Physical Applications of Geometric Algebra

Part II
Relativistic Physics

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Chapter 1

Spacetime Algebra

The geometric algebra of spacetime is called the *spacetime algebra* or STA. This forms the basis for most of the remainder of this course, where we will deal mainly with applications of geometric algebra to relativistic physics and gravitation. The algebra is constructed from four basis vectors, three spatial and one timelike. The spacelike and timelike vectors have opposite signs for their squares. Rotors in this algebra provide the simplest means of performing Lorentz transformations.

1.1 An Algebra for Spacetime

Special relativity is often introduced with the postulate that the speed of light is constant for all observers. From this one deduces the Lorentz transformation law before, finally, the concept of unifying space and time into a single spacetime is introduced. This partly mirrors the historical development of relativity. We will not follow this order. Instead, we jump straight to spacetime as the appropriate arena for relativistic physics. Our aim then is to construct the geometric algebra of spacetime. We start by recalling that the invariant interval of special relativity is

$$s^2 = c^2t^2 - \mathrm{x}^2 - \mathrm{y}^2 - \mathrm{z}^2, \tag{1.1}$$

where t is the time and x, y, and z are spatial (Cartesian) coordinates in some inertial frame. We adopt the 'particle physics' choice of signature. General relativists often flip all the signs. We work throughout in units where $c = 1$. It is clear that we must build our algebra from four vectors $\{e_0, e_i\}, i = 1 \ldots 3$ with the following properties:

$$e_0{}^2 = 1, \qquad e_0 \cdot e_i = 0, \qquad e_i \cdot e_j = -\delta_{ij}. \tag{1.2}$$

These are summarised as

$$\begin{aligned} e_\mu \cdot e_\nu &= \eta_{\mu\nu}, \quad \mu, \nu = 0 \ldots 3, \\ &= \mathrm{diag}(+ - - -) \, . \end{aligned} \tag{1.3}$$

1.1.1 The Bivector Algebra

There are $4 \times 3/2 = 6$ bivectors in our algebra. These fall into two classes; those that contain a timelike component (*e.g.* $e_i \wedge e_0$), and those that do not (*e.g.* $e_i \wedge e_j$). For any pair of vectors a and b with $a \cdot b = 0$ we have

$$(a \wedge b)^2 = abab = -abba = -a^2 b^2. \tag{1.4}$$

The two types of bivectors therefore have different signs of their squares. First, we have

$$(e_i \wedge e_j)^2 = -e_i{}^2 e_j{}^2 = -1, \tag{1.5}$$

which is the familiar result for Euclidean bivectors. Each of these generate rotations in a plane. For bivectors containing a timelike component, however, we have

$$(e_i \wedge e_0)^2 = -e_i{}^2 e_0{}^2 = +1. \tag{1.6}$$

Bivectors with positive square have a number of new properties. One immediate result we notice, for example, is that

$$\begin{aligned} \mathrm{e}^{\alpha e_1 e_0} &= 1 + \alpha e_1 e_0 + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} e_1 e_0 + \cdots \\ &= \mathrm{ch}(\alpha) + \mathrm{sh}(\alpha) e_1 e_0. \end{aligned} \tag{1.7}$$

This shows us that we are dealing with *hyperbolic geometry*. This will prove crucial to our treatment of Lorentz transformations. We have started to employ the useful abbreviations

$$\mathrm{ch}(\alpha) = \cosh\alpha, \qquad \mathrm{sh}(\alpha) = \sinh\alpha, \qquad \mathrm{th}(\alpha) = \tanh\alpha. \tag{1.8}$$

1.1.2 The Pseudoscalar

We define the (grade 4) pseudoscalar I by

$$I = e_0 e_1 e_2 e_3. \tag{1.9}$$

This defines a handedness for our algebra. The reason for this choice will emerge shortly. We still assume that e_1, e_2, e_3 form a right-handed orthonormal set, as usual

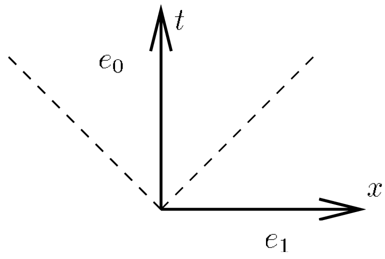


Figure 1.1: *A Spacetime Diagram.* Spacetime diagrams traditionally have the t-axis vertical, so the right-handed bivector for the plane is $e_1 e_0$.

for a 3D Cartesian frame. Traditionally, spacetime diagrams are drawn with the time axis vertical (see Figure 1.1). For these diagrams the right-handed bivector is, for example, $e_1 e_0$.

Since I is grade 4, it has

$$\tilde{I} = e_3 e_2 e_1 e_0 = I. \tag{1.10}$$

This makes it easy to compute the square of I :

$$I^2 = I\tilde{I} = (e_0 e_1 e_2 e_3)(e_3 e_2 e_1 e_0) = -1. \tag{1.11}$$

Multiplication of a bivector by I results in a multivector of grade $4 - 2 = 2$, so returns another bivector. This provides a map between the positive and negative square bivectors, *e.g.*

$$I e_1 e_0 = e_1 e_0 I = e_1 e_0 e_0 e_1 e_2 e_3 = -e_2 e_3. \tag{1.12}$$

If we define $B_i = e_i e_0$ then the bivector algebra can be written

$$\begin{aligned} B_i \times B_j &= \epsilon_{ijk} I B_k \\ (IB_i) \times (IB_j) &= -\epsilon_{ijk} I B_k \\ (IB_i) \times B_j &= -\epsilon_{ijk} B_k. \end{aligned} \tag{1.13}$$

As well as the four vectors, we also have four trivectors in our algebra. These are interchanged by a duality transformation,

$$e_1 e_2 e_3 = e_0 e_0 e_1 e_2 e_3 = e_0 I = -I e_0. \tag{1.14}$$

The pseudoscalar I *anticommutes* with vectors and trivectors, as we are in a space of even dimensions. As always, I commutes with all even-grade multivectors.

1.1.3 The Spacetime algebra

In many applications we are interested in physics in a single preferred orthonormal frame. We denote this frame by $\{\gamma_\mu\}$. Putting the preceding together, we arrive at an algebra with 16 terms:

$$\begin{array}{ccccc} 1 & \{\gamma_\mu\} & \{\gamma_\mu\wedge\gamma_\nu\} & \{I\gamma_\mu\} & I \\ \text{1 scalar} & \text{4 vectors} & \text{6 bivectors} & \text{4 trivectors} & \text{1 pseudoscalar} \end{array}. \tag{1.15}$$

This is the *spacetime algebra* or *STA*. We also introduce the following notation for the bivectors:

$$\boldsymbol{\sigma}_i = \gamma_i\gamma_0. \tag{1.16}$$

In the literature the symbol i is often used for the pseudoscalar. We have departed from this practice to avoid confusion with the i of quantum theory. Using the latter symbol presents a potential problem because of the fact that the pseudoscalar anticommutes with vectors. Occasionally we need to employ the *reciprocal* frame $\{\gamma^\mu\}$. These have $\gamma^0 = \gamma_0$ and $\gamma^i = -\gamma_i$.

The vector generators of the STA satisfy

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}. \tag{1.17}$$

These are the defining relations of the Dirac matrix algebra, though without an identity matrix on the right-hand side. It follows that the Dirac matrices define a representation of the STA. This also explains our notation of writing $\{\gamma_\mu\}$ for an orthonormal frame. But it must be remembered that the $\{\gamma_\mu\}$ are basis *vectors*, not a set of matrices in 'isospace'.

1.2 Frames and Trajectories

Suppose that $x(\lambda)$ describes a curve in spacetime, representing the worldline of some particle. The tangent vector to the curve is

$$x' = \frac{\partial x(\lambda)}{\partial\lambda} \tag{1.1}$$

There are two important cases to consider:

Timelike, $x'^2 > 0$

These are the trajectories of massive particles. In this case we introduce the preferred parameter along the curve, τ, defined so that

$$v = \partial_\tau x = \dot{x}, \qquad v^2 = 1. \tag{1.2}$$

The parameter τ is the proper time for the curve, and an observer moving along the curve measures this time. The unit timelike vector v then defines the four-velocity of the particle, and the associated instantaneous rest frame.

Null, $x'^2 = 0$

This defines a null trajectory, which are the paths taken by photons and other massless particles. We can no longer parameterise the curve in terms of proper time since this does not increase along the null curve. Photons do still carry an intrinsic clock, encoded in their phase, but this can tick at an arbitrary rate along a given null trajectory.

1.2.1 Relative Vectors

Now suppose that we are an observer at rest in some inertial frame. Our four-velocity is v. The inertial frame is generated by the unit timelike vector v, and a frame of rest vectors $\{e_i\}$ perpendicular to v. It is convenient to define $e_0 \equiv v$. Then a general event x can be decomposed in this frame as

$$x = te_0 + x^i e_i, \tag{1.3}$$

where the time coordinate is

$$t = x \cdot e_0 = x \cdot v \tag{1.4}$$

and spatial coordinates are

$$x^i = x \cdot e^i. \tag{1.5}$$

Suppose now that the event is a point on the worldline of an object at rest in our frame. The 3-d vector to this object is

$$x^i e_i = x \cdot e^\mu \, e_\mu - x \cdot e^0 \, e_0 = x - x \cdot v \, v = x \wedge v \, v. \tag{1.6}$$

Wedging with v projects onto the components of the vector x in the rest frame of v. The key quantity is the spacetime bivector $x \wedge v$. We call this the *relative* vector and write

$$\boldsymbol{x} = x \wedge v. \tag{1.7}$$

With these definitions we have

$$xv = x \cdot v + x \wedge v = t + \boldsymbol{x}. \tag{1.8}$$

The invariant distance now decomposes as

$$\begin{aligned} x^2 = xvvx &= (x \cdot v + x \wedge v)(x \cdot v + v \wedge x) \\ &= (t + \boldsymbol{x})(t - \boldsymbol{x}) = t^2 - \boldsymbol{x}^2, \end{aligned} \tag{1.9}$$

recovering the usual result. This is built into the definition of the STA.

1.2.2 The Even Subalgebra

Each inertial frame defines a set of relative vectors, which we model as spacetime bivectors. What algebraic properties do these have? To simplify things, we take the timelike velocity vector to be γ_0 so that the relative vectors are given by $\boldsymbol{\sigma}_i = \gamma_i \gamma_0$. These satisfy

$$\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = \tfrac{1}{2}(\gamma_i\gamma_0\gamma_j\gamma_0 + \gamma_j\gamma_0\gamma_i\gamma_0) = \tfrac{1}{2}(-\gamma_i\gamma_j - \gamma_j\gamma_i) = \delta_{ij}. \tag{1.10}$$

These act as vector generators for a 3-d algebra. This is the geometric algebra of the 3-d relative space in the rest frame defined by γ_0. Furthermore, the volume element of this 3-space is

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = (\gamma_1\gamma_0)(\gamma_2\gamma_0)(\gamma_3\gamma_0) = -\gamma_1\gamma_0\gamma_2\gamma_3 = \gamma_0\gamma_1\gamma_2\gamma_3 = I, \tag{1.11}$$

so the algebra of relative space shares the same pseudoscalar as spacetime. This was the reason for our earlier definition of I. Of course, we still have

$$\tfrac{1}{2}(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j\boldsymbol{\sigma}_i) = \epsilon_{ijk} I \boldsymbol{\sigma}_k, \tag{1.12}$$

so that both relative vectors and relative bivectors are spacetime bivectors. We have projected everything onto the even subalgebra of the STA.

$$\begin{array}{cccccc}
1 \cdots\cdots & \{\gamma_\mu\} \cdots\cdots & \{\boldsymbol{\sigma}_i, I\boldsymbol{\sigma}_i\} & \cdots\cdots\{I\gamma_\mu\} & \cdots\cdots I & 4-d \\
1 & \{\boldsymbol{\sigma}_i\} & \{I\boldsymbol{\sigma}_i\} & I & & 3-d
\end{array}$$

The 6 spacetime bivectors get split into relative vectors and relative bivectors. This split is *observer dependent*.

1.2.3 Conventions

Spacetime bivectors which are also used as relative vectors are written in bold. This conforms with our earlier usage of a bold face for 3-d vectors in the first half of this course.

There is a potential ambiguity here — how are we to interpret the expression $\boldsymbol{a}\wedge\boldsymbol{b}$? Our convention is that if all of the terms in an expression are bold, the dot and wedge symbols drop down to their 3-d meaning, otherwise they take their spacetime definition. This works pretty well in practice, though we will try to draw attention to the fact that this convention is in use.

1.2.4 Examples

i. Velocity

Suppose that an observer with constant velocity v measures the relative velocity of a particle with proper velocity $u(\tau)$, $u^2 = 1$. We have

$$uv = \partial_\tau[x(\tau)v] = \partial_\tau(t + \boldsymbol{x}), \tag{1.13}$$

so that

$$\partial_\tau t = u\cdot v, \tag{1.14}$$

and

$$\partial_\tau \boldsymbol{x} = u\wedge v. \tag{1.15}$$

The relative velocity is therefore

$$\boldsymbol{u} = \frac{\partial \boldsymbol{x}}{\partial t} = \frac{\partial \boldsymbol{x}}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{u\wedge v}{u\cdot v}. \tag{1.16}$$

If we form the Lorentz factor γ using

$$\begin{aligned} \gamma^{-2} &= 1 - \boldsymbol{u}^2 \\ &= 1 + (u\cdot v)^{-2}[(uv - u\cdot v)(vu - v\cdot u)] = (u\cdot v)^{-2}, \end{aligned} \tag{1.17}$$

we find that $\gamma = u\cdot v = \partial_\tau t$. It follows that we can decompose the velocity as

$$u = uvv = (u\cdot v + u\wedge v)v = \gamma(1 + \boldsymbol{u})v, \tag{1.18}$$

which shows a neat split into a part $\gamma\boldsymbol{u}v$ in the rest space of v, and a part γv along v.

ii. Momentum and Wave Vectors

Now suppose we observe a particle with mass m and velocity u. The energy-momentum vector is $p = mu$. The energy of the particle in the v-frame is $E = \gamma m$, which can be written as $E = p\cdot v$. The relative momentum is $\boldsymbol{p} = \gamma m\boldsymbol{u}$, so that $\boldsymbol{p} = p\wedge v$. It follows that

$$pv = p\cdot v + p\wedge v = E + \boldsymbol{p}. \tag{1.19}$$

From this we recover the invariant

$$m^2 = p^2 = pvvp = (E + \boldsymbol{p})(E - \boldsymbol{p}) = E^2 - \boldsymbol{p}^2. \tag{1.20}$$

Similarly, for a photon with wave-vector k we have

$$kv = k\cdot v + k\wedge v = \omega + \boldsymbol{k}, \tag{1.21}$$

where ω is the frequency in the v-frame, and $\boldsymbol{k}$ is the relative wave-vector. For photons in empty space $k^2 = 0$ so

$$0 = kvvk = (\omega + \boldsymbol{k})(\omega - \boldsymbol{k}) = \omega^2 - \boldsymbol{k}^2. \tag{1.22}$$

This recovers the dispersion relation $|\boldsymbol{k}| = \omega$, which holds in all frames.

This idea of projecting onto the even subalgebra to study physics in a rest frame is a very powerful technique. Our next task is to study Lorentz transformations to see how different observers see the same physics.

1.3 Lorentz Transformations

Lorentz Transformations are usually expressed in the form of a coordinate transformation, *e.g.* for relative motion along the x-axis

$$\begin{aligned} \mathrm{x}' &= \gamma(\mathrm{x} - \beta t) \qquad & t' &= \gamma(t - \beta \mathrm{x}) \\ \mathrm{x} &= \gamma(\mathrm{x}' + \beta t') \qquad & t &= \gamma(t' + \beta \mathrm{x}') \end{aligned} \tag{1.1}$$

where $\gamma = (1 - \beta^2)^{-1/2}$ and β is the scalar velocity between the two frames in units of c. The y and z coordinates are unchanged. Our first task is to manipulate these relations into a transformation law for vectors. The vector x has been decomposed in two frames, $\{e_\mu\}$ and $\{e'_\mu\}$, so that

$$x = x^\mu e_\mu = x^{\mu\prime} e'_\mu. \tag{1.2}$$

We then have, for example

$$t = e^0 \cdot x, \qquad t' = e^{0\prime} \cdot x. \tag{1.3}$$

Concentrating on the 0 and 1 components we have

$$t e_0 + \mathrm{x} e_1 = t' e'_0 + \mathrm{x}' e'_1, \tag{1.4}$$

and from this we derive the vector relations

$$e'_0 = \gamma(e_0 + \beta e_1), \quad e'_1 = \gamma(e_1 + \beta e_0). \tag{1.5}$$

These define the new frame in terms of the old ($e'_2 = e_2$ and $e'_3 = e_3$).

1.3.1 Rotor Form of a Lorentz Transformation

We saw earlier that bivectors with positive square lead to hyperbolic geometry. This suggests that we introduce an 'angle' α with

$$\tanh\alpha = \beta, \qquad (\beta < 1), \tag{1.6}$$

so that

$$\gamma = (1 - \tanh^2\alpha)^{-1/2} = \cosh\alpha. \tag{1.7}$$

The vector e_0' is now

$$\begin{aligned} e_0' &= \mathrm{ch}(\alpha)e_0 + \mathrm{sh}(\alpha)e_1 \\ &= [\mathrm{ch}(\alpha) + \mathrm{sh}(\alpha)e_1e_0]e_0 = \mathrm{e}^{\alpha\, e_1e_0}\, e_0, \end{aligned} \tag{1.8}$$

where we have expressed the scalar + bivector as an exponential. Similarly, we have

$$e_1' = \mathrm{ch}(\alpha)e_1 + \mathrm{sh}(\alpha)e_0 = \mathrm{e}^{\alpha\, e_1e_0}\, e_1. \tag{1.9}$$

Now recall that these are just two of four frame vectors, with the other pair untouched. Since e_1e_0 anticommutes with e_0 and e_1, but commutes with e_2 and e_3, we can express the relationship between the two frames as

$$e_\mu' = Re_\mu\tilde{R}, \quad e^{\mu\prime} = Re^\mu\tilde{R}, \quad R = \mathrm{e}^{\alpha\, e_1e_0/2}. \tag{1.10}$$

The same rotor prescription works for boosts as well as rotations! Now we really are treating spacetime as a unified entity.

1.3.2 The Restricted Lorentz Group

The transformation $a \mapsto Ra\tilde{R}$, with R a rotor, preserves causal ordering as well as parity. Transformations of this type are called 'proper orthochronous' transformations, and are elements of the *restricted Lorentz group*. (The full Lorentz group allows for the inclusion of reflections and inversions.) We can prove that rotor driven transformations are proper orthochronous by starting with the velocity γ_0 and transforming it to $v = R\gamma_0\tilde{R}$. We need the γ_0 component of v to be positive if causal ordering is to be preserved, that is

$$\gamma_0 \cdot v = \langle \gamma_0 R\gamma_0\tilde{R}\rangle > 0. \tag{1.11}$$

Decomposing in the γ_0-frame we can write

$$R = \alpha + \boldsymbol{a} + I\boldsymbol{b} + I\beta \tag{1.12}$$

and we find that

$$\langle \gamma_0 R\gamma_0\tilde{R}\rangle = \alpha^2 + \boldsymbol{a}^2 + \boldsymbol{b}^2 + \beta^2 > 0 \tag{1.13}$$

as required. Of the elements of the full Lorentz group, the proper orthochronous transformations are the most physically relevant.

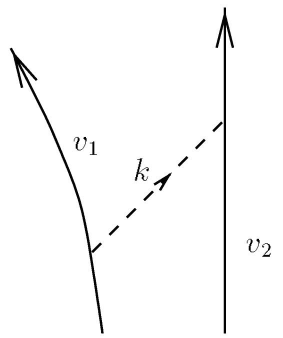


Figure 1.2: *Photon Emission and Absorption.* A photon is emitted by particle 1 and received by particle 2.

1.3.3 Examples

i. Addition of Velocities

As a simple example, suppose that we are in a frame with basis vectors $\{e_0, e_1\}$. We observe two objects flying apart with 4-velocities

$$v_1 = \mathrm{e}^{\alpha_1 e_1 e_0}\, e_0, \qquad v_2 = \mathrm{e}^{-\alpha_2 e_1 e_0}\, e_0. \tag{1.14}$$

What is the relative velocity they see for each other? We form

$$\frac{v_1 \wedge v_2}{v_1 \cdot v_2} = \frac{\langle \mathrm{e}^{(\alpha_1 + \alpha_2) e_1 e_0} \rangle_2}{\langle \mathrm{e}^{(\alpha_1 + \alpha_2) e_1 e_0} \rangle_0} = \frac{\sinh(\alpha_1 + \alpha_2) e_1 e_0}{\cosh(\alpha_1 + \alpha_2)}. \tag{1.15}$$

Both observers therefore measure a relative velocity of

$$\tanh(\alpha_1 + \alpha_2) = \frac{\tanh\alpha_1 + \tanh\alpha_2}{1 - \tanh\alpha_1 \tanh\alpha_2} \tag{1.16}$$

Addition of velocities is achieved by adding hyperbolic angles, which recovers the familiar formula.

ii. Photons and Redshifts

Often in studying the properties of electromagnetic waves we use the geometric optics approximation and work directly with null wave-vectors k. This provides for simple formulae for Doppler shifts and aberration. Suppose that two particles follow different worldlines and that particle 1 emits a photon which is received by particle 2 (see

Fig. 1.2). The frequency seen by particle 1 is $\omega_1 = v_1 \cdot k$, and by particle 2 is $\omega_2 = v_2 \cdot k$. The ratio of these describes the Doppler effect, often expressed as a redshift, z:

$$1 + z = \omega_1/\omega_2. \tag{1.17}$$

This can be applied in many ways. For example, suppose that the emitter is receding in the e_1 direction, and $v_2 = e_0$. We have

$$k = \omega_2(e_0 + e_1), \qquad v_1 = \cosh\alpha\, e_0 - \sinh\alpha\, e_1, \tag{1.18}$$

so that

$$1 + z = \frac{\omega_2(\cosh\alpha + \sinh\alpha)}{\omega_2} = \mathrm{e}^{\alpha}. \tag{1.19}$$

The velocity of the emitter in the e_0 frame is $\tanh\alpha$, and it is easy to check that

$$\mathrm{e}^{\alpha} = \left(\frac{1 + \tanh\alpha}{1 - \tanh\alpha}\right)^{1/2}, \tag{1.20}$$

recovering the standard expression for the relativistic Doppler effect. Aberration formulae can be obtained in the same way.

Chapter 2

Spacetime Dynamics

Lorentz transformations which preserve parity and causal structure can be described with rotors, and these provide the simplest way to gain insight into the structure of the Lorentz group. They quickly show, for example, that all transformations have two points on the celestial sphere which remain fixed. Dynamics in spacetime is traditionally viewed as a hard subject. This need not be the case, however. By parameterising the motion in terms of rotors many equations are considerably simplified, and can be solved in new ways. We illustrate this with a new formulation of the Lorentz force law, from which we obtain the general solution for the motion of a point particle in a constant electromagnetic field.

2.1 Spacetime Rotors

We saw in Section 1.3 that a restricted Lorentz transformation is generated by a rotor R, with $R\tilde{R} = 1$, in the usual way as $a \mapsto Ra\tilde{R}$. Every rotor in spacetime can be written in terms of a bivector as

$$R = \pm\, \mathrm{e}^{B/2}\,. \tag{2.1}$$

(The minus sign is rarely required, and does not affect the vector transformation law.) We can understand many of the features of spacetime transformations and rotors through the properties of the bivector B.

2.1.1 Invariant Decomposition

The rotor R can be decomposed in a Lorentz invariant way by first writing

$$B^2 = \langle B^2\rangle_0 + \langle B^2\rangle_4 = \rho\, \mathrm{e}^{I\phi}\,, \tag{2.2}$$

and we will assume that $\rho \neq 0$. (The case of a null bivector is treated slightly differently.) We now define

$$\hat{B} = \rho^{-1/2}\, \mathrm{e}^{-I\phi/2}\, B, \tag{2.3}$$

which is a unit timelike bivector since

$$\hat{B}^2 = \rho^{-1}\, \mathrm{e}^{-I\phi}\, B^2 = 1. \tag{2.4}$$

With this we can now write

$$B = \rho^{1/2}\, \mathrm{e}^{I\phi/2}\, \hat{B} = \alpha\hat{B} + \beta I\hat{B}, \tag{2.5}$$

which decomposes B into a pair of bivector blades $\alpha\hat{B}$ and $\beta I\hat{B}$. Since

$$\hat{B}(I\hat{B}) = (I\hat{B})\hat{B} = I, \tag{2.6}$$

the separate bivector blades commute, which is possible now that we are in 4 dimensions. The rotor R now decomposes into

$$R = \mathrm{e}^{\alpha\hat{B}/2}\, \mathrm{e}^{\beta I\hat{B}/2} = \mathrm{e}^{\beta I\hat{B}/2}\, \mathrm{e}^{\alpha\hat{B}/2} \tag{2.7}$$

exhibiting an *invariant* split into a boost and a rotation. The boost is generated by $\hat{B}$ and the rotation by $I\hat{B}$.

2.1.2 Fixed Points

For every timelike bivector $\hat{B}$, $\hat{B}^2 = 1$, we can construct a pair of null vectors $n_\pm$ satisfying

$$\hat{B}\cdot n_\pm = \pm n_\pm. \tag{2.8}$$

These are necessarily null, since

$$(\hat{B}\cdot n_\pm)\cdot n_\pm = 0 = \pm n_\pm^2. \tag{2.9}$$

The two null vectors can also be chosen so that

$$n_+\wedge n_- = 2\hat{B}, \tag{2.10}$$

so that they form a null basis for the timelike plane defined by $\hat{B}$ (see Fig. 2.1).

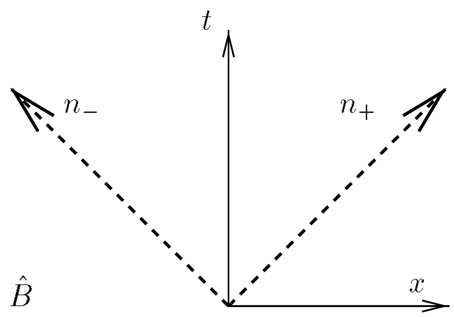


Figure 2.1: *A timelike plane.* Any timelike plane $\hat{B}$, $\hat{B}^2 = 1$ contains two null vectors n_+ and n_-. These can be normalised so that $n_+\wedge n_- = 2\hat{B}$.

Taking the wedge product of Eq. (2.8) with $\hat{B}$ gives $\hat{B}\wedge n_\pm = 0$, so the null vectors $n_\pm$ anticommute with $\hat{B}$ and therefore commute with $I\hat{B}$. The effect of the Lorentz transformation on $n_\pm$ is therefore

$$\begin{aligned} Rn_\pm\tilde{R} &= \mathrm{e}^{\alpha\hat{B}/2}\, n_\pm\, \mathrm{e}^{-\alpha\hat{B}/2} = \mathrm{e}^{\alpha\hat{B}}\, n_\pm \\ &= \mathrm{ch}(\alpha) n_\pm + \mathrm{sh}(\alpha)\hat{B}\cdot n_\pm = \mathrm{e}^{\pm\alpha}\, n_\pm. \end{aligned} \tag{2.11}$$

The two null vectors are therefore just scaled — their direction is unchanged. It follows that every Lorentz transformation has two invariant null directions. For the case where the bivector generator itself is null, $B^2 = 0$, there is a single invariant null vector n which is the unique (up to scaling) null solution of $B\cdot n = 0$.

2.1.3 The Celestial Sphere

One way to visualise the effect of Lorentz transformations is through their effect on the past light cone (see Fig. 2.2). Each null vector on the past light cone maps to a point on the sphere S^- — the *celestial sphere* for the observer. Suppose then that light is received along the null vector n, with the observer's velocity chosen to be γ_0. The relative vector in the γ_0 frame is $n\wedge\gamma_0$. This has magnitude

$$(n\wedge\gamma_0)^2 = (n\cdot\gamma_0)^2 - n^2\gamma_0^2 = (n\cdot\gamma_0)^2. \tag{2.12}$$

We therefore define the unit relative vector $\boldsymbol{n}$ by the projective formula

$$\boldsymbol{n} = \frac{n\wedge\gamma_0}{n\cdot\gamma_0}. \tag{2.13}$$

Different observers passing through the same point see different celestial spheres. If a second observer has velocity $v = R\gamma_0\tilde{R}$, the unit relative vectors in this observer's

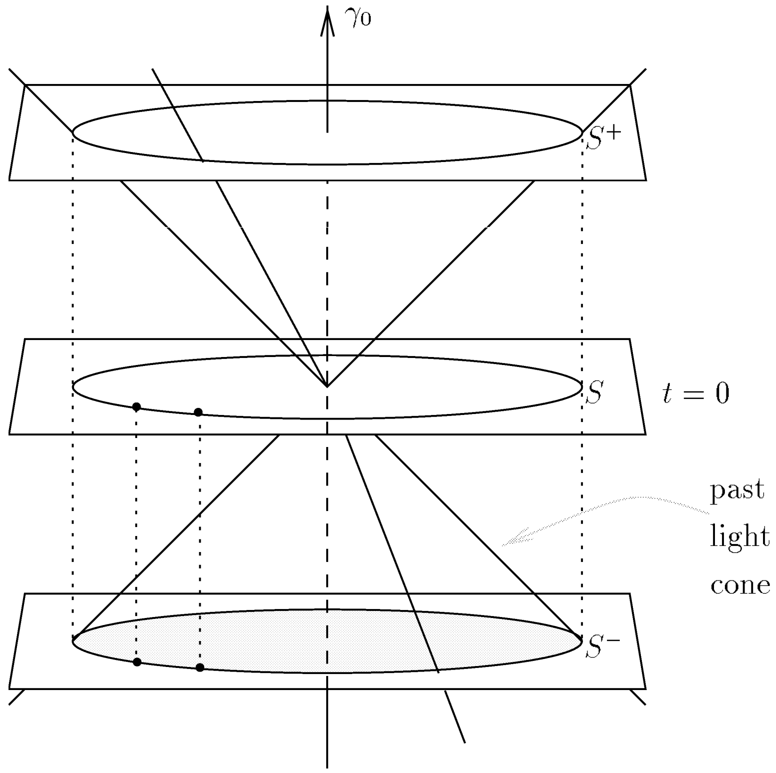


Figure 2.2: *The Celestial Sphere.* Each observer sees events in their past light cone, which can be viewed as defining a sphere.

frame are formed from $n\wedge v/n\cdot v$. These can be brought to the γ_0 frame for comparison by forming

$$\boldsymbol{n}' = \tilde{R}\frac{n\wedge v}{n\cdot v}R = \frac{n'\wedge\gamma_0}{n'\cdot\gamma_0} \tag{2.14}$$

where $n' = \tilde{R}nR$. The effects of Lorentz transformations can be visualised simply by moving around points on the celestial sphere with the map $n \mapsto \tilde{R}nR$. We know immediately, then, that two points remain invariant so are the same for both observers.

2.1.4 Pure Boosts and Observer Splits

Suppose we are travelling with velocity u and want to boost to velocity v. We seek the rotor for this which contains no additional rotational factors. We have

$$v = Lu\tilde{L} \tag{2.15}$$

with $La_\perp \tilde{L} = a_\perp$ for any vector outside the $u\wedge v$ plane. It is clear that the appropriate bivector for the rotor is $u\wedge v$, and as this anticommutes with u and v we have

$$v = Lu\tilde{L} = L^2 u \qquad \Longrightarrow \; L^2 = vu \tag{2.16}$$

The solution to this is

$$L = \frac{1 + vu}{[2(1 + u\cdot v)]^{1/2}} = \exp\left(\frac{\alpha}{2}\frac{v\wedge u}{|v\wedge u|}\right) \tag{2.17}$$

where the angle α is defined by $\cosh(\alpha) = u\cdot v$.

Now suppose that we start in the γ_0 frame and some arbitrary rotor R takes this to $v = R\gamma_0\tilde{R}$. We know that the pure boost for this transformation is

$$L = \frac{1 + v\gamma_0}{[2(1 + v\cdot\gamma_0)]^{1/2}} = \exp\left(\frac{\alpha}{2}\frac{v\wedge\gamma_0}{|v\wedge\gamma_0|}\right), \tag{2.18}$$

where $v\cdot\gamma_0 = \mathrm{ch}(\alpha)$. Now define the further rotor U by

$$U = \tilde{L}R, \qquad U\tilde{U} = \tilde{L}R\tilde{R}L = 1. \tag{2.19}$$

This satisfies

$$U\gamma_0\tilde{U} = \tilde{L}vL = \tilde{L}L\gamma_0\tilde{L}L = \gamma_0, \tag{2.20}$$

so $U\gamma_0 = \gamma_0 U$. We must therefore have $U = e^{I\boldsymbol{b}/2}$, where $I\boldsymbol{b}$ is a relative bivector, and U generates a pure rotation in the γ_0 frame. We now have

$$R = LU \tag{2.21}$$

which decomposes R into a relative rotation and boost. Unlike earlier, this decomposition is frame dependent, and in general L and U do not commute.

2.2 Spacetime Rotor Equations

A spacetime trajectory $x(\tau)$ has a future-pointing velocity vector $\dot{x} = v$, where the overdots denote ∂_τ. The velocity is normalised to $v^2 = 1$ by parameterising the curve in terms of the proper time. This suggests an analogy with rigid body dynamics. We write

$$v = R\gamma_0\tilde{R}, \tag{2.1}$$

which keeps v future-pointing and normalised. This moves all of the dynamics into the rotor $R = R(\tau)$, and this is the key idea which simplifies much of relativistic dynamics.

2.2.1 The Proper Acceleration

The first quantity we need to find is the acceleration

$$\dot{v} = \partial_\tau (R\gamma_0 \tilde{R}) = \dot{R}\gamma_0 \tilde{R} + R\gamma_0 \dot{\tilde{R}}. \tag{2.2}$$

But we know that $\dot{R}\tilde{R} = -R\dot{\tilde{R}}$ is a bivector, so we have

$$\dot{v} = \dot{R}\tilde{R}R\gamma_0 \tilde{R} + R\gamma_0 \tilde{R}R\dot{\tilde{R}} = \dot{R}\tilde{R}v - v\dot{R}\tilde{R} = 2(\dot{R}\tilde{R})\cdot v. \tag{2.3}$$

This equation is consistent with the fact that $v\cdot\dot{v} = 0$, which follows from $v^2 = 1$.

We now have

$$\dot{v}v = 2(\dot{R}\tilde{R})\cdot v\, v. \tag{2.4}$$

The bivector $\dot{v}v$ is the acceleration seen in the instantaneous rest frame. We call this object $\dot{v}v$ the *acceleration bivector*. Eq. (2.4) determines the projection of $\dot{R}\tilde{R}$ parallel to v in terms of the kinematics of v. The remaining freedom in $\dot{R}\tilde{R}$ corresponds to an additional rotation in R which does not change v. For the purposes of determining the velocity and trajectory of the particle the component of $\dot{R}\tilde{R}$ perpendicular to v is of no relevance. However, in some applications it is useful to attach physical significance to the rotated vectors $\{e_i\} = R\gamma_i \tilde{R}$, $i = 1\ldots 3$ which span the instantaneous rest space of v. In this case, the dynamics of the e_i can be used to determine the component of $\dot{R}\tilde{R}$ which is not fixed by v alone.

2.2.2 Fermi Transport

The vectors $\{e_i\}$ are carried along the trajectory by the rotor R, so that $e_i\cdot v = 0$. They are said to be *Fermi transported* if their transformation from one instance to the next is a pure boost in the v frame. One can think of this as the vectors $\{e_i\}$ remaining as constant as possible, subject to the constraint $e_i\cdot v = 0$. The direction defined by the angular momentum of an inertial guidance gyroscope (supported at its centre of mass so there are no torques) is Fermi transported along the path of the gyroscope through spacetime.

To ensure Fermi transport of $R\gamma_i \tilde{R}$ we need to ensure that the rotor we work with correctly describes pure boosts from one instance to the next (see Fig. 2.3). To first order we have

$$v(\tau + \delta\tau) = v(\tau) + \delta\tau\, \dot{v}. \tag{2.5}$$

The pure boost between $v(\tau)$ and $v(\tau + \delta\tau)$ is represented by the rotor

$$L = \frac{1 + v(\tau + \delta\tau)v(\tau)}{[2(1 + v(\tau + \delta\tau)\cdot v(\tau))]^{1/2}} = 1 + \tfrac{1}{2}\delta\tau\, \dot{v}v, \tag{2.6}$$

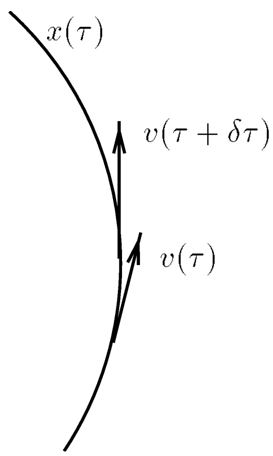


Figure 2.3: *The Proper Boost.* The change in velocity from τ to $\tau+\delta\tau$ should be described by a rotor solely in the $\dot{v}\wedge v$ plane.

to first order in $\delta\tau$. But since

$$R(\tau+\delta\tau) = R(\tau) + \delta\tau\dot{R}(\tau) = (1+\delta\tau\dot{R}\tilde{R})R(\tau), \tag{2.7}$$

The additional rotation that takes our frame $\{e_i\}$ from τ to $\tau+\delta\tau$ is described by the rotor $1+\delta\tau\dot{R}\tilde{R}$ to first-order. Equating this to the pure boost L, we find that the correct expression to ensure Fermi transport of the $\{e_i\}$ is

$$\dot{R}\tilde{R} = \tfrac{1}{2}\dot{v}v, \tag{2.8}$$

which is sensible. The bivector describing the change in the rotor is simply the acceleration bivector.

Under Fermi transport the $\{e_i\}$ frame vectors satisfy

$$\dot{e}_i = 2(\dot{R}\tilde{R})\cdot e_i = -e_i\cdot(\dot{v}v). \tag{2.9}$$

This gives rise to the definition of the *Fermi derivative*

$$\frac{Da}{D\tau} = \dot{a} + a\cdot(\dot{v}v). \tag{2.10}$$

The Fermi derivative of a vector vanishes if the vector is Fermi transported along the worldline. The derivative preserves both the magnitude a^2 and $a\cdot v$. The former holds because

$$\frac{d}{d\tau}(a^2) = -2a\cdot\big(a\cdot(\dot{v}\wedge v)\big) = 0. \tag{2.11}$$

Conservation of $a\cdot v$ is also straightforward to check:

$$\begin{aligned}\frac{d}{d\tau}(a\cdot v) &= -\big(a\cdot(\dot{v}v)\big)\cdot v + a\cdot\dot{v}\\ &= -a\cdot\dot{v} + a\cdot v\,\dot{v}\cdot v + a\cdot\dot{v} = 0.\end{aligned} \tag{2.12}$$

It follows that if a starts perpendicular to v it remains so. In the case where $a\cdot v = 0$ the Fermi derivative takes on the simple form

$$\frac{Da}{D\tau} = \dot{a} + a\cdot\dot{v}\, v = \dot{a} - \dot{a}\cdot v\, v = \dot{a}\wedge v\, v. \tag{2.13}$$

This is the projection of $\dot{a}$ perpendicular to v, as expected. In Section 2.3.4 we show how Thomas precession comes about from the idea of Fermi transport.

2.3 The Lorentz Force Law

We are all familiar with the non-relativistic form of the Lorentz force law,

$$\frac{d\boldsymbol{p}}{dt} = q(\boldsymbol{E} + \boldsymbol{v}\boldsymbol{\times}\boldsymbol{B}), \tag{2.1}$$

with all relative vectors expressed in the γ_0 frame. The bold cross symbol here denotes the vector cross product. We seek a relativistic version of this law. The quantity $\boldsymbol{p}$ on the left-hand side is the relative vector $p\wedge\gamma_0$. Since $dt = \gamma d\tau$, we must multiply through by $\gamma = v\cdot\gamma_0$ to convert the derivative into one with respect to proper time. The first term on the right-hand side then becomes

$$\begin{aligned} v\cdot\gamma_0\, \boldsymbol{E} &= \tfrac{1}{4}\big(\boldsymbol{E}(v\gamma_0 + \gamma_0 v) + (v\gamma_0 + \gamma_0 v)\boldsymbol{E}\big) \\ &= \tfrac{1}{4}\big((\boldsymbol{E}v - v\boldsymbol{E})\gamma_0 - \gamma_0(\boldsymbol{E}v - v\boldsymbol{E})\big) \\ &= (\boldsymbol{E}\cdot v)\wedge\gamma_0. \end{aligned} \tag{2.2}$$

Recall at this point that $\boldsymbol{E}$ is a spacetime *bivector* and is built from the $\boldsymbol{\sigma}_k = \gamma_k\gamma_0$, so $\boldsymbol{E}$ *anticommutes* with γ_0. The magnetic term is

$$\begin{aligned} -v\cdot\gamma_0\, \boldsymbol{v}\cdot(I\boldsymbol{B}) &= -(v\wedge\gamma_0)\times(I\boldsymbol{B}) \\ &= \tfrac{1}{4}\big(I\boldsymbol{B}(v\gamma_0 - \gamma_0 v) - (v\gamma_0 - \gamma_0 v)I\boldsymbol{B}\big) \\ &= \tfrac{1}{4}\big((I\boldsymbol{B}v - vI\boldsymbol{B})\gamma_0 - \gamma_0(I\boldsymbol{B}v - vI\boldsymbol{B})\big) \\ &= \big((I\boldsymbol{B})\cdot v\big)\wedge\gamma_0, \end{aligned} \tag{2.3}$$

and here we use the fact that γ_0 *commutes* with the combination $I\boldsymbol{B}$.

We can now write Eq. (2.1) in the form

$$\frac{d\boldsymbol{p}}{d\tau} = \dot{p}\wedge\gamma_0 = q[(\boldsymbol{E} + I\boldsymbol{B})\cdot v]\wedge\gamma_0. \tag{2.4}$$

We now define the *Faraday bivector* F by

$$F = \boldsymbol{E} + I\boldsymbol{B}. \tag{2.5}$$

This is the covariant form of the electromagnetic field strength. It unites the electric and magnetic fields into a single spacetime structure. We study this in greater detail in Chapter 3. Our equation is now

$$\dot{p}\wedge\gamma_0 = q(F\cdot v)\wedge\gamma_0. \tag{2.6}$$

The rate of working on the particle is $q\boldsymbol{E}\cdot\boldsymbol{v}$, so

$$\frac{dp_0}{dt} = q\boldsymbol{E}\cdot\boldsymbol{v}. \tag{2.7}$$

Here, $p_0 = p\cdot\gamma_0$ is the particle's energy in the γ_0 frame. Multiplying through by $v\cdot\gamma_0$, we find

$$\dot{p}\cdot\gamma_0 = q\boldsymbol{E}\cdot(v\wedge\gamma_0) = q(F\cdot v)\cdot\gamma_0. \tag{2.8}$$

In the final step we have used $(I\boldsymbol{B})\cdot(v\wedge\gamma_0) = 0$. Adding this equation to Eq. (2.6), and multiplying on the right by γ_0, we find

$$\dot{p} = qF\cdot v. \tag{2.9}$$

Recalling that $p = mv$, we arrive at the relativistic form of the *Lorentz force law*,

$$m\dot{v} = qF\cdot v. \tag{2.10}$$

This is *manifestly* Lorentz covariant, because no particular frame is picked out. The acceleration bivector is

$$\dot{v}v = \frac{q}{m}F\cdot v\, v = \frac{q}{m}(F\cdot v)\wedge v = \frac{q}{m}\boldsymbol{E}_v \tag{2.11}$$

where $\boldsymbol{E}_v$ is the relative electric field in the v frame. A charged point particle only responds to the electric field in its instantaneous frame.

2.3.1 Rotor Form of the Lorentz Force Law

Now suppose that we parameterise the velocity with a rotor. We have

$$\dot{v} = 2(\dot{R}\tilde{R})\cdot v = \frac{q}{m}F\cdot v. \tag{2.12}$$

The simplest form of the rotor equation comes from equating the projected terms to get

$$\dot{R} = \frac{q}{2m}FR \tag{2.13}$$

This is not the most general possibility as we could include an extra multiple of $F\wedge v\, v$. The rotor determined by Eq. (2.13) will not, in general, describe Fermi transport of the $R\gamma_i\tilde{R}$ vectors. However, Eq. (2.13) is sufficient to determine the velocity of the particle, and is certainly the simplest form of rotor equation to work with. How does this help us solve the equations of motion? One immediate advantage is that the equations are now first order:

$$\dot{x} = v = Rv_0\tilde{R}, \qquad 2m\dot{R} = qFR, \tag{2.14}$$

(we usually take $v_0 = \gamma_0$). These are numerically very robust.

2.3.2 Example — Constant Field

This is very easy now! We can immediately integrate the rotor equation to give

$$R = \exp(\frac{q}{2m} F\tau). \tag{2.15}$$

To proceed and recover the trajectory we form the invariant decomposition of F. We first write

$$F^2 = \langle F^2 \rangle_0 + \langle F^2 \rangle_4 = \rho\, \mathrm{e}^{I\beta} \tag{2.16}$$

so that

$$F = \rho^{1/2}\, \mathrm{e}^{I\beta/2}\, \hat{F} = \alpha \hat{F} + I\beta\hat{F} \tag{2.17}$$

where $\hat{F}^2 = 1$. (If F is null a slightly different procedure is followed.) We now have

$$R = \exp(\frac{q}{2m}\alpha\hat{F}\tau)\exp(\frac{q}{2m}I\beta\hat{F}\tau). \tag{2.18}$$

Next we decompose the initial velocity v_0 into components in and out of the $\hat{F}$ plane,

$$v_0 = \hat{F}^2 v_0 = \hat{F}\,\hat{F}\cdot v_0 + \hat{F}\,\hat{F}\wedge v_0 = v_{0||} + v_{0\perp}. \tag{2.19}$$

Now $v_{0||} = \hat{F}\,\hat{F}\cdot v_0$ anticommutes with $\hat{F}$, and $v_{0\perp}$ commutes with $\hat{F}$, so

$$\dot{x} = \exp(\frac{q}{m}\alpha\hat{F}\tau)v_{0||} + \exp(\frac{q}{m}I\beta\hat{F}\tau)v_{0\perp}. \tag{2.20}$$

This integrates immediately to give the particle history

$$x - x_0 = \frac{\mathrm{e}^{q\alpha\hat{F}\tau/m} - 1}{q\alpha/m}\hat{F}\cdot v_0 - \frac{\mathrm{e}^{q\beta I\hat{F}\tau/m} - 1}{q\beta/m}(I\hat{F})\cdot v_0 \tag{2.21}$$

The first term gives linear acceleration and the second is periodic and drives rotational motion. This is as expected, because in the $v_{0||}$ frame, $\hat{F}$ is an electric field, and in the $v_{0\perp}$ frame, $\hat{F}$ is a magnetic field.

2.3.3 The Gyromagnetic Moment

For a particle with spin, the gyromagnetic ratio g is determined in non-relativistic physics by the precession of the relative spin vector $\boldsymbol{s}$ in a magnetic field $\boldsymbol{B}$:

$$\dot{\boldsymbol{s}} = g\frac{q}{2m}\boldsymbol{s}\times\boldsymbol{B} = g\frac{q}{2m}(I\boldsymbol{B})\cdot\boldsymbol{s}. \tag{2.22}$$

We want to extend this definition of g to relativistic scenarios. We start by introducing the spin vector of the particle s, which is perpendicular to the velocity v. For a particle at rest in the γ_0 frame we have $s = \boldsymbol{s}\gamma_0$. The particle's spin will interact with the magnetic field only in the instantaneous rest frame, so we should regard Eq. (2.22) as referring to this frame.

Recalling that $I\boldsymbol{B} = (F + \gamma_0 F \gamma_0)/2$, we find that

$$\begin{aligned}(I\boldsymbol{B})\cdot\boldsymbol{s} &= \tfrac{1}{4}\big((F + \gamma_0 F \gamma_0)s\gamma_0 - s\gamma_0(F + \gamma_0 F \gamma_0)\big)\\ &= \tfrac{1}{4}\big((Fs - sF)\gamma_0 - \gamma_0(Fs - sF)\big)\\ &= \tfrac{1}{2}\big((F\cdot s)\gamma_0 - \gamma_0(F\cdot s)\big)\\ &= (F\cdot s)\wedge\gamma_0.\end{aligned} \tag{2.23}$$

So, for a particle at rest in the γ_0 frame, equation (2.22) can be written

$$\frac{ds}{dt} = g\frac{q}{2m}(F\cdot s)\wedge\gamma_0\,\gamma_0. \tag{2.24}$$

To write down an equation which is valid for arbitrary velocity we must replace the two factors of γ_0 on the right-hand side with the velocity v. On the left-hand side we need the derivative of s which preserves $s\cdot v = 0$. This is the Fermi derivative of Section 2.2.2, which tells us that the relativistic form of the spin precession equation is

$$\dot{s} + s\cdot(\dot{v}v) = g\frac{q}{2m}(F\cdot s)\wedge v\,v. \tag{2.25}$$

This equation tells us how much the spin vector rotates, relative to a Fermi-transported frame, which is physically sensible. We can eliminate the acceleration bivector $\dot{v}v$ by using the relativistic Lorentz force law to find

$$\begin{aligned}\dot{s} &= g\frac{q}{2m}(F\cdot s)\wedge v\,v - \frac{q}{m}s\cdot(F\cdot v\,v)\\ &= \frac{q}{2m}\big(g(F\cdot s)\wedge v + 2(F\cdot s)\cdot v\big)v\\ &= \frac{q}{m}F\cdot s + (g-2)\frac{q}{2m}(F\cdot s)\wedge v\,v.\end{aligned} \tag{2.26}$$

This is called the Bargmann-Michel-Telegdi (BMT) equation.

For the value $g = 2$, we find the very simple equation

$$\dot{s} = \frac{q}{m}F\cdot s, \tag{2.27}$$

which has the same form as the Lorentz force law, Eq. (2.10). In this sense, $g = 2$ is the most natural value of the gyromagnetic ratio of a point particle in relativistic physics. Ignoring quantum corrections, this is indeed found to be the value for an electron. Quantum corrections tell us that for an electron $g = 2(1 + \alpha/2\pi + \ldots)$. The corrections are due to the fact that the electron is never truly isolated and constantly interacts with virtual particles from the quantum vacuum.

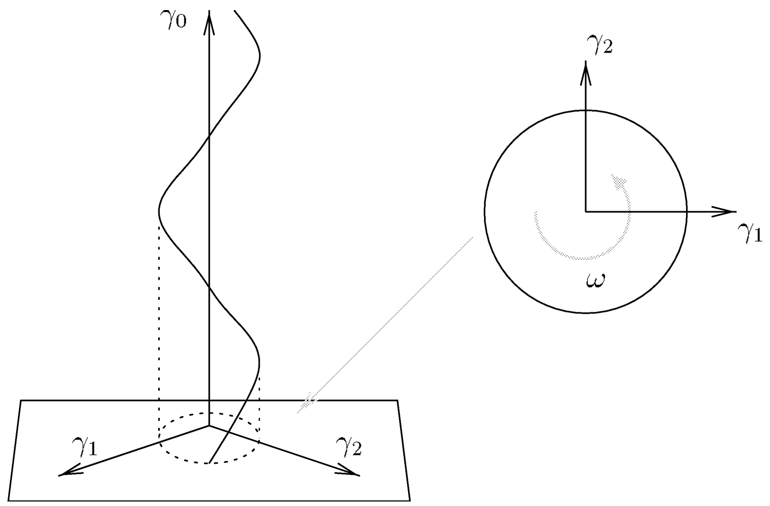


Figure 2.4: *Thomas Precession.* The particle follows a helical worldline, rotating at a constant rate in the γ_0 frame.

Given a velocity v and a spin vector s, $v\cdot s = 0$, we can always find a rotor such that

$$v = R\gamma_0\tilde{R}, \quad s = R\gamma_3\tilde{R}. \tag{2.28}$$

For these we have

$$\dot{v} = 2(\dot{R}\tilde{R})\cdot v, \quad \dot{s} = 2(\dot{R}\tilde{R})\cdot s. \tag{2.29}$$

For a particle with $g = 2$, this pair of equations reduces to the single rotor equation of (2.13). The simple form of this equation also justifies the claim that $g = 2$ is the natural, relativistic value of the gyromagnetic ration. This also means that once we have solved the rotor equation, we have simultaneously solved for the trajectory and precession of a classical relativistic particle with $g = 2$.

2.3.4 Worked Example — Thomas Precession

As an application of the rotor formulation of spacetime dynamics, we derive the Thomas precession of a particle moving in a circular orbit (Fig. 2.4). You may already have met Thomas precession in the context of spin-orbit coupling in the Hydrogen atom.

The worldline of a particle describing a circle of radius a at angular frequency ω is

$$x(\tau) = t(\tau)\gamma_0 + a[\cos(\omega t)\gamma_1 + \sin(\omega t)\gamma_2], \tag{2.30}$$

and the velocity is

$$v = \partial_\tau x = \dot{t}\left(\gamma_0 + a\omega[-\sin(\omega t)\gamma_1 + \cos(\omega t)\gamma_2]\right). \tag{2.31}$$

(Throughout we use dots to denote differentiation with respect to proper time τ). The relative velocity $\boldsymbol{v} = v\wedge\gamma_0/v\cdot\gamma_0$ has magnitude $|\boldsymbol{v}| = a\omega$. We therefore introduce the hyperbolic angle α, with

$$\tanh\alpha = a\omega, \qquad \dot{t} = \cosh\alpha. \tag{2.32}$$

The velocity is now

$$v = \mathrm{ch}(\alpha)\gamma_0 + \mathrm{sh}(\alpha)[-\sin(\omega t)\gamma_1 + \cos(\omega t)\gamma_2] = \mathrm{e}^{\alpha\boldsymbol{n}/2}\,\gamma_0\,\mathrm{e}^{-\alpha\boldsymbol{n}/2} \tag{2.33}$$

where

$$\boldsymbol{n} = -\sin(\omega t)\boldsymbol{\sigma}_1 + \cos(\omega t)\boldsymbol{\sigma}_2. \tag{2.34}$$

This form of time-dependence in the rotor is inconvenient to work with. To simplify, we write

$$\boldsymbol{n} = \mathrm{e}^{-\omega t I\boldsymbol{\sigma}_3}\,\boldsymbol{\sigma}_2 = R_\omega\boldsymbol{\sigma}_2\tilde{R}_\omega \tag{2.35}$$

where $R_\omega = \exp(-\omega t I\boldsymbol{\sigma}_3/2)$. We now have

$$\mathrm{e}^{\alpha\boldsymbol{n}/2} = \exp(\alpha R_\omega\boldsymbol{\sigma}_2\tilde{R}_\omega/2) = R_\omega\,\mathrm{e}^{\alpha\boldsymbol{\sigma}_2/2}\,\tilde{R}_\omega = R_\omega R_\alpha\tilde{R}_\omega \tag{2.36}$$

where $R_\alpha = \exp(\alpha\boldsymbol{\sigma}_2/2)$. The velocity is now given by

$$v = R_\omega R_\alpha\tilde{R}_\omega\gamma_0 R_\omega\tilde{R}_\alpha\tilde{R}_\omega = R_\omega R_\alpha\gamma_0\tilde{R}_\alpha\tilde{R}_\omega, \tag{2.37}$$

where the final expression follows because R_ω commutes with γ_0.

We can now see that the rotor for the motion must have the form

$$R = R_\omega R_\alpha\Phi, \tag{2.38}$$

where Φ is a rotor that commutes with γ_0. We want R to describe Fermi transport of the $\{R\gamma_i\tilde{R}\}$, so we must have $\dot{v}v = 2\dot{R}\tilde{R}$. We begin by forming the acceleration bivector $\dot{v}v$. We can simplify this derivation by writing $v = R_\omega v_\alpha\tilde{R}_\omega$, where $v_\alpha = R_\alpha\gamma_0\tilde{R}_\alpha$. We then get

$$\begin{aligned}\dot{v}v &= R_\omega[2(\tilde{R}_\omega\dot{R}_\omega)\cdot v_\alpha\, v_\alpha]\tilde{R}_\omega = -\omega\,\mathrm{ch}(\alpha)R_\omega[(I\boldsymbol{\sigma}_3)\cdot v_\alpha\, v_\alpha]\tilde{R}_\omega\\ &= \omega\,\mathrm{sh}(\alpha)\mathrm{ch}(\alpha)R_\omega[-\mathrm{ch}(\alpha)\boldsymbol{\sigma}_1 + \mathrm{sh}(\alpha)I\boldsymbol{\sigma}_3]\tilde{R}_\omega.\end{aligned} \tag{2.39}$$

We also form the rotor equivalent $2\dot{R}\tilde{R}$, which is

$$\begin{aligned}2\dot{R}\tilde{R} &= 2\dot{R}_\omega\tilde{R}_\omega + 2R_\omega R_\alpha\dot{\Phi}\tilde{\Phi}\tilde{R}_\alpha\tilde{R}_\omega\\ &= -\omega\mathrm{ch}(\alpha)I\boldsymbol{\sigma}_3 + 2R_\omega R_\alpha\dot{\Phi}\tilde{\Phi}\tilde{R}_\alpha\tilde{R}_\omega.\end{aligned} \tag{2.40}$$

Equating these we find that

$$\begin{aligned}2\dot{\Phi}\tilde{\Phi} &= \omega\mathrm{ch}^2(\alpha)\tilde{R}_\alpha[-\mathrm{sh}(\alpha)\boldsymbol{\sigma}_1 + \mathrm{ch}(\alpha)I\boldsymbol{\sigma}_3]R_\alpha\\ &= \mathrm{ch}^2(\alpha)\omega I\boldsymbol{\sigma}_3.\end{aligned} \tag{2.41}$$

The solution with $\Phi = 1$ at $t = 0$ is $\Phi = \exp[\mathrm{ch}(\alpha)\omega t \boldsymbol{I\sigma}_3/2]$, so the full rotor is

$$R = \mathrm{e}^{-\omega t \boldsymbol{I\sigma}_3/2} \, \mathrm{e}^{\alpha \boldsymbol{\sigma}_2/2} \, \mathrm{e}^{\mathrm{ch}(\alpha)\omega t \boldsymbol{I\sigma}_3/2}. \tag{2.42}$$

This form of the rotor ensures that the $e_i = R\gamma_i\tilde{R}$ are Fermi transported. At a given point on the circle we can compare the $\{e_i\}$ from one period to the next. We find that $e_3 = \gamma_3$ is constant during the motion, but e_1 and e_2 precess. Forming e_1 at $t = 2\pi/\omega$, we find

$$e_1(2\pi/\omega) = \mathrm{e}^{\alpha \boldsymbol{\sigma}_2/2} \, \mathrm{e}^{2\pi \mathrm{ch}(\alpha) \boldsymbol{I\sigma}_3} \, \gamma_1 \, \mathrm{e}^{-\alpha \boldsymbol{\sigma}_2/2}. \tag{2.43}$$

Dotting this with the initial vector $e_1(0)$ we see that the vector has precessed through an angle

$$\theta = 2\pi(\cosh\alpha - 1). \tag{2.44}$$

This shows that the effect is of order $|\boldsymbol{v}|^2/c^2$.

Chapter 3

Electromagnetism

The spacetime vector derivative and the geometric product enable us to unite all four of Maxwell's equations into a single equation. This is one of the most impressive results in geometric algebra. Unlike the separate gradient and curl operators, the vector derivative is invertible and this leads to a number of simplifications. As an application, we look at the derivation of the fields due to a point source. We also derive expressions for the field energy and the Poynting vector, and introduce the important idea of the field stress-energy tensor.

3.1 Maxwell's Equations

The four Maxwell equations (in natural units $c = \epsilon_0 = \mu_0 = 1$) are

$$\begin{array}{ll} \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 & \boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho \\ \boldsymbol{\nabla}\times\boldsymbol{E} = -\partial_t\boldsymbol{B} & \boldsymbol{\nabla}\times\boldsymbol{B} = \boldsymbol{J} + \partial_t\boldsymbol{E} \end{array} \tag{3.1}$$

where as usual we employ the symbol $\times$ for the vector cross product. The 3D vector derivative operator here is

$$\boldsymbol{\nabla} = \boldsymbol{\sigma}_i \frac{\partial}{\partial x_i} = \boldsymbol{\sigma}_i\partial_i. \tag{3.2}$$

The fact that we now have the geometric product available for the $\boldsymbol{\sigma}_i$ vectors suggests that we should consider uniting pairs of equations. This will lead us to a manifestly covariant expression of the equations. First we take the two equations for $\boldsymbol{E}$ and write these (in the 3D geometric algebra) as

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho, \quad \boldsymbol{\nabla}\wedge\boldsymbol{E} = -\partial_t(I\boldsymbol{B}). \tag{3.3}$$

These combine to give the single equation

$$\boldsymbol{\nabla}\boldsymbol{E} = \rho - \partial_t(I\boldsymbol{B}). \tag{3.4}$$

Unlike the separate divergence and curl operations, the vector derivative operator in $\boldsymbol{\nabla}\boldsymbol{E}$ has an inverse. That is, there is a Green's function associated directly with $\boldsymbol{\nabla}$. This is outside the scope of this course, but if you look back over your electromagnetism notes you should be able to spot the form of this Green's function.

A similar manipulation combines the $\boldsymbol{B}$-field equations into

$$\boldsymbol{\nabla}\boldsymbol{B} = I(\boldsymbol{J} + \partial_t\boldsymbol{E}). \tag{3.5}$$

If we multiply both sides of this equation by I we arrive at the equation

$$\boldsymbol{\nabla}(I\boldsymbol{B}) = -\boldsymbol{J} - \partial_t\boldsymbol{E}. \tag{3.6}$$

This is a combination of a (spatial) bivector and pseudoscalar equation, whereas Eq. (3.4) contains only scalar and vector parts. It follows that we can combine all of these equations into the single multivector equation

$$\boldsymbol{\nabla}(\boldsymbol{E} + I\boldsymbol{B}) + \partial_t(\boldsymbol{E} + I\boldsymbol{B}) = \rho - \boldsymbol{J}, \tag{3.7}$$

which is beginning to look pretty good! We have not lost any information in writing this, since each of the separate Maxwell equations can be recovered by picking out terms of a given grade.

Recalling the derivation of the relativistic form of the Lorentz force law from Section 2.3, we define the *electromagnetic field strength* F by

$$F = \boldsymbol{E} + I\boldsymbol{B}. \tag{3.8}$$

This is a spacetime bivector. In terms of this we have

$$\boldsymbol{\nabla}F + \partial_t F = \rho - \boldsymbol{J}. \tag{3.9}$$

To convert this to covariant form, we introduce the spacetime current J, which has

$$\rho = J\cdot\gamma_0, \qquad \boldsymbol{J} = J\wedge\gamma_0. \tag{3.10}$$

It follows that

$$\rho - \boldsymbol{J} = \gamma_0\cdot J + \gamma_0\wedge J = \gamma_0 J. \tag{3.11}$$

We now pre-multiply Eq. (3.9) by γ_0 to assemble the equation

$$\gamma_0(\partial_t + \boldsymbol{\nabla})F = J. \tag{3.12}$$

The differential operator on the left-hand side is (recalling that $\gamma^0 = \gamma_0$ and $\gamma^i = -\gamma_i$)

$$\gamma_0\partial_t + \gamma_0\gamma_i\gamma_0\partial_i = \gamma^0\partial_t + \gamma^i\partial_i = \gamma^\mu\partial_\mu = \nabla. \tag{3.13}$$

This defines the *spacetime vector derivative*

$$\nabla = \gamma^\mu \partial_\mu, \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}. \tag{3.14}$$

The spacetime split of the vector derivative is

$$\nabla\gamma_0 = (\gamma^0\partial_t + \gamma^i\partial_i)\gamma_0 = \partial_t - \boldsymbol{\sigma}_i\partial_i = \partial_t - \boldsymbol{\nabla}. \tag{3.15}$$

The minus sign here is in contrast to $x\gamma_0 = t + \boldsymbol{x}$ and is due to the Lorentzian metric. One has to take care to remember this. It becomes obvious when forming

$$\nabla x = 4 = \gamma_0 \nabla x \gamma_0 = \gamma_0 \nabla (t + \boldsymbol{x}) \tag{3.16}$$

which tells us that we must have $\gamma_0\nabla = \partial_t + \boldsymbol{\nabla}$. Hence $\nabla\gamma_0 = (\gamma_0\nabla)^\sim = \partial_t - \boldsymbol{\nabla}$.

In terms of the spacetime vector derivative, all 4 Maxwell equations can be united in the single, manifestly covariant equation

$$\nabla F = J. \tag{3.17}$$

These can be separated into 2 spacetime equations for the vector and trivector parts,

$$\nabla\cdot F = J, \quad \nabla\wedge F = 0. \tag{3.18}$$

In tensor language, these correspond to the pair of spacetime equations

$$\partial_\mu F^{\mu\nu} = J^\nu, \qquad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0. \tag{3.19}$$

This is as compact a formulation of the Maxwell equations as tensor algebra can achieve. (The same is also true of the popular language of *differential forms*.) Only geometric algebra enables us to combine the pair of covariant equations (3.18) into the single equation $\nabla F = J$. This is more than a mere cosmetic trick — this unified equation offers a number of significant improvements. In particular, the ∇ operator (like the $\boldsymbol{\nabla}$ operator) is invertible — there is a Green's function for it. This simplifies diffraction theory and directly encodes Huygen's principle (outside this course). In addition, first order equations are numerically more robust than second order equations, so are preferable for numerical computation.

The wave theory of electromagnetism is recovered by introducing the *vector potential* A, defined so that

$$F = \nabla\wedge A. \tag{3.20}$$

It then follows automatically that

$$\nabla\wedge F = \nabla\wedge(\nabla\wedge A) = \gamma^\mu\wedge\gamma^\nu\wedge\left(\frac{\partial^2 A}{\partial x^\mu \partial x^\nu}\right) = 0, \tag{3.21}$$

which holds due to the antisymmetry of the exterior product. We have some *gauge* freedom in the choice of A, as we can always add the gradient of a scalar field to it. The most natural way to soak up this freedom is to impose the *Lorentz condition* $\nabla\cdot A = 0$, so that $F = \nabla A$. We then recover the familiar wave equation

$$\nabla^2 A = J. \tag{3.22}$$

3.2 The Electromagnetic Field Strength

The spacetime bivector $F = \boldsymbol{E} + I\boldsymbol{B}$ is the *electromagnetic field strength*, also called the Faraday bivector. It is a covariant spacetime bivector. Its components in the $\{\gamma^\mu\}$ frame give rise to the tensor

$$F^{\mu\nu} = \gamma^\nu \cdot (\gamma^\mu \cdot F) = (\gamma^\nu \wedge \gamma^\mu) \cdot F. \tag{3.1}$$

These are the components of a rank-2 antisymmetric tensor which, written out as a matrix, has entries

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}. \tag{3.2}$$

This form is often presented in textbooks on relativistic electrodynamics. The big disadvantage of this matrix form is that the the natural complex structure is hidden.

Writing $F = \boldsymbol{E} + I\boldsymbol{B}$ decomposes F into the sum of a relative vector $\boldsymbol{E}$ and a relative bivector $I\boldsymbol{B}$. The separate $\boldsymbol{E}$ and $I\boldsymbol{B}$ fields are recovered from

$$\begin{aligned} \boldsymbol{E} &= \tfrac{1}{2}(F - \gamma_0 F \gamma_0) \\ I\boldsymbol{B} &= \tfrac{1}{2}(F + \gamma_0 F \gamma_0). \end{aligned} \tag{3.3}$$

This shows clearly how the split into $\boldsymbol{E}$ and $I\boldsymbol{B}$ fields depends on the observer velocity (γ_0 here). Observers in relative motion see different fields. For example, suppose a second observer has velocity $v = R\gamma_0\tilde{R}$ and constructs the rest frame basis vectors

$$\gamma'_\mu = R\gamma_\mu\tilde{R}. \tag{3.4}$$

This observer measures components of an electric field to be

$$E'_i = (\gamma'_i\gamma'_0) \cdot F = (R\boldsymbol{\sigma}_i\tilde{R}) \cdot F = \boldsymbol{\sigma}_i \cdot (\tilde{R}FR). \tag{3.5}$$

The effect of a Lorentz transformation can therefore be seen by taking F to $\tilde{R}FR$. The fact that bivectors are subject to the same rotor transformation law as vectors make it easy to recover the standard formulae.

3.2.1 Observers in Relative Motion

Suppose that in the γ_0 frame some stationary charge configuration sets up the field

$$F = \boldsymbol{E} = E_x\boldsymbol{\sigma}_1 + E_y\boldsymbol{\sigma}_2. \tag{3.6}$$

A second observer has velocity $\tanh(\alpha)$ in the γ_1 direction, so

$$R = \mathrm{e}^{\alpha\boldsymbol{\sigma}_1/2}. \tag{3.7}$$

This observer measures the $\boldsymbol{\sigma}_i$ components of

$$\tilde{R}FR = \mathrm{e}^{-\alpha\boldsymbol{\sigma}_1/2} F \mathrm{e}^{\alpha\boldsymbol{\sigma}_1/2} = E_x\boldsymbol{\sigma}_1 + E_y \mathrm{e}^{-\alpha\boldsymbol{\sigma}_1} \boldsymbol{\sigma}_2 \tag{3.8}$$

which gives

$$E'_x = E_x, \quad E'_y = \mathrm{ch}(\alpha)E_y, \quad B'_z = -\mathrm{sh}(\alpha)E_y. \tag{3.9}$$

This approach is *much* simpler than working with tensors.

3.2.2 Invariants

A further useful result for the F field is to construct its Lorentz invariant terms. We form the quantity

$$F^2 = \langle FF\rangle + \langle FF\rangle_4 = \alpha + I\beta. \tag{3.10}$$

But if we also form

$$(\tilde{R}FR)(\tilde{R}FR) = \tilde{R}FFR = \alpha + I\beta, \tag{3.11}$$

we see that the result is invariant. So both the scalar and pseudoscalar terms are Lorentz invariant — that is, independent of the frame in which they are measured. In the γ_0 frame these are

$$\alpha = \langle(\boldsymbol{E}+I\boldsymbol{B})(\boldsymbol{E}+I\boldsymbol{B})\rangle = \boldsymbol{E}^2 - \boldsymbol{B}^2 \tag{3.12}$$

and

$$\beta = -\langle I(\boldsymbol{E}+I\boldsymbol{B})(\boldsymbol{E}+I\boldsymbol{B})\rangle = 2\boldsymbol{E}\cdot\boldsymbol{B}. \tag{3.13}$$

The former yields the Lagrangian density for the electromagnetic field. The latter is seen less often, and at first it is quite surprising to learn that $\boldsymbol{E}\cdot\boldsymbol{B}$ is a full Lorentz invariant, rather than just being invariant under rotations.

3.3 Fields from a Point Charge

As an application of the power of the STA formulation of electromagnetism, we now give a compact formula for the fields of an arbitrarily moving charge. A charge q moves along a world-line $x_0(\tau)$ (see Fig. 3.1). An observer at spacetime position x receives an electromagnetic influence from the point where the charge's worldline intersects the observer's past light-cone. The vector

$$X \equiv x - x_0(\tau) \tag{3.1}$$

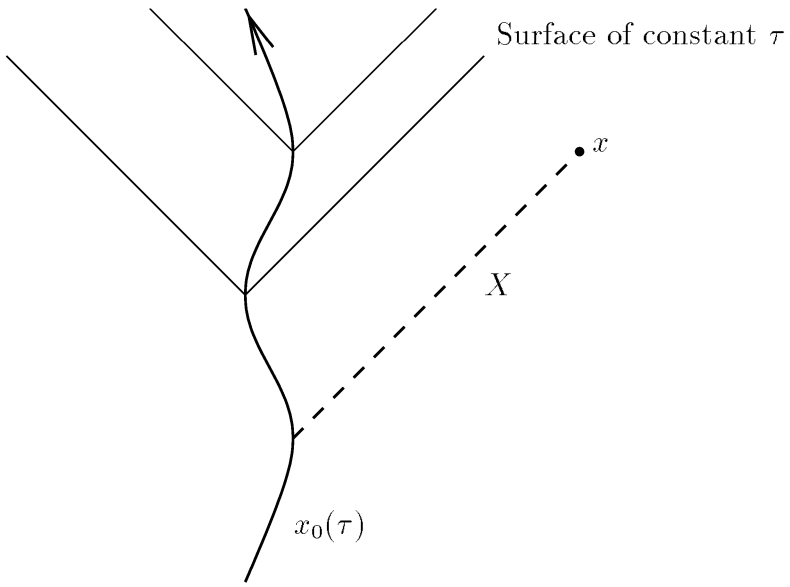


Figure 3.1: *Field from a moving point charge.* The charge follows the trajectory $x_0(\tau)$. $X = x - x_0(\tau)$ is the null vector connecting the point x to the worldline. The time τ can be viewed as a scalar field with each value of τ extended out over the forward null cone.

is the separation vector down the light-cone, joining the observer to this intersection point. This vector must be null, $X^2 = 0$. For every spacetime position x there is a unique value of the proper time along the charge's world-line for which the vector connecting x to the world-line is null. We can write $\tau = \tau(x)$, and treat τ as a scalar field.

The Liénard-Wiechert potential for the retarded field from the charge is

$$A = \frac{q}{4\pi} \frac{v}{X\cdot v}, \tag{3.2}$$

where $v = \dot{x}_0$ is the velocity of the charge at the retarded position $x_0(\tau)$, and X is the null vector connecting $x_0(\tau)$ to the observer's position x. It is not difficult to check that the field of Eq. (3.2) reproduces the Coulomb potential for a charge at rest.

3.3.1 The Field Strength

We now differentiate the potential of Eq. (3.2) to find the Faraday bivector. First, we differentiate the equation $X^2 = 0$ to obtain

$$\gamma^\mu(\partial_\mu X)\cdot X = \gamma^\mu\,(\gamma_\mu - \partial_\mu \tau \partial_\tau x_0)\cdot X = X - \nabla\tau(v\cdot X) = 0. \tag{3.3}$$

It follows that

$$\nabla\tau = \frac{X}{X\cdot v}. \tag{3.4}$$

The gradient of τ points in the direction of constant τ! This is a peculiarity of null surfaces and is one reason why one has to be careful when defining the normal vector to a surface in mixed signature spaces. In finding an expression for $\nabla\tau$ we have demonstrated how the particle proper time can be treated as a spacetime scalar field. Feynman and Wheeler call this an *adjunct* field. It carries information, but does not exist in any physical sense.

To differentiate A we need $\nabla(X\cdot v)$. Using the results already established we have

$$\nabla(X\cdot v) = \gamma^\mu(\partial_\mu X)\cdot v + \nabla\tau\, X\cdot(\partial_\tau v) = v - \nabla\tau + \nabla\tau\, X\cdot\dot{v} \tag{3.5}$$

where $\dot{v} = \partial_\tau v$. We now evaluate ∇A as follows:

$$\begin{aligned} \nabla A &= \frac{q}{4\pi}\left(\frac{\nabla v}{X\cdot v} - \frac{1}{(X\cdot v)^2}\nabla(X\cdot v)v\right) \\ &= \frac{q}{4\pi}\left(\frac{X\dot{v}}{(X\cdot v)^2} - \frac{1}{(X\cdot v)^2} - \frac{(X\, X\cdot\dot{v} - X)v}{(X\cdot v)^3}\right) \\ &= \frac{q}{4\pi}\left(\frac{X\wedge\dot{v}}{(X\cdot v)^2} + \frac{X\wedge v - X\cdot\dot{v}\, X\wedge v}{(X\cdot v)^3}\right). \end{aligned} \tag{3.6}$$

The bracketed term is a pure bivector, so $\nabla\cdot A = 0$ and the A field of Eq. (3.2) is in the Lorentz gauge.

We can gain some insight into the expression for F by writing

$$X\cdot v\, X\wedge\dot{v} - X\cdot\dot{v}\, X\wedge v = -X\wedge[X\cdot(\dot{v}\wedge v)] = \tfrac{1}{2}X\dot{v}\wedge vX, \tag{3.7}$$

which uses the fact that $X^2 = 0$. Writing $\Omega_v = \dot{v}\wedge v$ for the acceleration bivector of the particle, we arrive at the compact formula

$$F = \frac{q}{4\pi}\frac{X\wedge v + \frac{1}{2}X\Omega_v X}{(X\cdot v)^3}. \tag{3.8}$$

This displays a clean split into a velocity term proportional to $1/(\text{distance})^2$ and a long-range radiation term proportional to $1/(\text{distance})$. (The distance here is $X\cdot v$. This is just the distance between the events x and $x_0(\tau)$ as measured in the rest frame of the charge at its retarded position.) The first term in Eq. (3.8) is exactly the Coulomb field in the rest frame of the charge, and the radiation term,

$$F_{\text{rad}} = \frac{q}{4\pi}\frac{\frac{1}{2}X\Omega_v X}{(X\cdot v)^3}, \tag{3.9}$$

is proportional to the rest-frame acceleration projected down the null-vector X. One can go on now to show that, away from the worldline, F satisfies the free-field equation $\nabla F = 0$. The details are left as a (voluntary) exercise.

3.3.2 Uniformly Moving Charge

A charge with constant velocity v has trajectory

$$x_0(\tau) = v\tau, \tag{3.10}$$

where we have chosen an origin so that the particle passes through this point at $\tau = 0$. The intersection of $x_0(\tau)$ with the past lightcone through x is determined by

$$(x - v\tau)^2 = 0 \quad \implies \tau = v\cdot x - [(v\cdot x)^2 - x^2]^{1/2}. \tag{3.11}$$

We have chosen the earlier root to ensure that the intersection is on the past lightcone. We now form $X\cdot v$ to find

$$X\cdot v = (x - v\tau)\cdot v = [(v\cdot x)^2 - x^2]^{1/2}, \tag{3.12}$$

which we can write as $|x\wedge v|$ since

$$|x\wedge v|^2 = x\cdot[v\cdot(x\wedge v)] = (x\cdot v)^2 - x^2. \tag{3.13}$$

The acceleration bivector vanishes since v is constant, and $X\wedge v = x\wedge v$. It follows that the Faraday bivector is simply

$$F = \frac{q}{4\pi}\frac{x\wedge v}{|x\wedge v|^3}. \tag{3.14}$$

The Faraday bivector decomposes in the γ_0 frame into electric and magnetic fields. Using

$$x\wedge v = \langle x\gamma_0\gamma_0 v\rangle_2 = \gamma\langle(t+\boldsymbol{x})(1-\boldsymbol{v})\rangle_2 = \gamma(\boldsymbol{x} - \boldsymbol{v}t) - \gamma\boldsymbol{x}\wedge\boldsymbol{v}, \tag{3.15}$$

where $\boldsymbol{v}$ is the relative velocity, we have

$$\boldsymbol{E} = \frac{q\gamma}{4\pi d^3}(\boldsymbol{x} - \boldsymbol{v}t) \tag{3.16}$$

$$\boldsymbol{B} = \frac{q\gamma}{4\pi d^3}I\boldsymbol{x}\wedge\boldsymbol{v}. \tag{3.17}$$

Here, the effective distance d is

$$d^2 = \gamma^2(|\boldsymbol{v}|t - \boldsymbol{v}\cdot\boldsymbol{x}/|\boldsymbol{v}|)^2 + \boldsymbol{x}^2 - (\boldsymbol{x}\cdot\boldsymbol{v})^2/\boldsymbol{v}^2. \tag{3.18}$$

Note that the electric field points towards the actual position of the charge at time t, and not its retarded position at time τ.

3.3.3 Circular Orbits — Non-Examinable

For a particle moving in a circle in the $\boldsymbol{I\sigma}_3$ plane,

$$x_0(\tau) = \mathrm{ch}(\alpha)\tau\gamma_0 + a[\cos(\omega\tau)\gamma_1 + \sin(\omega\tau)\gamma_2]. \tag{3.19}$$

The angular velocity ω is measured with respect to proper time τ. The speed of the particle is $|\boldsymbol{v}| = \tanh\alpha$, where $\mathrm{sh}(\alpha) = a\omega$. The velocity of the particle is

$$v = \mathrm{ch}(\alpha) + \mathrm{sh}(\alpha)[-\sin(\omega\tau)\gamma_1 + \cos(\omega\tau)\gamma_2], \tag{3.20}$$

and the acceleration bivector is

$$\Omega_v = -\omega\mathrm{sh}(\alpha)\{\mathrm{ch}(\alpha)[\cos(\omega\tau)\boldsymbol{\sigma}_1 + \sin(\omega\tau)\boldsymbol{\sigma}_2] - \mathrm{sh}(\alpha)\boldsymbol{I\sigma}_3\}. \tag{3.21}$$

The null condition $(x-x_0)^2 = 0$ gives

$$t = \tau\mathrm{ch}(\alpha) + \sqrt{\{\boldsymbol{x}^2 + a^2 - 2a[\mathrm{x}\cos(\omega\tau) + \mathrm{y}\sin(\omega\tau)]\}}, \tag{3.22}$$

where $\boldsymbol{x} = \mathrm{x}\boldsymbol{\sigma}_1 + \mathrm{y}\boldsymbol{\sigma}_2 + \mathrm{z}\boldsymbol{\sigma}_3$. Eq. (3.22) is an implicit equation for τ, and is simple to solve numerically. With τ determined, one can use the expressions for the retarded velocity and acceleration to plot field lines for various values of the angular velocity. These are shown in Figures 3.2 and 3.3 in the $\boldsymbol{I\sigma}_3$ plane. They display many interesting features, some of which are described in the figure captions.

3.4 Field Momentum and the Stress-Energy Tensor

The energy density contained in an electromagnetic field is

$$\mathcal{E} = \tfrac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2), \tag{3.1}$$

and the momentum density is just the Poynting vector

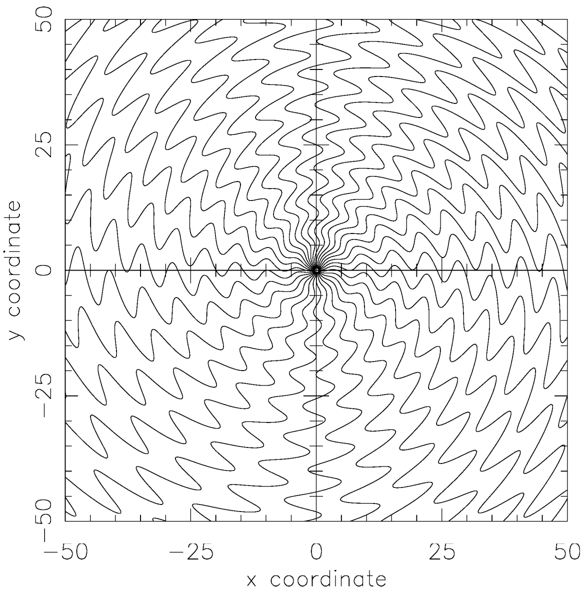
$$\boldsymbol{P} = \boldsymbol{E}\times\boldsymbol{B} = -\boldsymbol{E}\cdot(I\boldsymbol{B}). \tag{3.2}$$

For free fields ($J = 0$), the integrals of these quantities over space ought to be the components of a spacetime 4-vector P_{tot}. This suggests forming the density

$$\begin{aligned}(\mathcal{E} + \boldsymbol{P})\gamma_0 &= \tfrac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2)\gamma_0 + \tfrac{1}{2}(I\boldsymbol{B}\boldsymbol{E} - \boldsymbol{E}I\boldsymbol{B})\gamma_0 \\ &= \tfrac{1}{2}(\boldsymbol{E} + I\boldsymbol{B})(\boldsymbol{E} - I\boldsymbol{B})\gamma_0 \\ &= \tfrac{1}{2}F(-\gamma_0 F\gamma_0)\gamma_0 = -\tfrac{1}{2}F\gamma_0 F. \end{aligned} \tag{3.3}$$

Integrating over space we have

$$P_{\mathrm{tot}} = -\frac{1}{2}\int |d^3x|\, F\gamma_0 F, \tag{3.4}$$



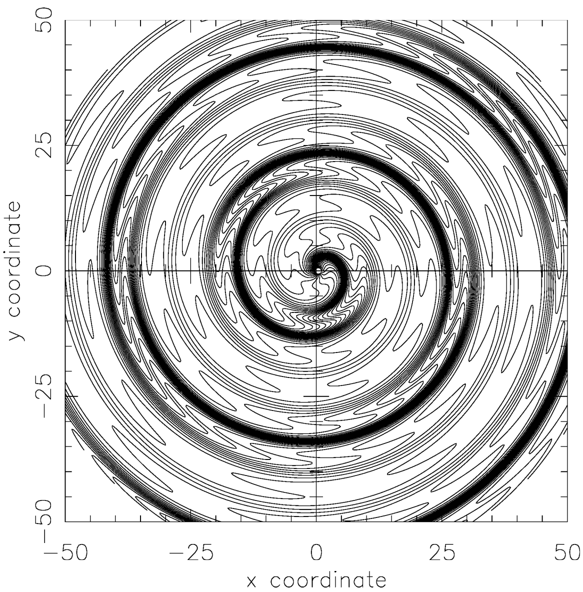


Figure 3.2: *Field lines of a rotating charge.* The top diagram has $\alpha = 0.1$ and the particle velocity $\tanh(\alpha)$ is low. A gentle wavy pattern of field lines is produced, characteristic of electromagnetic waves. At intermediate velocities (bottom diagram, with $\alpha = 0.4$) a complicated structure emerges as the field lines start to concentrate together.

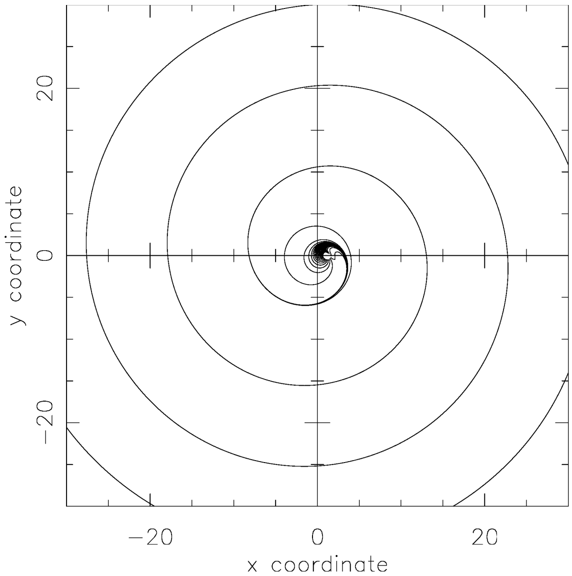


Figure 3.3: *Synchrotron Radiation.* By $\alpha = 1$ the field lines concentrate into pure synchrotron pulses as the radiation is focussed into the direction of motion of the particle.

which we can also write as

$$P_{\text{tot}} = -\frac{1}{2}\int dA\, FnF, \tag{3.5}$$

where the integral is over the spacelike hypersurface $t =$ constant. The scalar measure $dA = |d^3x|$ and $n = \gamma_0$ is the normal vector to the surface. The total 4-momentum P_{tot} is independent of the hypersurface over which the integral in Eq. (3.5) is performed provided the fields fall off sufficiently rapidly at spatial infinity: P_{tot} is a covariant (non-local) property of the field configuration. To see this, consider comparing P_{tot} computed on two spacelike hypersurfaces. For localised fields, we can join these hypersurfaces at infinity with timelike hypersurfaces and extend the domain of integration over the entire closed surface, ∂_V. The difference in P_{tot} now takes the form

$$\Delta P_{\text{tot}} = -\frac{1}{2}\int_{\partial V} dA\, FnF. \tag{3.6}$$

For each component of ΔP_{tot} we have

$$\gamma_\mu\cdot(\Delta P_{\text{tot}}) = -\frac{1}{2}\int_{\partial V} dA\,\langle\gamma_\mu FnF\rangle = -\frac{1}{2}\int_{\partial V} dA\, n\cdot(F\gamma_\mu F). \tag{3.7}$$

The divergence theorem enables us to convert the surface integral to an integral over the enclosed volume V to get

$$\gamma_\mu\cdot(\Delta P_{\text{tot}}) = -\frac{1}{2}\int_V |d^4x|\,\nabla\cdot(F\gamma_\mu F). \tag{3.8}$$

To complete the proof we use

$$\nabla\cdot(F\gamma_\mu F) = \langle \nabla F\gamma_\mu F\rangle - \langle F\gamma_\mu(\nabla F)^{\sim}\rangle \tag{3.9}$$

and since $\nabla F = 0$ in vacuum, we have $\Delta P_{\text{tot}} = 0$. This establishes the result that P_{tot} is independent of the hypersurface.

The construction $-\frac{1}{2}FaF$ is the *stress-energy tensor* of the electromagnetic field. We write this as

$$\mathsf{T}(a) = -\tfrac{1}{2}FaF. \tag{3.10}$$

The stress-energy tensor $\mathsf{T}(a)$ returns the flux of 4-momentum across the hypersurface perpendicular to a. We continue to use a component-free formulation of linear functions, preferring to encode what happens directly to the input vector a. $\mathsf{T}(a)$ is the relativistic extension of the stress tensor, and it is as fundamental to fields as momentum is to point particles. It is instructive to contrast the neat STA form of Eq. (3.10) with the tensor formula

$$\mathsf{T}^{\mu}{}_{\nu} = \tfrac{1}{4}\delta^{\mu}_{\nu}F^{\alpha\beta}F_{\alpha\beta} + F^{\mu\alpha}F_{\alpha\nu}. \tag{3.11}$$

There is little doubt which form best captures the geometric content of the tensor!

All relativistic fields, classical or quantum, have a stress-energy tensor which contains information about the distribution of energy in the fields (and acts as a source of gravity). We can illustrate some general properties of these using electromagnetism as an example. The first property is that the stress-energy tensor is (usually) symmetric. For example, we have

$$a\cdot\mathsf{T}(b) = -\tfrac{1}{2}\langle aFbF\rangle = -\tfrac{1}{2}\langle FaFb\rangle = \mathsf{T}(a)\cdot b. \tag{3.12}$$

The stress-energy tensor can have a non-symmetric contribution in the presence of quantum spin.

The second property is that the energy density $v\cdot\mathsf{T}(v)$ is positive for any timelike vector v. Matter which does not satisfy this property is said to be 'exotic'. The third main property of stress-energy tensors is that they give rise to conserved currents:

$$\nabla\cdot\mathsf{T}(\gamma_\mu) = 0 \qquad \mu = 0\ldots 3. \tag{3.13}$$

We have already proved this for the electromagnetic case.

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Chapter 4

The Dirac Equation

The relativistic wave equation for a spin-1/2 particle is the *Dirac equation*. This is a first order wave equation, which is necessary to achieve an equation which is Lorentz invariant and which has a future-pointing conserved current. The Dirac matrices which appear in the wave equation constitute a matrix representation of the basis vectors of 4-d spacetime. Since the algebra of the Dirac matrices is isomorphic to the spacetime algebra, it is no surprise that Dirac theory finds a natural expression in geometric algebra. The theory of the Dirac equation is a large subject and we will only touch briefly on a few of its properties here.

4.1 Relativistic Quantum Spin

The relativistic quantum mechanics of a spin-1/2 particle is described by the *Dirac theory*. The Dirac matrix operators are

$$\hat{\gamma}_0 = \begin{pmatrix} \mathsf{I} & 0 \\ 0 & -\mathsf{I} \end{pmatrix}, \quad \hat{\gamma}_k = \begin{pmatrix} 0 & -\hat{\sigma}_k \\ \hat{\sigma}_k & 0 \end{pmatrix} \quad \text{and} \quad \hat{\gamma}_5 = \begin{pmatrix} 0 & \mathsf{I} \\ \mathsf{I} & 0 \end{pmatrix}, \tag{4.1}$$

where $\hat{\gamma}_5 = -i\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$ and I is the 2×2 identity matrix. These matrices act on Dirac spinor fields, which have 4 complex components (8 real degrees of freedom) at a point. We follow an analogous procedure to the Pauli case covered in Chapter 4 of Handout 1, and map these spinors onto elements of the 8-dimensional even subalgebra of the STA. Dirac spinors can be visualised as decomposing into 'upper' and 'lower' components,

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix}, \tag{4.2}$$

where $|\phi\rangle$ and $|\eta\rangle$ are a pair of 2-component spinors. We already know how to represent these as multivectors ϕ and η, which lie in the space of scalars + relative bivectors.

Our map from the Dirac spinor onto an element of the full 8-dimensional subalgebra is simply

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix} \leftrightarrow \psi = \phi + \eta\boldsymbol{\sigma}_3. \tag{4.3}$$

The action of the Dirac matrix operators now becomes,

$$\begin{aligned} \hat{\gamma}_\mu|\psi\rangle &\leftrightarrow \gamma_\mu\psi\gamma_0 \quad (\mu = 0,\ldots 3) \\ i|\psi\rangle &\leftrightarrow \psi\, \boldsymbol{I\sigma}_3 \\ \hat{\gamma}_5|\psi\rangle &\leftrightarrow \psi\boldsymbol{\sigma}_3. \end{aligned}$$

Verifying that this map is consistent is a matter of routine computation. One thing to note is that we now have two 'reference' vectors that can appear on the right-hand side of ψ: γ_0 and γ_3. That is, the relative vector $\boldsymbol{\sigma}_3$ used in the Pauli theory has been decomposed into a spacelike and timelike direction. Since $\boldsymbol{I\sigma}_3$ and γ_0 commute,

$$i\hat{\gamma}_\mu|\psi\rangle \leftrightarrow \gamma_\mu\psi\gamma_0\boldsymbol{I\sigma}_3 = \gamma_\mu\psi\boldsymbol{I\sigma}_3\gamma_0 \leftrightarrow \hat{\gamma}_\mu i|\psi\rangle, \tag{4.4}$$

and our use of right multiplication by $\boldsymbol{I\sigma}_3$ for the complex structure remains consistent.

4.2 The Dirac Equation

The Dirac equation is the quantum mechanical wave equation for spin-1/2 particles. We need to construct a relativistic wave equation for the spinor field $\psi(x)$, where ψ is an element of the 8-dimensional even subalgebra of the STA. To write down a covariant equation, the only objects we can place on the left-hand side of ψ are scalars, pseudoscalars, and the vector derivative. The simplest equation we could write down is therefore

$$\nabla\psi = 0. \tag{4.1}$$

Remarkably, this equation does describe the behaviour of fermions — it is the wave equation for a *neutrino*. Any solution to this decomposes into two separate solutions by writing

$$\psi = \psi\tfrac{1}{2}(1+\boldsymbol{\sigma}_3) + \psi\tfrac{1}{2}(1-\boldsymbol{\sigma}_3) = \psi_+ + \psi_-, \tag{4.2}$$

since

$$\nabla\psi = 0 \implies \nabla\psi_\pm = 0. \tag{4.3}$$

The separate solutions ψ_+ and ψ_- are the right-handed and left-handed helicity eigenstates. For neutrinos, nature only appears to make use of the left-handed solutions. A more complete treatment of this subject involves the *electroweak* theory.

4.2.1 Aside — The Cauchy Riemann Equations

We have seen the importance of the vector derivative in space and spacetime. It is instructive to see the form of this derivative in two dimensions. With $\boldsymbol{e}_1, \boldsymbol{e}_2$ a pair of orthonormal vectors we have

$$\boldsymbol{\nabla} = \boldsymbol{e}_1 \partial_x + \boldsymbol{e}_2 \partial_y. \tag{4.4}$$

If this acts on the 'complex' field $\psi = u + \boldsymbol{e}_1\boldsymbol{e}_2 v$ the result is

$$\boldsymbol{\nabla}\psi = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\boldsymbol{e}_1 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\boldsymbol{e}_2. \tag{4.5}$$

The terms here are those that appear in the *Cauchy-Riemann* equations. It follows that in two dimensions the geometric algebra equation

$$\boldsymbol{\nabla}\psi = 0 \tag{4.6}$$

is equivalent to demanding that ψ satisfies the Cauchy-Riemann equations. But the theory of complex analytic functions is restricted to two dimensions, whereas we can write down an equation like (4.6) in any space. It is this equation, therefore, which generalises the concept of analytic functions to higher dimensions. Remarkably, this apparently rather mathematical reasoning leads us directly to the spacetime Dirac equation for a massless fermion!

4.2.2 The Massive Dirac Equation

The formal operator identification of $i\partial_\mu$ with p_μ tells us that any wavefunction for a free massive particle should satisfy $\nabla^2\psi = -m^2\psi$. We therefore need to add a term to the right-hand side of Eq. (4.1) which is linear in the particle mass m and which generates $-m^2\psi$ on squaring the operator. If we think about plane-wave states with momentum p, we arrive at an equation of the form

$$p\psi = m\psi a_0 \tag{4.7}$$

where a_0 is some multivector to be determined. It is immediately clear that a_0 must have odd grade, and must square to $+1$. The obvious candidate is γ_0, so that ψ contains a rotor to transform γ_0 to the velocity p/m. We are therefore led to the equation

$$\nabla\psi \boldsymbol{I}\boldsymbol{\sigma}_3 = m\psi\gamma_0 \tag{4.8}$$

or, post-multiplying by $\boldsymbol{I}\boldsymbol{\sigma}_3$,

$$\nabla\psi = -m\psi I\gamma_3. \tag{4.9}$$

This is the *Dirac equation* in its STA form. The more common matrix/spinor form,

$$(i\hat{\gamma}_\mu\partial_\mu + m)|\psi\rangle = 0, \tag{4.10}$$

is recovered by converting ψ back to a column spinor, and writing

$$\nabla\psi\gamma_0 \leftrightarrow \hat{\gamma}_\mu\partial_\mu|\psi\rangle \tag{4.11}$$

for the vector derivative.

4.2.3 Dirac Observables

Our simple reasoning has led us to a first-order wave equation for the spinor wavefunction ψ. To construct a probabilistic interpretation for the theory we must find a conserved current, which is future-pointing and timelike. We follow a similar procedure to that in non-relativistic quantum mechanics. Multiplying the Dirac equation (4.9) by $\gamma_0\tilde{\psi}$ on the right gives

$$(\nabla\psi)\gamma_0\tilde{\psi} = -m\psi \boldsymbol{I\sigma}_3\tilde{\psi}. \tag{4.12}$$

The object on the right is reverse antisymmetric, so we can eliminate it by adding Eq. (4.12) to its reverse:

$$(\nabla\psi)\gamma_0\tilde{\psi} + \psi\gamma_0(\nabla\psi)^\sim = 0. \tag{4.13}$$

Finally, we take the scalar part to find

$$\begin{aligned} 0 &= \langle \nabla\psi\gamma_0\tilde{\psi} + \psi\gamma_0(\nabla\psi)^\sim\rangle \\ &= \langle \nabla(\psi\gamma_0\tilde{\psi})\rangle \\ &= \nabla\cdot\langle\psi\gamma_0\tilde{\psi}\rangle_1. \end{aligned} \tag{4.14}$$

The object $\psi\gamma_0\tilde{\psi}$ is odd grade and reverse symmetric, and so can only have a vector part. It is the *Dirac current* $J = \psi\gamma_0\tilde{\psi}$, which we have shown to be conserved: $\nabla\cdot J = 0$.

To establish the nature of the current J, we first form the quantity $\psi\tilde{\psi}$ which is even and reverse symmetric. As it can only contain scalar and pseudoscalar parts, we write

$$\psi\tilde{\psi} = \rho\, \mathrm{e}^{I\beta}\,, \tag{4.15}$$

with $\rho \geq 0$ and β scalars. For $\rho \neq 0$ (which excludes the massless helicity eigenstates $\psi_\pm$) we can define a spacetime rotor R by

$$R = \psi\rho^{-1/2}\,\mathrm{e}^{-I\beta/2}\,, \qquad R\tilde{R} = 1. \tag{4.16}$$

We have now decomposed the spinor ψ into

$$\psi = \rho^{1/2}\, \mathrm{e}^{I\beta/2}\, R \tag{4.17}$$

which separates out a density ρ and the rotor R. The remaining factor of β is curious. It turns out that plane-wave particle states have $\beta = 0$, whereas antiparticle states have $\beta = \pi$ (see Section 4.2.4). The picture for bound state wavefunctions is more complicated, however.

With this decomposition of ψ, the current becomes

$$J = \psi\gamma_0\tilde{\psi} = \rho\, \mathrm{e}^{I\beta/2}\, R\gamma_0\tilde{R}\, \mathrm{e}^{I\beta/2} \;= \rho R\gamma_0\tilde{R}. \tag{4.18}$$

So the rotor is now an instruction to rotate γ_0 onto the direction of the current, and the density ρ dilates $R\gamma_0\tilde{R}$ to return the current. The decomposition $J = \rho R\gamma_0\tilde{R}$ confirms that J is indeed timelike and future-pointing.

The time component of J in the γ_0 frame, say, is

$$J_0 = \gamma_0 \cdot J = \langle \gamma_0\tilde{\psi}\gamma_0\psi\rangle => 0 \tag{4.19}$$

which is *positive definite* (see Eq. (1.13)). This is interpreted as a probability density for locating the electron, and localised wave functions are usually normalised such that

$$\int |d^3x|\, J_0 = 1. \tag{4.20}$$

The normalisation condition is preserved in time since the current J is conserved. Arriving at a relativistic theory with a consistent probabilistic interpretation was Dirac's original goal.

Another important observable in the Dirac theory is the spin vector $s = \rho R\gamma_3\tilde{R}$. This vector is orthogonal to the current J, and so may be interpreted as representing the intrinsic spin of the particle. This form for the spin vector of a particle with gyromagnetic ratio $g = 2$ was also suggested by the classical model of spin given in Section 2.3.3 in connection with the Lorentz force law.

4.2.4 Plane-Wave States

A positive energy plane-wave state is defined by

$$\psi = \psi_0\, \mathrm{e}^{-\boldsymbol{I\sigma}_3 p\cdot x} \tag{4.21}$$

where ψ_0 is a constant spinor. The Dirac equation (4.8) tells us that ψ_0 satisfies

$$p\psi_0 = m\psi_0\gamma_0, \tag{4.22}$$

and post-multiplying by $\tilde{\psi}_0$ we see that

$$p\psi_0\tilde{\psi}_0 = mJ. \tag{4.23}$$

Writing $\psi_0\tilde{\psi}_0 = \rho\,\mathrm{e}^{i\beta}$, and noting that both p and J are vectors, we see that we must have $\exp(i\beta) = \pm 1$. For positive energy states the timelike component of p is positive, as is the timelike component of J, so we take the positive solution $\beta = 0$. It follows that ψ_0 is then simply a rotor with a normalisation constant. Splitting the rotor into a pure boost L and a pure rotation U in the γ_0 frame (see Section 2.1.4),

$$R = LU, \tag{4.24}$$

we see that the boost L taking $m\gamma_0$ onto the momentum has

$$p = mL\gamma_0\tilde{L} = mL^2\gamma_0, \tag{4.25}$$

which we know from Eq. (1.18) of Handout 10 is solved by

$$L = \frac{m + p\gamma_0}{[2m(m + p\cdot\gamma_0)]^{1/2}} = \frac{E + m + \boldsymbol{p}}{[2m(E+m)]^{1/2}}, \tag{4.26}$$

where we have employed the spacetime split $p\gamma_0 = E + \boldsymbol{p}$.

Negative energy solutions have a phase factor of $\mathrm{e}^{+I\boldsymbol{\sigma}_3 p\cdot x}$, with $E = \gamma_0\cdot p > 0$. For these we have $-p\psi\tilde{\psi} = mJ$ so it is clear that we now need $\beta = \pi$. Positive and negative energy plane wave states can therefore be summarised by

$$\begin{array}{rl} \text{positive energy} & \psi^{(+)}(x) = L(p)U\,\mathrm{e}^{-I\boldsymbol{\sigma}_3 p\cdot x} \\ \text{negative energy} & \psi^{(-)}(x) = L(p)UI\,\mathrm{e}^{I\boldsymbol{\sigma}_3 p\cdot x} \end{array} \tag{4.27}$$

with $L(p)$ given by Eq. (4.26). These are fundamental components in *scattering theory.*

4.2.5 The Hydrogen atom

To convert the Dirac equation $\nabla\psi I\boldsymbol{\sigma}_3 = m\psi\gamma_0$ into Hamiltonian form we pre-multiply by γ_0, which produces (in natural units)

$$i\partial_t\psi = -\boldsymbol{\nabla}\psi I\boldsymbol{\sigma}_3 + m\gamma_0\psi\gamma_0 = -i\boldsymbol{\nabla}\psi + m\bar{\psi}. \tag{4.28}$$

Here i is a convenient abbreviation for multiplication on the right by $I\boldsymbol{\sigma}_3$, and the bar operation is defined by

$$\bar{\psi} = \gamma_0\psi\gamma_0. \tag{4.29}$$

This flips the sign of the relative vector and pseudoscalar terms in ψ. The right-hand side of equation (4.28) is the free-space Dirac Hamiltonian. In the presence of an attractive Coulomb potential we must add a term going as

$$eV(r) = -\frac{Ze^2}{4\pi\epsilon_0 r} = -\frac{Z\alpha}{r} \tag{4.30}$$

where $\alpha \approx 1/137$ is the dimensionless fine structure constant and Z is the atomic charge. If we look for stationary states of energy E, the equation we need to solve becomes

$$E\psi = -i\boldsymbol{\nabla}\psi - \frac{Z\alpha}{r}\psi + m\bar{\psi}. \tag{4.31}$$

To solve this equation we start by assuming that the ground state wavefunction is spherically-symmetric. We should therefore be able to build the wavefunction out of combinations of real and imaginary combinations of scalars and the position vector $\boldsymbol{x}$. At this point it is convenient to introduce a standard set of polar coordinates (r, θ, ϕ), and from these we define the unit polar vectors

$$\begin{aligned}
\boldsymbol{\sigma}_r &= \sin\theta(\cos\phi\,\boldsymbol{\sigma}_1 + \sin\phi\,\boldsymbol{\sigma}_2) + \cos\theta\,\boldsymbol{\sigma}_3 \\
\boldsymbol{\sigma}_\theta &= \cos\theta(\cos\phi\,\boldsymbol{\sigma}_1 + \sin\phi\,\boldsymbol{\sigma}_2) - \sin\theta\,\boldsymbol{\sigma}_3 \\
\boldsymbol{\sigma}_\phi &= -\sin\phi\,\boldsymbol{\sigma}_1 + \cos\phi\,\boldsymbol{\sigma}_2.
\end{aligned} \tag{4.32}$$

For our candidate spinor we take

$$\psi = u(r) + \boldsymbol{\sigma}_r v(r) I\boldsymbol{\sigma}_3 \tag{4.33}$$

where u and v are "complex", so contain scalar and $I\boldsymbol{\sigma}_3$ terms only.

For the action of $\boldsymbol{\nabla}$ we need

$$\boldsymbol{\nabla} = \boldsymbol{\sigma}_r\frac{\partial}{\partial r} + \frac{1}{r}\boldsymbol{\sigma}_\theta\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\boldsymbol{\sigma}_\phi\frac{\partial}{\partial\phi}. \tag{4.34}$$

It follows that

$$\boldsymbol{\nabla}\boldsymbol{\sigma}_r = \frac{1}{r}\boldsymbol{\sigma}_\theta^2 + \frac{1}{r\sin\theta}\boldsymbol{\sigma}_\phi\sin\theta\boldsymbol{\sigma}_\phi = \frac{2}{r}. \tag{4.35}$$

The Dirac equation for our candidate spinor therefore reduces to

$$\begin{aligned}
E(u + \boldsymbol{\sigma}_r v I\boldsymbol{\sigma}_3) = &-\boldsymbol{\sigma}_r(u' + \boldsymbol{\sigma}_r v' I\boldsymbol{\sigma}_3)I\boldsymbol{\sigma}_3 - \frac{2}{r}vI\boldsymbol{\sigma}_3 I\boldsymbol{\sigma}_3 \\
&- \frac{Z\alpha}{r}(u + \boldsymbol{\sigma}_r v I\boldsymbol{\sigma}_3) + m(u - \boldsymbol{\sigma}_r v I\boldsymbol{\sigma}_3)
\end{aligned} \tag{4.36}$$

Equating the scalar and $I\boldsymbol{\sigma}_3$ parts we obtain

$$Eu = v' + \frac{2}{r}v - \frac{Z\alpha}{r}u + mu \tag{4.37}$$

and the $\boldsymbol{\sigma}_r$ terms give

$$Ev = -u' - \frac{Z\alpha}{r}v - mv \tag{4.38}$$

This results in a pair of coupled complex equations for the variables u and v. All of the coefficients are real, however, so both the real and imaginary parts of u and v satisfy the same equations. As we have the freedom to set an overall, constant phase, we can use this to set u (and hence v) to be real.

To solve our pair of coupled real equations, we employ the familiar technique of separating out the large and small r behaviour. First at large r we have

$$v' \approx (E-m)u, \qquad u' \approx -(E+m)v. \tag{4.39}$$

It follows that, for example,

$$v'' \approx (m^2 - E^2)v. \tag{4.40}$$

Since we are looking for bound states we must have $E < m$, so that the solutions to this have exponentially growing and decaying modes. Physically we must restrict to the decaying mode, so, with δ given by

$$\delta = \sqrt{m^2 - E^2} \tag{4.41}$$

the large r dependence of u and v will go as $\mathrm{e}^{-\delta r}$.

For the small r behaviour the equations reduce to

$$ru' \approx -Z\alpha v, \qquad rv' \approx Z\alpha u - 2v. \tag{4.42}$$

These can be solved by setting $u = u_0 r^\beta$ and $v = v_0 r^\beta$. The equations then reduce to

$$\beta u_0 = -Z\alpha v_0, \qquad \beta v_0 = Z\alpha u_0 - 2v_0, \tag{4.43}$$

and in matrix form we have

$$\begin{pmatrix} \beta & Z\alpha \\ -Z\alpha & \beta+2 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.44}$$

This equation can only have non-zero solutions if the matrix on the left has zero determinant. This implies that

$$\beta^2 + 2\beta + (Z\alpha)^2 = 0 \tag{4.45}$$

hence

$$\beta = -1 \pm \sqrt{1-(Z\alpha)^2}. \tag{4.46}$$

To choose the appropriate sign we need to consider the density at the origin. The Dirac current is given by

$$\psi\gamma_0\tilde{\psi} = (u + \boldsymbol{\sigma}_r v I\boldsymbol{\sigma}_3)\gamma_0(u + I\boldsymbol{\sigma}_3 v\boldsymbol{\sigma}_r) = (u^2+v^2)\gamma_0 + 2uv\sin\theta\boldsymbol{\sigma}_\phi\gamma_0. \tag{4.47}$$

so the density in the γ_0-frame is $u^2 + v^2$. The integral of this over all space must be finite. Near the origin we have

$$\int d^3x J\cdot\gamma_0 \sim 4\pi \int r^2 r^{2\beta} dr, \tag{4.48}$$

so we require that $2\beta > -3$. For small Z this forces us to chose the positive root and set

$$\beta = \beta_+ = -1 + \sqrt{1 - (Z\alpha)^2}. \tag{4.49}$$

For $Z > 118$ something unexpected happens: the negative branch becomes permissible. This has some potentially devastating consequences for the nature of the bound state spectrum. In practice, replacing the point source of charge with a more realistic, smeared out model of the nucleus pushes the problem up to higher Z values. As we shall shortly see, these higher values hide a second problem.

So far, our ground state solution has the general form

$$u = U(r) r^{\beta_+} \mathrm{e}^{-\delta r}, \qquad v = V(r) r^{\beta_+} \mathrm{e}^{-\delta r}. \tag{4.50}$$

On substituting this into our original equation, we find that we can solve the equations completely with U and V equal to the constants u_0 and v_0. Since this represents the simplest solution, with a single peak in the density, we expect this to be the ground state. The equations also fix the energy by imposing an additional algebraic constraint. This comes from the fact that our two asymptotic equations impose two constraints on the ratio of u_0 and v_0. Using the first of equations (4.43), which set constraints at the origin, we have

$$v_0 = -\frac{\beta_+}{Z\alpha} u_0. \tag{4.51}$$

Out at infinity, on the other hand, equation (4.38) gives

$$\delta u_0 = (E + m) v_0. \tag{4.52}$$

Satisfying both constraints simultaneously requires

$$\frac{E + m}{\delta} = -\frac{Z\alpha}{\beta_+} = \frac{Z\alpha}{1 - \sqrt{1 - (Z\alpha)^2}}. \tag{4.53}$$

The left-hand side can be written

$$\frac{E + m}{\delta} = \frac{m^2 - E^2}{(m - E)\delta} = \frac{\delta/m}{1 - E/m} = \frac{\delta/m}{1 - \sqrt{1 - (\delta/m)^2}}, \tag{4.54}$$

so equating both sides we must have $\delta = mZ\alpha$. It follows that

$$E = m\sqrt{1 - (Z\alpha)^2} \tag{4.55}$$

which provides a neat, simple expression for the ground state energy. Removing the rest mass contribution the binding energy is

$$E' = m(\sqrt{1 - (Z\alpha)^2} - 1) \approx -\tfrac{1}{2} m (Z\alpha)^2, \tag{4.56}$$

where we have assumed that $Z\alpha$ is a lot less than 1. Re-inserting the dimensional constants recovers the familiar expression for the non-relativistic ground state energy.

The result for the ground state energy is the lowest energy of the general formula

$$E^2 = m^2\Big(1 - \frac{(Z\alpha)^2}{n^2 + 2n\nu + \kappa^2}\Big). \tag{4.57}$$

Here n is a non-negative integer, $\kappa = l + 1$ is a positive integer, and $\nu = [(l+1)^2 - (Z\alpha)^2]^{1/2}$. Since $\alpha \approx 1/137$ is small, for low Z we can approximate to give

$$E \approx m\left[1 - \frac{(Z\alpha)^2}{2}\frac{1}{n^2 + 2n(l+1) + (l+1)^2}\right] \tag{4.58}$$

Subtracting off the rest-mass contribution to the energy we recover non-relativistic formula

$$E_{NR} = -m\frac{(Z\alpha)^2}{2}\frac{1}{(n+l+1)^2} = -\frac{mZ^2e^4}{32\pi^2\epsilon_0^2\hbar^2}\frac{1}{n'^2} \tag{4.59}$$

where $n' = n + l + 1$ is the familiar principal quantum number. Expanding to next order we find that

$$E_{NR} = -m\frac{(Z\alpha)^2}{2n'^2} - m\frac{(Z\alpha)^4}{2n'^4}\left(\frac{n'}{l+1} - \frac{3}{4}\right). \tag{4.60}$$

The first relativistic correction shows that the binding energy is increased slightly from the non-relativistic value, and also introduces some dependence on the angular quantum number l. This lifts some degeneracies present in the non-relativistic solution. The various corrections contributing to the energy levels are shown in Fig. 4.1. A more complete analysis also requires replacing the electron mass m by the reduced mass of the two-body system. This introduces corrections of the same order of the relativistic corrections, but only affects the overall scale.

The case of large Z is interesting. As Z approaches 137 we see that the energy becomes undefined. Another way to see that $Z > 137$ is unphysical is from the current in equation (4.47). The current defines a relative velocity of

$$\boldsymbol{v} = \frac{2uv}{u^2 + v^2}\sin\theta\boldsymbol{\sigma}_\phi = Z\alpha\sin\theta\boldsymbol{\sigma}_\phi. \tag{4.61}$$

If $Z\alpha > 1$ the velocity in the $\theta = \pi/2$ plane is greater than the speed of light, which is also unphysical. In practice, nuclei with $Z > 137$ are unstable and the large electromagnetic fields at the nucleus are sufficient to generate pair production. This has been observed in high-energy scattering experiments.

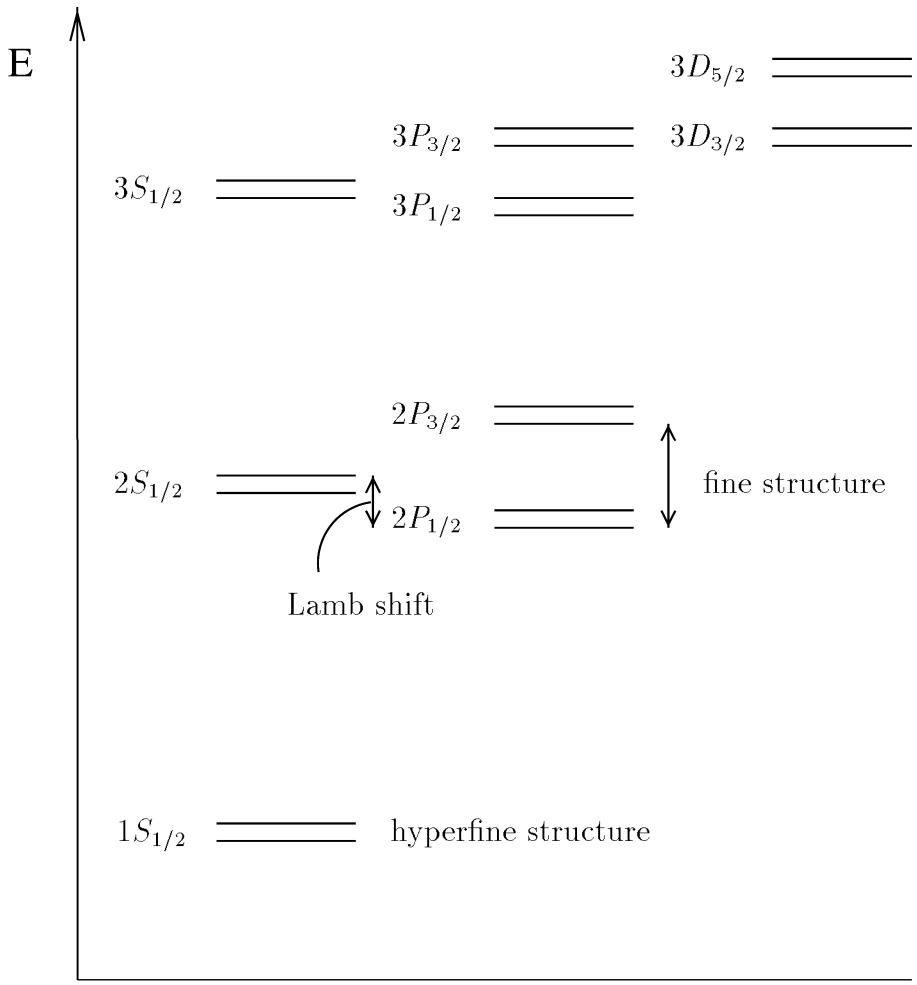

Figure 4.1: *Hydrogen atom energy levels.* The diagram illustrates how various degeneracies are broken by relativistic and spin effects. The Dirac equation accounts for the fine structure. The hyperfine structure is due to interaction with the magnetic moment of the nucleus. The Lamb shift is explained by quantum field theory. It lifts the degeneracy between $S_{1/2}$ and $P_{1/2}$ states.