2(0) - e_2(T))\right\}. \tag{5.25}$$

It is reassuring to note that in the free-frame limit $\omega \to 0$, we recover (5.21), since

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} e^{a \wedge b/\epsilon} = (2\pi)^2 \delta^2(a) \delta^2(b). \tag{5.26}$$

Extending these results to higher dimensions will require extensions of complex analysis to accomodate non-commuting bivectors. This may not be easy to implement, but we hope these preliminary results have demonstrated that it is a worthwhile exercise.

6 Conclusions

We have shown how Grassmann algebra can be naturally embedded within geometric algebra, and how this simplifies many of the manipulations encountered in applications of Grassmann variables. This has many conceptual advantages through the association of a natural geometric picture to previously abstract entities, and this makes many results easier to understand and to interpret.

The 3-dimensional Grassmann oscillator was presented as a detailed application of this idea, and a number of interesting concepts have emerged — multivector Lagrangians and their associated symmetries; multivector realisations of the super-Jacobi identities. The system was then quantised along standard lines. The canonical approach was shown to result in a *classical* system, which casts doubt on some of the claims originally made for the pseudoclassical mechanics. The path-integral quantization was shown to appear unnatural, and a possible alternative (based purely on geometric algebra) was outlined.

Throughout, we have emphasised two key points. Grassmann calculus is richer when formulated within geometric algebra, and Clifford algebras in general (and the Pauli algebra in particular) are just as relevant to classical as to quantum mechanics. The boundaries between classical, pseudoclassical and quantum mechanics are therefore less clearly defined as might have been thought previously, and this was illustrated by the 'quantization' of a pseudoclassical system apparently yielding a classical system.

In future work we will extend these ideas to supersymmetry and twistor theory, though these are only two of a number of possible applications which were touched on in the text. We suggest that further elaboration of the ideas developed throughout this paper will be significant for other applications involving Grassmann algebras. For example, many of the structures studied in [23] (super-Lie algebras, super-Hilbert spaces) have natural multivector expressions, and the cyclic cohomology groups of Grassmann algebras described in [24] can be formulated in terms of the multilinear function theory set out in [14]. It is our hope that others will follow these avenues, and thus implement a critical reappraisal of the status of Grassmann variables in mathematics and physics.

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A The Grassmann Fourier Transform

In Grassmann algebra one defines Fourier integral transformations between anticommuting spaces $\{\zeta_k\}$ and $\{\rho_k\}$ by [2]

$$\begin{aligned} G(\zeta) &= \int \exp\{i \textstyle\sum \zeta_k \rho^k\} H(\rho) d\rho^n \dots d\rho^1 \\ H(\rho) &= \epsilon^n \int \exp\{-i \textstyle\sum \zeta_k \rho^k\} G(\zeta) d\zeta_n \dots d\zeta_1, \end{aligned} \tag{A.1}$$

where $\epsilon^n = 1$ for n even and i for n odd. The factors of i are irrelevant, and can be dropped, so that (A.1) becomes

$$\begin{aligned} G(\zeta) &= \int \exp\{\textstyle\sum \zeta_k \rho^k\} H(\rho) d\rho^n \dots d\rho^1 \\ H(\rho) &= (-1)^n \int \exp\{-\textstyle\sum \zeta_k \rho^k\} G(\zeta) d\zeta_n \dots d\zeta_1. \end{aligned} \tag{A.2}$$

We will first translate this into geometric algebra to find an equivalent expression, and then show how the geometric algebra expression can be manipulated into a much clearer form, demonstrating that there is some simple geometry at work. We introduce a pair of anticommuting copies of the same frame, $\{e_k\}$, $\{f_k\}$, so that

$$e_i \cdot e_j = f_i \cdot f_j \tag{A.3}$$

$$e_i \cdot f_j = 0; \tag{A.4}$$

hence the full set $\{e_k, f_k\}$ generate a $2n$-dimensional Clifford algebra. The translation now proceeds by replacing

$$\zeta_k \leftrightarrow e_k, \qquad \rho^k \leftrightarrow f^k, \tag{A.5}$$

where the $\{\rho^k\}$ have been replaced by elements of the reciprocal frame $\{f^k\}$. This must satisfy

$$e^i \cdot e^j = f^i \cdot f^j. \tag{A.6}$$

We next define the bivector

$$J = \sum_i e_i \wedge f^i = \sum_i e^i \wedge f_i, \tag{A.7}$$

where the equality of the two expressions for J follows from (A.3). It is now a simple matter to expand a vector in the $\{e_k, f^k\}$ basis, and prove that

$$J\cdot(J\cdot a) = -a, \tag{A.8}$$

for any vector a in the $2n$-dimensional algebra. The bivector J thus clearly plays the rôle of a complex structure (this in itself is a good reason for ignoring the scalar i). Equation (A.8) can be extended to give

$$e^{J\theta/2} a e^{-J\theta/2} = \cos\theta a + \sin\theta J\cdot a, \tag{A.9}$$

hence $e^{J\pi/2}$ anticommutes with all vectors. Consequently it can only be a multiple of the pseudoscalar and, since it has unit magnitude, we can define the orientation such that

$$e^{J\pi/2} = I. \tag{A.10}$$

This definition implies that

$$E_n F^n = E^n F_n = I. \tag{A.11}$$

Finally, we introduce the notation

$$C_k = \frac{1}{k!}\langle J^k\rangle_{2k}. \tag{A.12}$$

The formulae (A.2) now translate to

$$\begin{aligned} G(e) &= \textstyle\sum_{j=0}^n (C_j H(f))\cdot F_n \\ H(f) &= (-1)^n \textstyle\sum_{j=0}^n (\tilde{C}_j G(e))\cdot E^n, \end{aligned} \tag{A.13}$$

where we adopt the convention that these expressions are zero if the $C_j H$ or $\tilde{C}_j G$ terms have grade less than n. Since G and H only contain terms constructed from the $\{e_k\}$ and $\{f^k\}$ respectively, (A.13) can be written as

$$\begin{aligned} G(e) &= \textstyle\sum_{j=0}^n (C_{n-j} \wedge \langle H(f)\rangle_j)\cdot F_n \\ H(f) &= \textstyle\sum_{j=0}^n (-1)^j (\langle G(e)\rangle_j \wedge C_{n-j})\cdot E^n. \end{aligned} \tag{A.14}$$

So far we have only derived a formula analogous to (A.2), but we can now go much further. Using

$$e^{J\theta} = \cos^n(\theta) + \cos^{n-1}(\theta)\sin(\theta)C_1 + \ldots + \sin^n(\theta)I, \tag{A.15}$$

to decompose $e^{J(\theta+\pi/2)} = e^{J\theta}I$ in two ways, it can be seen that

$$C_{n-r} = (-1)^r C_r I = (-1)^r I C_r, \tag{A.16}$$

and hence (using some simple duality relations) (A.14) become

$$\begin{aligned} G(e) &= \textstyle\sum_{j=0}^n C_j \cdot H_j E_n \\ H(f) &= (-1)^n \textstyle\sum_{j=0}^n G_j \cdot C_j F^n. \end{aligned} \tag{A.17}$$

Finally, since G and H are pure in the $\{e_k\}$ and $\{f^k\}$ respectively, the effect of dotting with C_k is simply to interchange these. For vectors this is achieved by dotting with J, but from (A.9) this can also be achieved by a rotation through $\pi/2$, which extends simply via outermorphism, so that

$$\begin{aligned} C_j \cdot H_j &= e^{J\pi/4} H_j e^{-J\pi/4} \\ G_j \cdot C_j &= e^{-J\pi/4} G_j e^{J\pi/4}. \end{aligned} \tag{A.18}$$

We have now arrived at the following equivalent expressions for (A.13):

$$\begin{aligned} G(e) &= e^{J\pi/4} H(f) e^{-J\pi/4} E_n \\ H(f) &= (-1)^n e^{-J\pi/4} G(e) e^{J\pi/4} F^n. \end{aligned} \tag{A.19}$$

Thus, the Grassmann Fourier transformations have been reduced to rotations through $\pi/2$ in the planes specified by J, followed by an (uninteresting) duality transformation. Proving the 'inversion' theorem (i.e. that the above expressions are consistent), amounts to no more than carrying out a rotation, followed by its inverse,

$$\begin{aligned} G(e) &= e^{J\pi/4}((-1)^n e^{-J\pi/4} G(e) e^{J\pi/4} F^n) e^{-J\pi/4} E_n \\ &= G(e) E^n E_n = G(e), \end{aligned} \tag{A.20}$$

which is far simpler than any proof carried out in Grassmann algebra [1].