Physical Applications of Geometric Algebra

Part III
Gauge Theories

Chris Doran and Anthony Lasenby

www.mrao.cam.ac.uk/∼clifford/ptIIIcourse

Chapter 1

Electromagnetism as a Gauge Theory

The fundamental forces of nature can all be described in terms of *gauge theories*. In the early part of the 20th century physicists noticed that electromagnetic interactions arise from demanding invariance of quantum wave equations under local changes of phase. There the position remained until the fifties, when Yang and Mills showed how to construct theories based on more complicated, non-commuting groups. This is the basis for the *standard model* of the electroweak and strong interactions. In this section we analyse how electromagnetism arises as a gauge theory in the context of Dirac theory.

1.1 Phase Invariance

Consider the free-particle Dirac equation,

$$\nabla \psi I \boldsymbol{\sigma}_3 = m \psi \gamma_0. \tag{1.1}$$

Since $I\boldsymbol{\sigma}_3$ commutes with γ_0, a *global symmetry* of this equation is the transformation

$$\psi \mapsto \psi' = \psi \, \mathrm{e}^{I\boldsymbol{\sigma}_3 \theta}, \tag{1.2}$$

where θ is a constant. This is a symmetry because if Eq. (1.1) holds for ψ, it also holds for ψ'. The symmetry is 'global' because θ has the same value everywhere in space and time. The quantity $\exp(I\boldsymbol{\sigma}_3\theta)$ is the STA version of a phase factor. It can also be viewed as a rotor, corresponding to rotations in the $\gamma_2\gamma_1$ plane through angle 2θ. To exhibit the relationship between electromagnetism and other gauge theories we write this phase change as the rotor R, so

$$\psi' = \psi R = \psi \, \mathrm{e}^{I\boldsymbol{\sigma}_3 \theta}. \tag{1.3}$$

Suppose now that θ is not a constant, but depends on spacetime position x, $\theta = \theta(x)$. In this case ψ' will no longer be a solution of the equation if ψ is, since

$$\nabla\psi' = (\nabla\psi)R + (\nabla\theta)\psi R I\boldsymbol{\sigma}_3 \tag{1.4}$$

and so $\nabla\psi' \neq m\psi'\gamma_0$. Hence Eq. (1.2) is not a *local* symmetry of equation (1.1) as θ cannot be varied arbitrarily from point to point. So why do we want Eq. (1.2) to work as a local symmetry? The answer lies in the structure of the physical statements that can be extracted from the Dirac theory. There are two main types:

1. The values of *observables*, such as the current J or spin s. These are bilinear in ψ, such as $J = \psi\gamma_0\tilde{\psi}$, and so are invariant under phase changes.
2. Statements about the equality of two spinor expressions, for example

$$\psi = \psi_1 + \psi_2. \tag{1.5}$$

   This might decompose ψ into two orthogonal eigenstates of some operator.

In both cases, if all spinors pick up the same locally-varying phase factor (rotor) then the physical predictions are unchanged.

1.2 Covariant Derivatives

Now that we have understood the motivation, we must find how to modify Eq. (1.1) in order that phase changes become a local symmetry. We first return to the component form of ∇ as

$$\nabla = \gamma^\mu \partial_\mu. \tag{1.6}$$

This separates out the vector and derivative characteristics of ∇. (A coordinate-free development of the ideas of this section does exist, but will not be used here.) The equation for ψ' now includes the term

$$\nabla\psi' = \gamma^\mu \left(\partial_\mu\psi R + \psi\partial_\mu R\right). \tag{1.7}$$

We clearly need to modify the ∇ operator to be able to cancel out the term in the derivative of R. If ψ satisfies the original equation, we find that ψ' satisfies

$$\nabla\psi' I\boldsymbol{\sigma}_3 - \gamma^\mu\psi'(\tilde{R}\partial_\mu R)I\boldsymbol{\sigma}_3 = m\psi'\gamma_0. \tag{1.8}$$

The term $\tilde{R}\partial_\mu R$ is a *bivector*, as we know from all our work on rotor equations. To remove this term we must *add* a new, bivector-valued term to the partial derivative.

This new field is called a *connection*, and is usually written Ω_μ. We add this term to the partial derivative operator to define a *covariant derivative* D_μ, where

$$D_\mu \psi = \partial_\mu \psi + \tfrac{1}{2}\psi\Omega_\mu. \tag{1.9}$$

(the factor 1/2 is inserted for later convenience). Our modified Dirac equation now reads

$$\gamma^\mu D_\mu \psi \boldsymbol{I\sigma}_3 = m\psi\gamma_0, \tag{1.10}$$

and this is the equation we need to ensure has the desired local symmetry properties.

The behaviour we require of Eq. (1.10) is that under a local rotation, the pair ψ and Ω_μ should transform in such a way as to generate a new solution from the original solution. With $\psi' = \psi R$ the transformed equation must read

$$\begin{aligned}\gamma^\mu D'_\mu \psi' \boldsymbol{I\sigma}_3 &= \gamma^\mu(\partial_\mu \psi' + \tfrac{1}{2}\psi'\Omega'_\mu)\boldsymbol{I\sigma}_3 \\ &= \gamma^\mu(\partial_\mu\psi R + \psi\partial_\mu R + \tfrac{1}{2}\psi R\Omega'_\mu)\boldsymbol{I\sigma}_3 = m\psi'\gamma_0.\end{aligned} \tag{1.11}$$

For this to hold, given that $m\psi\gamma_0 = \gamma^\mu D_\mu\psi\boldsymbol{I\sigma}_3$, we require that

$$\tfrac{1}{2}R\Omega'_\mu + \partial_\mu R = \tfrac{1}{2}\Omega_\mu R \tag{1.12}$$

so

$$\Omega'_\mu = \tilde{R}\Omega_\mu R - 2\tilde{R}\partial_\mu R. \tag{1.13}$$

This defines the transformation properties of the connection field Ω_μ under local changes of gauge. The connection Ω_μ is bivector-valued, and under a change of gauge it picks up an extra bivector term from the derivative of the rotor R. In general, Ω_μ is also rotated by R, but this does not change its grade.

In writing the transformation law (1.13) we are *not* asserting that the connection can be written in the form $-2\tilde{R}\partial_\mu R$. If it could, we could find a gauge where the connection vanished and we would be back where we started — with no new physical effects. The essence of the gauging process is that the Ω_μ are *arbitrary*, position-dependent bivector functions. They transform to pick up the derivative of a rotor, but they are not restricted to this form. Indeed, it is the *difference* between Ω_μ and the derivative of a rotor field that gives rise to physical effects. We will see how to quantify this difference in Section (1.4). Since the gauge fields bring with them new dynamical degrees of freedom, further equations must be found to solve for them. These new equations must also be invariant under local changes of gauge, and this requirement places stringent restrictions on the nature of physical interactions.

1.3 The Minimally Coupled Dirac Equation

Returning to electromagnetism, we are concerned with the restricted class of rotations which take place wholly in the $\gamma_2\gamma_1$ plane. In this case, writing $R = \exp(\boldsymbol{I\sigma}_3\theta)$, we have

$$-2\tilde{R}\partial_\mu R = -2\,\mathrm{e}^{-\boldsymbol{I\sigma}_3\theta}\,\partial_\mu\theta\,\mathrm{e}^{+\boldsymbol{I\sigma}_3\theta}\,\boldsymbol{I\sigma}_3 = -2\partial_\mu\theta\boldsymbol{I\sigma}_3. \tag{1.14}$$

The connection we introduce must be restricted to this general form, so we can write

$$\Omega_\mu = \alpha A_\mu \boldsymbol{I\sigma}_3, \tag{1.15}$$

where A_μ are the components of a spacetime vector, and α is some coupling constant. Under gauge transformations, the A_μ coefficients transform as

$$\alpha A_\mu \mapsto \alpha A'_\mu = \alpha A_\mu - 2\partial_\mu\theta, \tag{1.16}$$

or, in coordinate-free notation

$$\alpha A \mapsto \alpha A' = \alpha A - 2\nabla\theta. \tag{1.17}$$

At this point we are back on familiar ground. The connection coefficients A_μ are just the coefficients of the electromagnetic vector potential A, and under a local change of phase A picks up the derivative of the phase variable. This has no physically observable consequences because the electromagnetic field strength $F = \nabla\wedge A$ is unchanged if A is replaced by $A + \nabla\phi$.

We are now in a position to reassemble our full, covariant Dirac equation. We have

$$\gamma^\mu D_\mu\psi = \gamma^\mu(\partial_\mu\psi + \tfrac{1}{2}\alpha A_\mu\psi\boldsymbol{I\sigma}_3) = \nabla\psi + \tfrac{1}{2}\alpha A\psi\boldsymbol{I\sigma}_3, \tag{1.18}$$

where we see that that connection combines with the frame vectors γ^μ to assemble a vector A multiplying ψ from the left. The Hamiltonian from this operator contains a new term in $\alpha\gamma_0 A/2$, and the scalar part of this is $\alpha V/2$. It is clear that for an electron we require $\alpha = 2e$, where $e = -|e|$ is the (signed) charge of the electron, so the '*minimally coupled*' Dirac equation is

$$\nabla\psi\boldsymbol{I\sigma}_3 - eA\psi = m\psi\gamma_0. \tag{1.19}$$

The equation is *minimally coupled* because by adding an interaction term solely in A we are making the simplest possible modification to the original equation. The minimal coupling principle can be summarised as 'replace all partial derivatives with covariant derivatives'. The resulting equations will be gauge-invariant. There is nothing preventing us, in principle, from adding extra gauge invariant terms. We could, for example, add further terms in F, or F^2 multiplying ψ, and the equation would still be gauge invariant. It appears, however, that nature does not employ this possibility. Why this should be so is far from clear.

1.4 Field Strength

The introduction of gauge fields introduces new physical degrees of freedom, and these need to satisfy their own dynamical equations. The key to constructing these is the field strength. This encodes the content of the gauge fields which is not generated by gauge transformations alone.

The field strength tensor is found in general by commuting covariant derivatives. This defines the object

$$\begin{aligned}[D_\mu, D_\nu]\psi &= D_\mu(\partial_\nu\psi + \tfrac{1}{2}\psi\Omega_\nu) - D_\nu(\partial_\mu\psi + \tfrac{1}{2}\psi\Omega_\mu) \\ &= \tfrac{1}{2}\psi(\partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu - \Omega_\mu\times\Omega_\nu),\end{aligned} \tag{1.20}$$

where we recall that $A\times B = (AB - BA)/2$. Despite the fact that we formed commutators of derivatives on ψ, all of the derivatives of ψ have cancelled. The resulting object is the gauge field strength, $\mathsf{F}_{\mu\nu}$,

$$\mathsf{F}_{\mu\nu} = \partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu - \Omega_\mu\times\Omega_\nu. \tag{1.21}$$

This is bivector-valued, and is antisymmetric on its indices μ, ν. This can be viewed as a linear mapping of bivectors onto bivectors. The transformation properties of the field strength are easily established from

$$[D'_\mu, D'_\nu]\psi' = \tfrac{1}{2}\psi'\mathsf{F}'_{\mu\nu}. \tag{1.22}$$

This involves terms going as (setting $\psi' = \psi R$)

$$D'_\mu(D'_\nu\psi') = D'_\mu(D_\nu\psi R) = (D_\mu D_\nu\psi)R, \tag{1.23}$$

and it follows that

$$R\mathsf{F}'_{\mu\nu} = \mathsf{F}_{\mu\nu}R. \tag{1.24}$$

The field strength tensor therefore satisfies the covariant transformation law

$$\mathsf{F}'_{\mu\nu} = \tilde{R}\mathsf{F}_{\mu\nu}R. \tag{1.25}$$

This is the transformation law for the field strength in a general Yang-Mills gauge theory.

Specialising to the case of electromagnetism, where $\Omega_\mu = 2eA_\mu \boldsymbol{I\sigma}_3$, we find that the term multiplying ψ is

$$\begin{aligned}&e(\partial_\mu A_\nu \boldsymbol{I\sigma}_3 - \partial_\nu A_\mu \boldsymbol{I\sigma}_3) - 2e^2 A_\mu A_\nu \boldsymbol{I\sigma}_3\times\boldsymbol{I\sigma}_3 \\ &= e(\partial_\mu A_\nu - \partial_\nu A_\mu)\boldsymbol{I\sigma}_3 = e(\gamma_\nu\wedge\gamma_\mu)\cdot(\nabla\wedge A)\boldsymbol{I\sigma}_3.\end{aligned} \tag{1.26}$$

This is a function which maps the bivector $\gamma_\nu\wedge\gamma_\mu$ linearly onto a pure phase term. The electromagnetic case has two unique features which simplify the nature of the field strength. The first is that the commutator term $\Omega_\mu\times\Omega_\nu$ vanishes, because all the bivector terms lie in the same plane. This means that the field strength is *linear* in A. This is not the case for more complicated groups, and is one reason why the strong and gravitational interactions are so complicated to compute. The second feature is that the transformation law (1.25) has no effect, since the bivector $I\boldsymbol{\sigma}_3$ is unaffected by rotations in its own plane. These two features of electromagnetism mean it is convenient to lose the mapping nature of the field strength and instead work directly with the bivector $F = \nabla \wedge A$. For more complicated systems this is not appropriate.

In forming the commutator of covariant derivatives we have extracted the correct field strength, $F = \nabla \wedge A$. This encodes the physically measurable content of the electromagnetic field, and vanishes if A is a pure gauge field, $A = \nabla\phi$. The field equations for electromagnetism are written directly in terms of F, and as such are guaranteed to be gauge covariant. (If fact, due to the special nature of electromagnetism, the equations are gauge *in*variant.) The relevant equation is $\nabla F = J$, where the current J is also gauge invariant.

1.5 Electroweak Interactions

The two main observables for the Dirac spinor ψ are the current $J = \psi\gamma_0\tilde{\psi}$ and the spin vector $s = \psi\gamma_3\tilde{\psi}$. If one looks for rotors which leave both of these invariant, we are restricted to

$$R\gamma_0\tilde{R} = \gamma_0, \qquad R\gamma_3\tilde{R} = \gamma_3. \tag{1.27}$$

The only rotors satisfying both these equations are those for rotations in the $I\boldsymbol{\sigma}_3$ plane and, as we have just seen, gauging these gives rise to electromagnetic interactions.

If we remove the restriction about the spin current, and just look for transformations which leave J invariant, we introduce a wider set of gauge fields. Now our only restriction is $R\gamma_0\tilde{R} = \gamma_0$, which defines the group of spatial rotors in the rest-frame of γ_0. The group of rotors in three dimensions form a special group denoted SU(2). This is the group of 2×2 complex unitary matrices with determinant $+1$. The fact that spatial rotors form this group can be seen by writing them in terms of the Pauli matrices.

The group SU(2) is one of the main building blocks of the *electroweak theory*. The second main ingredient is a further group of phase transformations, sometimes simply denoted as U(1). This further group can be incorporated by extending the allowed transformations to include exponentials of the pseudoscalar I. These also leave the current unchanged as

$$\psi\, \mathrm{e}^{I\beta/2}\, \gamma_0\, \mathrm{e}^{I\beta/2}\, \tilde{\psi} = \psi\gamma_0\tilde{\psi}. \tag{1.28}$$

Duality transformations such as this are not symmetries of the full, massive Dirac equation, but they are symmetries of the *massless* equation. The electroweak theory is constructed from an abstract internal set of SU(2) and U(1) transformations, applied to the massless Dirac theory. Mass is then introduced in a gauge-invariant way through a further coupling to a new field, called the *Higgs field*. The details of this are outside the scope of this course, but it is highly suggestive that the gauge groups of electroweak theory can be justified simply from the symmetries of the Dirac current.

Chapter 2

Gravity as a Gauge Theory

Building on the success of gauge theories such as the standard model of particle physics, many physicists and mathematicians have attempted to formulate *general relativity* (GR) as a gauge theory. These attempts have met with mixed success. By the sixties it was established that GR *could* be formulated as a gauge theory, but the equations obtained always ended up looking extremely complicated. Certainly more so than those from the traditional view of gravity arising from spacetime curvature. Geometric algebra provides a solution to this problem. Utilising the full structure of the spacetime algebra (STA), it is possible to construct gravity as a gauge theory in a formalism that is actually *easier* to understand and work with than the curved-space viewpoint.

2.1 Gauge Principles for Gravitation

Our aim is to model gravitational interactions in terms of (gauge) fields defined in the STA. Already, this is a radical departure from GR. The STA is the geometric algebra of *flat* spacetime, and the introduction of fields cannot alter this basic property. What then are we to make of the standard viewpoint that spacetime is curved? The answer is that all of the arguments which lead to this conclusion involve light paths, or measuring rods, or such like, and all of these processes are also modelled by fields defined in the STA. Since all physical quantities correspond to fields, the *absolute* position and orientation of particles or fields in the STA is not measurable. The only predictions that can be extracted are relative relations between fields. Ensuring that this property is true locally means there is no conflict with any of the principles by which one is traditionally led to GR.

The preceding considerations become clearer if we consider relations between quantum fields. Suppose that $\psi_1(x)$ and $\psi_2(x)$ are spinor fields. A physical statement could be a simple relation of equality,

$$\psi_1(x) = \psi_2(x). \tag{2.1}$$

But all this statement says is that at a point where one field has a particular value, then the second field has the same value. This statement is completely independent of where we choose to place the fields in the STA. And, more importantly, it is totally independent of where we choose to locate other values of the fields. We could equally well introduce two new fields

$$\psi_1'(x) = \psi_1(x'), \quad \psi_2'(x) = \psi_2(x'), \tag{2.2}$$

where x' is an arbitrary function of position x. The statement $\psi_1'(x) = \psi_2'(x)$ contains precisely the same physical content as the original equation.

The same picture emerges if both fields are acted on by a spacetime rotor at a given point, giving rise to new fields

$$\psi_1' = R\psi_1, \quad \psi_2' = R\psi_2. \tag{2.3}$$

Again, the statement $\psi_1' = \psi_2'$ has the same physical content as the original equation. Similar considerations apply to the observables formed from ψ, such as the vector $J = \psi\gamma_0\tilde{\psi}$. Replacing ψ by ψ' produces the new vector $J' = RJ\tilde{R}$. However, any physically-defined directions in spacetime must also be derived from fields, so these directions will rotate in the same way as J. Only the relative orientation of physical vectors is measurable; the absolute direction of vectors in the STA has no physical significance. We now have a clear mathematical statement of the invariance properties we want to establish. The next task is to study the form of the gauge fields needed to enforce this invariance.

2.2 Position Invariance

Suppose that $x' = f(x)$ is some arbitrary (differentiable) map between spacetime position vectors. The transformation we are interested in is where the field $\psi(x)$ is moved around to the new field

$$\psi'(x) = \psi(x'). \tag{2.4}$$

Since physical events must be identified by what happens there, e.g. by the values of fields at that point, the map $x' = f(x)$ can be interpreted as a reparameterisation of physical events by position vectors in the STA. With this interpretation, the event is originally represented by x', but this gets mapped to x under the reparameterisation. We call this this operation of moving fields around in a general manner a '*displacement*'.

As with electromagnetism, we now need to consider the behaviour of the derivatives of ψ. To help with this, we introduce the new coordinates

$$x^{\mu\prime} = \gamma^\mu \cdot x'. \tag{2.5}$$

In terms of these we can write $\psi'(x^\mu) = \psi(x^{\mu\prime})$. The derivatives of the displaced field are now

$$\partial_\mu \psi' = \partial_\mu \psi(x^{\nu\prime}) = (\partial_\mu x^{\nu\prime})\partial'_\nu \psi(x^{\lambda\prime}). \tag{2.6}$$

The Jacobian factor of $\partial_\mu x^{\nu\prime}$ is an unwanted term which must be removed by adding a suitable gauge field. Since this term acts directly on the partial derivatives it cannot be removed by introducing a bivector connection. The only alternative is that we must allow the γ^μ vectors to transform in such a way to remove the unwanted term. The gauge process for displacements is therefore to replace the constant γ^μ frame with a set of 4 vector gauge fields $g^\mu(x)$,

$$\gamma^\mu \mapsto g^\mu(x). \tag{2.7}$$

These are defined to transform in such a way as to remove the unwanted extra term. (The details of this are not discussed here.) At first, replacing the frame vectors with arbitrary fields appears to be a very strange step to take. But if we think about this, converting the γ^μ frame vectors to gauge fields is *precisely* what we want to do. With the vectors now treated as gauge fields in the STA, they cease to be physically observable, in the same way that the electromagnetic vector potential A is not observable. This removes any dependence on a fixed, global inertial frame, so achieving one of the key goals of general relativity. And in achieving this we have not had to set foot outside the spacetime algebra. Introducing these new gauge fields will also entail the introduction of new physical degrees of freedom, which are described in later sections.

2.3 Rotations

The second symmetry we require is that our wave equation should be invariant under the transformation

$$\psi \mapsto \psi' = R\psi, \tag{2.8}$$

where R is an arbitrary, position-dependent rotor in spacetime. (We refer to the rotor R as generating rotations, understanding that boosts are now a special case of a rotation.) So far, our Dirac equation has been modified to read

$$g^\mu(x)\partial_\mu \psi I \boldsymbol{\sigma}_3 = m\psi\gamma_0. \tag{2.9}$$

If R is a constant rotor, this equation is only invariant under $\psi \mapsto R\psi$ if we also transform the g^μ vectors as

$$g^\mu(x) \mapsto g^\mu(x)' = Rg^\mu(x)\tilde{R}. \tag{2.10}$$

Since the g^μ are (gauge) vector fields we have introduced, we are free to set their transformation properties under rotations. We therefore define the g^μ to transform according to (2.10) under arbitrary local rotations.

The final step is to modify the partial derivatives in Eq. (2.9) to ensure invariance under local rotations. Now we are back in the familiar territory of Chapter 1, but with the important difference that the rotor now acts to the left of ψ. We replace the partial derivatives with the covariant derivatives

$$D_\mu \psi = \partial_\mu \psi + \tfrac{1}{2}\Omega_\mu \psi. \tag{2.11}$$

The gauge fields have the transformation law

$$\Omega_\mu \mapsto \Omega'_\mu = R\Omega_\mu \tilde{R} - 2\partial_\mu R \tilde{R}. \tag{2.12}$$

Since R is an arbitrary rotor there is now no constraint on the blades that Ω_μ can contain, so Ω_μ has $6 \times 4 = 24$ degrees of freedom.

Our final, full covariant Dirac equation now reads

$$g^\mu D_\mu \psi I\boldsymbol{\sigma}_3 = m\psi\gamma_0. \tag{2.13}$$

This incorporates all the gravitational gauge fields and is the appropriate, relativistic equation for a Fermion in a gravitational background. The equation is covariant under the simultaneous replacements

$$\psi \mapsto R\psi, \quad g^\mu \mapsto Rg^\mu \tilde{R}, \quad \Omega_\mu \mapsto R\Omega_\mu \tilde{R} - 2\partial_\mu R\tilde{R}. \tag{2.14}$$

The equation is also fully covariant under local displacements. This is slightly harder to prove, and the details of this are not covered here.

2.4 Covariant Derivatives for Observables

Having seen what the covariant derivative of a spinor looks like, it is a simple matter to establish a formula for the derivative of the observables formed from a spinor. In general, these observables have the form

$$M = \psi\Gamma\tilde{\psi}, \tag{2.15}$$

where Γ is a constant multivector formed from combinations of γ_0, γ_3 and $I\boldsymbol{\sigma}_3$. The observable M inherits its transformation properties from the spinor ψ, so under displacements it transforms as

$$M(x) \mapsto M'(x) = M(x') \tag{2.16}$$

and under rotations it transforms as

$$M \mapsto M' = RM\tilde{R}. \tag{2.17}$$

Multivectors with these transformation properties are said to be *covariant.*

If we now form the derivatives of M we get

$$\partial_\mu M = (\partial_\mu \psi)\Gamma\tilde{\psi} + \psi\Gamma(\partial_\mu\psi)^{\sim}. \tag{2.18}$$

This immediately tells us how to construct a covariant derivative for M. We simply replace spinor directional derivatives with their covariant version and form

$$\begin{aligned}(D_\mu\psi)\Gamma\tilde{\psi} + \psi\Gamma(D_\mu\psi)^{\sim} &= \partial_\mu\psi\Gamma\tilde{\psi} + \psi\Gamma(\partial_\mu\psi)^{\sim} + \tfrac{1}{2}\Omega_\mu\psi\Gamma\tilde{\psi} - \tfrac{1}{2}\psi\Gamma\tilde{\psi}\Omega_\mu \\ &= \partial_\mu(\psi\Gamma\tilde{\psi}) + \Omega_\mu \times (\psi\Gamma\tilde{\psi}). \end{aligned} \tag{2.19}$$

The covariant derivative for observables therefore takes the form

$$D_\mu M = \partial_\mu M + \Omega_\mu \times M. \tag{2.20}$$

This is the form appropriate for acting on covariant multivectors, including observables formed from spinors. In keeping with standard notation we use the same universal symbol D_μ for the covariant derivative, and let the type of object being differentiated dictate the explicit form of D_μ.

The bivector commutator in (2.20) has two important properties. This first is that it is grade preserving, so the full D_μ operator preserves grade. The second is that

$$\Omega_\mu \times (MN) = (\Omega_\mu \times M)N + M(\Omega_\mu \times N). \tag{2.21}$$

This ensures that D_μ is a *derivation*, that is, it satisfies Leibniz' rule

$$D_\mu(MN) = (D_\mu M)N + M(D_\mu N). \tag{2.22}$$

Both of these are necessary properties for D_μ to be a suitable generalisation of a directional derivative.

2.5 The Gravitational Field Equations

In forming a Dirac equation with all the desired local symmetries we have been forced to introduce two new gauge fields. A set of four vector fields g^μ and a set of bivector fields Ω_μ. In total this is $4 \times 4 + 4 \times 6 = 40$ degrees of freedom! We are now faced with the daunting task of constructing field equations for all of these variables. For the rotational gauge field these equations will involve the field strength, but the equations for our generalised vectors are not so obvious. To form suitable equations we start by introducing the dual vectors g_μ defined by

$$g^\mu \cdot g_\nu = \delta^\mu_\nu. \tag{2.23}$$

(Note that this employs the standard STA inner product — there is no 'curved space' product to worry about.) Suppose that we have a general curvilinear coordinate system x^μ, with frame vectors

$$e^\mu = \nabla x^\mu, \quad e_\mu = \partial_\mu x. \tag{2.24}$$

In terms of these we can write $\nabla = e^\mu \partial_\mu$. The e_μ vectors satisfy

$$\partial_\mu e_\nu - \partial_\nu e_\mu = [\partial_\mu, \partial_\nu] x = 0. \tag{2.25}$$

The g_μ vectors must satisfy some covariant extension to this equation, so we focus attention on the object

$$D_\mu g_\nu - D_\nu g_\mu = \mathsf{S}_{\mu\nu}. \tag{2.26}$$

This defines a covariant vector, $\mathsf{S}_{\mu\nu}$, which is antisymmetric on its 2 indices. The simplest possible field equation we could write down is to set the right-hand side to zero, so that

$$D_\mu g_\nu - D_\nu g_\mu = 0. \tag{2.27}$$

This is borne out by a Lagrangian analysis, which shows that $\mathsf{S}_{\mu\nu}(x)$ is proportional to the *quantum spin density* at x. For any given solution of the field equations, the quantum spin density will be of the order of $\hbar/L^3$, where L is some characteristic length defined by the fields. Since $\mathsf{S}_{\mu\nu}$ has dimensions of (length)$^{-1}$, the constant of proportionality between $\mathsf{S}_{\mu\nu}$ and the quantum spin density has the same dimensions as G/c^3, where G is Newton's constant and c is the speed of light. These are the natural constants to describe the scale of gravitational interactions in a relativistic theory, so $\mathsf{S}_{\mu\nu}$ should be of the order of $\hbar G/(cL)^3$. The combination $\hbar G/c^3$ has dimensions of (length)^2. Its square root is the *Planck length* which evaluates to 1.6×10^{-35}m. The Planck length sets the fundamental length scale for quantum gravitational effects — it is clearly far smaller than any scale currently accessible to experiment. This dimensional argument shows that, even in the presence of quantum spin, $L\mathsf{S}_{\mu\nu}$ will be tiny on length scales large compared to the Planck length, and can be legitimately set to zero.

In setting $\mathsf{S}_{\mu\nu}$ to zero, we might expect this to imply that the g_μ are pure gauge fields. But in forming $D_\mu g_\nu - D_\nu g_\mu$ we are coupling in the Ω_μ field, so cannot set the g_μ to be a coordinate frame. It is precisely this coupling which generates some dynamics. It is also Eq. (2.27) that ensures that the equations we derive are (locally) equivalent to those of GR! The revealing feature of this approach is that GR is only recovered in the limit where quantum interactions are ignored. This has a number of implications for attempts to unite quantum theory and gravity.

2.6 Gravitational Field Strength

We follow the standard prescription of commuting covariant derivatives to form the field strength. This gives

$$[D_\mu, D_\nu]\psi = \tfrac{1}{2}\mathsf{R}_{\mu\nu}\psi \tag{2.28}$$

where

$$\mathsf{R}_{\mu\nu} = \partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu + \Omega_\mu \times \Omega_\nu. \tag{2.29}$$

This is a bivector-valued function, and is antisymmetric on the indices μ, ν. This is the gauge theory version of the *Riemann tensor*. We can view this as a linear map from the bivector $g_\mu \wedge g_\nu$ onto the space of bivectors. Since there are no restrictions on the blades present in $\mathsf{R}_{\mu\nu}$, this linear function has $6 \times 6 = 36$ degrees of freedom.

We started with 40 degrees of freedom, and so far have introduced the field equations

$$D_\mu g_\nu - D_\nu g_\mu = 0. \tag{2.30}$$

The left-hand side here is vector-valued and antisymmetric on the two indices. This equation therefore contains $4 \times 6 = 24$ separate scalar equations. We therefore have a further 16 equations to find, so cannot write down an equation directly for $\mathsf{R}_{\mu\nu}$. The solution, which again is justified by a rigorous Lagrangian analysis, is to introduce the contracted function

$$\mathsf{R}_\nu = g^\mu \cdot \mathsf{R}_{\mu\nu}. \tag{2.31}$$

This is a vector-valued function with a single index, which can be viewed as a linear mapping from vectors to vectors. This potentially has 16 degrees of freedom, and is the gauge theory analog of the *Ricci tensor*.

We still need one further ingredient before writing down our second field equation. We carry out a final contraction to define the *Ricci scalar*

$$R = g^\mu \cdot \mathsf{R}_\mu. \tag{2.32}$$

This quantity is gauge invariant and forms the Lagrangian for an action principle leading to the gravitational field equations. The second field equation can now be written

$$\mathsf{R}_\mu - \tfrac{1}{2}R g_\mu = 8\pi G \mathsf{T}_\mu. \tag{2.33}$$

The right-hand side denotes the stress-energy tensor for the matter fields. This equation therefore relates the gravitational field strength to the energy content of the fields present. In relativistic physics, *energy* is a source of gravitation, and not just mass.

In the absence of any matter (appropriate, for example, when describing the gravitational fields in the vacuum region outside a star), the gravitational field equations reduce to

$$\mathsf{R}_\mu - \tfrac{1}{2}R g_\mu = 0. \tag{2.34}$$

If we contract this equation we find that

$$g^{\mu}\cdot\mathsf{R}_{\mu} - \tfrac{1}{2}g^{\mu}\cdot g_{\mu}R = R - 2R = -R = 0, \tag{2.35}$$

so our vacuum equations can be more compactly stated as

$$\mathsf{R}_{\mu} = 0. \tag{2.36}$$

Equations (2.27) and (2.33) complete our set of gravitational field equations. The challenge now is to solve them!

2.7 Relationship to General Relativity Non-Examinable

This section is included for general interest, particularly for those who have already attended a course in GR. Clearly, many of the structures introduced so far have counterparts in GR, but equally there are some key ingredients apparently missing. The first of these is the metric. This is defined simply by

$$g_{\mu\nu} = g_{\mu}\cdot g_{\nu}. \tag{2.37}$$

This construction is highly revealing. It shows that the metric behaves as a *scalar* under the rotation gauge group. Since GR is traditionally formulated entirely from the metric, it is blind to the existence of the rotation gauge group. This simple fact held back the development of gravity as a gauge theory for many years. The second key ingredient is the *Christoffel connection.* This is recovered through the definition

$$\Gamma^{\lambda}_{\mu\nu} = g^{\lambda}\cdot(D_{\mu}g_{\nu}). \tag{2.38}$$

This is automatically antisymmetric on its indices μ and ν. This definition is also the obvious covariant extension of the role of the Christoffel symbol in flat space for holding information about the derivatives of a curvilinear set of frame vectors.

The full tensor forms of the Riemann and Ricci tensors are equally simply recovered:

$$R^{\mu}{}_{\nu\rho\sigma} = (g^{\mu}\wedge g_{\nu})\cdot\mathsf{R}_{\sigma\rho} \tag{2.39}$$

$$R_{\mu\nu} = g_{\mu}\cdot\mathsf{R}_{\nu}. \tag{2.40}$$

A final question relates to the symmetries of these various tensors. Some of the symmetries of the Riemann tensor are now obvious as they come directly from the bivector nature of $\mathsf{R}_{\mu\nu}$. A final set of symmetries require slightly more work. We start with the result that, for covariant multivectors,

$$[D_{\mu}, D_{\nu}]M = \mathsf{R}_{\mu\nu}\times M. \tag{2.41}$$

A special case of this is

$$[D_\mu, D_\nu]g_\lambda = \mathsf{R}_{\mu\nu}\cdot g_\lambda. \tag{2.42}$$

But we can also write

$$D_\mu(D_\nu g_\lambda) - D_\nu(D_\mu g_\lambda) = D_\mu(D_\lambda g_\nu) - D_\nu(D_\lambda g_\mu) \tag{2.43}$$

$$= [D_\mu, D_\lambda]g_\nu - [D_\nu, D_\lambda]g_\mu + D_\lambda(D_\mu g_\nu - D_\nu g_\mu) \tag{2.44}$$

so, rearranging, we find that

$$g_\lambda\cdot\mathsf{R}_{\mu\nu} + g_\nu\cdot\mathsf{R}_{\lambda\mu} + g_\mu\cdot\mathsf{R}_{\nu\lambda} = 0. \tag{2.45}$$

This vector vanishes for all antisymmetric combinations of 3 indices, which results in $4\times 4 = 16$ scalar equations. These reduce the number of independent degrees of freedom in the Riemann tensor from 36 to the familiar 20 of GR. Contracting this equation with g^λ establishes

$$(g^\lambda\wedge g_\nu)\cdot\mathsf{R}_{\lambda\mu} + (g^\lambda\wedge g_\mu)\cdot\mathsf{R}_{\nu\lambda} = -g_\nu\cdot\mathsf{R}_\mu + g_\mu\cdot\mathsf{R}_\nu = 0 \tag{2.46}$$

This establishes that the Ricci tensor is symmetric, $R_{\mu\nu} = R_{\nu\mu}$. It follows that the stress-energy tensor must also be symmetric, which turns out to be the case in the absence of spin. Much of the apparatus of GR is based on this assumption that quantum spin effects can be ignored. This assumption is necessary to set

$$D_\mu g_\nu - D_\nu g_\mu = 0, \tag{2.47}$$

without which much of the elegant structure of GR evaporates! In fact, quantum spin is a well-observed fact of the physical world, even if it rarely leads to measurable gravitational effects. Many of the recent developments in GR have consisted of attempts to incorporate spin effects while retaining as much as possible of the curved space GR formalism. This has met with mixed success, and some researchers are now considering the alternative of reverting to understanding gravity as a (gauge) field theory in a flat (unobservable) background spacetime.

Chapter 3

Point Particle Trajectories

The gauging arguments we have applied to spinor fields also apply to particle trajectories, and enable us to write down a covariant equation for a point particle in a gravitational background. This equation exposes the link between the gauge theory approach to gravity and General Relativity (GR).

3.1 Trajectories and Tangents

A path in spacetime can be written as $x(\lambda)$, where λ is some parameter along the trajectory. Our gauge principles imply that the actual positions along this trajectory can have no significance. But if the path is irrelevant, what happens to the velocity? The velocity is usually encoded in the tangent vector, but this cannot have any relevance if the path is essentially arbitrary. The answer is to convert the tangent vector to a suitable gauge-covariant vector. To see how, we write

$$\frac{dx}{d\lambda} = \frac{dx^\mu}{d\lambda}\gamma_\mu. \tag{3.1}$$

A similar argument to that applied to the Dirac equation tells us that we must replace the γ_μ vectors with the gauge fields $g_\mu(x)$. We therefore define the covariant velocity vector

$$v = \frac{dx^\mu}{d\lambda} g_\mu(x). \tag{3.2}$$

Under displacements of the trajectory the velocity vector v is simply carried from one position to the next (details not covered here). Under rotations v inherits the transformation law of the $g_\mu(x)$ vectors. We know that

$$g_\mu \cdot g^\nu = \delta^\nu_\mu \tag{3.3}$$

and under arbitrary rotations $g^\mu \mapsto R g^\mu \tilde{R}$. It follows that the appropriate transformation law for the g_μ is also

$$g_\mu \mapsto g'_\mu = R g_\mu \tilde{R}. \tag{3.4}$$

The velocity vector v must transform the same way, so under rotations

$$v \mapsto v' = R v \tilde{R}. \tag{3.5}$$

The vector v is therefore a genuine covariant object. Its absolute direction in the STA has no significance, since it is gauge dependent, but the relationship of v to other covariant fields does produce invariant physical predictions.

3.2 Inertial Observers and Free Fall

In the absence of gravitational effects, inertial observers are those whose velocity vector is constant. The equation defining such an observer can therefore be written

$$\partial_\lambda(\partial_\lambda x(\lambda)) = 0. \tag{3.6}$$

To include gravitational effects we simply make this equation covariant. The equation will then define the paths of observers who are not acted on by any force other than gravity. These are observers in *free fall*, and they generalise the notion of an inertial observer to gravity. To form the covariant version of (3.6) we first replace $\partial_\lambda x(\lambda)$ with the vector v. As $v \mapsto R v \tilde{R}$ under rotations, we must also convert the partial derivative of v to a covariant derivative. To see how to do this we first write

$$\partial_\lambda = \frac{\partial x^\mu}{\partial \lambda} \partial_\mu. \tag{3.7}$$

This tells us precisely what to do. We simply replace the ∂_μ by the covariant derivative, D_μ, producing the equation

$$\frac{\partial x^\mu}{\partial \lambda}(\partial_\mu v + \Omega_\mu \cdot v) = \partial_\lambda v + \frac{\partial x^\mu}{\partial \lambda} \Omega_\mu \cdot v = 0. \tag{3.8}$$

On contracting this equation with v, and using $(\Omega_\mu \cdot v) \cdot v = 0$, we find that v^2 is constant. We therefore choose the parameter for the trajectory such that

$$v^2 = 1. \tag{3.9}$$

This defines the *proper time* for an observer along the trajectory. This definition is gauge invariant, and reduces to the familiar relativistic definition in the absence of gravitational fields. If τ denotes the proper time, our trajectory equation can be written

$$\dot{v} + \dot{x}^\mu \Omega_\mu \cdot v = 0, \qquad v^2 = 1, \tag{3.10}$$

where the overdots denote differentiation with respect to proper time τ.

3.3 The Metric and GR

The equation (3.10) has one immediate feature — it is independent of the mass of the particle. It follows that, in the absence of external forces, all particles follow the same paths, whatever their mass. This is the gauge theory implementation of the *equivalence principle*. It is also possible to derive this result by taking the appropriate classical limit of the Dirac theory. This is a particularly attractive feature of the gauge theory viewpoint, as it shows that all the key features of GR are derived from a single principle — that of local gauge invariance applied to the Dirac theory. The equivalence principle is a consequence of the gauge theory approach, rather than having to be introduced as an independent physical principle.

Since $v^2 = 1$ defines the proper time parameter, the proper time (or proper distance) along a path is defined by the invariant integral

$$\begin{aligned} s &= \int_{\tau_1}^{\tau_2} \sqrt{|v^2|}\, d\tau \\ &= \int_{\tau_1}^{\tau_2} |\dot{x}^\mu \dot{x}^\nu g_\mu \cdot g_\nu|^{1/2}\ d\tau \\ &= \int_{x_1}^{x_2} |g_\mu \cdot g_\nu dx^\mu\, dx^\nu|^{1/2}. \end{aligned} \tag{3.11}$$

A comparison with the equivalent formula in GR enables us to read off the *metric* as

$$g_{\mu\nu} = g_\mu \cdot g_\nu. \tag{3.12}$$

One can show that the equation of motion (3.10) is obtained by looking for paths which minimise the proper distance between points. In GR the equivalent equation is called the *geodesic equation*. Equation (3.10) is therefore the gauge theory analog of the geodesic equation, and both approaches define the same trajectories. This goes some way to establishing the equivalence between the gauge theory of gravity developed here and GR. The remaining step is to prove that a set of gauge fields $g_\mu(x)$ satisfying the gauge field equations define a metric through (3.12) which satisfies the Einstein equations. This is indeed the case, provided spin effects are ignored, but this important result is not easy to prove and is not covered here.

Chapter 4

Spherically-Symmetric Gravitational Fields

A solution to the field equations in gauge theory gravity amounts, in principle, to specifying the g_μ vectors. These in turn define the connection bivectors Ω_μ through the equations

$$D_\mu g_\nu - D_\nu g_\mu = 0. \tag{4.1}$$

These give 24 equations for the 24 degrees of freedom in Ω_μ, and can be solved by linear algebra alone. Once the bivectors are known, we simply calculate the field strength and contract it to find R_μ and the Ricci scalar R. These in turn specify a stress-energy tensor, and we simply assert that we have solved the field equations for the resulting matter distribution! Unfortunately, this is back-to-front, as we really want to specify the stress-energy tensor first and find the gravitational fields it generates. This is far more difficult and today only a handful of exact solutions are known. Of these, the most important is that for a spherically-symmetric point source, which is known as the Schwarzschild solution. This is the gravitational analog of the Coulomb potential in electromagnetism. In this final chapter we focus on a number of properties of this important physical solution.

4.1 The Solution

In writing down the g_μ vectors for a given solution we have considerable gauge freedom at our disposal. Both displacements and rotations affect the appearance of the g_μ, and we can take advantage of this by making gauge choices to simplify the g_μ as far as possible. The simplest form yet found for the Schwarzschild solution has

$$g_0 = \gamma_0 + \sqrt{(2GM/r)}e_r, \quad g_i = \gamma_i, \tag{4.2}$$

where $i = 1 \ldots 3$. These vectors differ from the original γ_μ frame through the addition of a single term to γ_0. This term is $\surd(2GM/r)e_r$, where

$$e_r = \frac{1}{r} x^i \gamma_i \tag{4.3}$$

is the unit radial vector and r is the (Euclidean) distance from the origin. The quantity $\surd(2GM/r)$ defines the Newtonian free fall velocity for observers at rest at infinity. We shall see that aspects of the Newtonian behaviour are retained in the full, relativistic treatment.

From here on we will work in natural units where $G = c = \hbar = 1$. The factors of G can be easily reintroduced, since they always go with the mass M. Various arguments can be employed to justify the form of the vectors in (4.2), but only direct calculation can confirm that these define a vacuum solution. We start by forming the reciprocal frame to the g_μ, which is given by

$$g^0 = \gamma^0, \quad g^i = \gamma^i - \surd(2M/r)(x^i/r)\gamma^0. \tag{4.4}$$

The connection bivectors are found to be

$$\Omega_0 = \frac{M}{r^2}\boldsymbol{\sigma}_r \tag{4.5}$$

$$\Omega_i = -\frac{1}{2r}\left(\frac{2M}{r}\right)^{1/2}(2\boldsymbol{\sigma}_i - 3\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_r\,\boldsymbol{\sigma}_r), \tag{4.6}$$

where

$$\boldsymbol{\sigma}_r = e_r\gamma_0. \tag{4.7}$$

Verifying that these solve (4.1) is left as an exercise. Notice that the Ω_0 term, which governs acceleration, goes as the Newtonian expression GM/r^2. The following form of the connection bivectors is convenient in calculations:

$$\Omega_\mu = \frac{-1}{2r}\left(\frac{2M}{r}\right)^{1/2}\left(2g_\mu\wedge\gamma_0 + 3g_\mu\cdot e_r\,\boldsymbol{\sigma}_r\right). \tag{4.8}$$

The next step is compute the terms in the Riemann tensor. This is laborious, and best done with the aid of a symbolic algebra package, but the end result is strikingly simple,

$$\mathsf{R}_{\mu\nu} = -\frac{M}{2r^3}(g_\mu\wedge g_\nu + 3\boldsymbol{\sigma}_r g_\mu\wedge g_\nu\boldsymbol{\sigma}_r). \tag{4.9}$$

The immediate question, then, is why is this a solution?

The vacuum equations are

$$g^\mu\cdot\mathsf{R}_{\mu\nu} = 0. \tag{4.10}$$

An advantage of the form of (4.9) is that we can verify these equations in a handful of simple steps. We first compute

$$g^\mu \cdot (g_\mu \wedge g_\nu) = 4g_\nu - \delta^\mu_\nu g_\mu = 3g_\nu. \tag{4.11}$$

For the remaining term we need

$$\begin{aligned} g^\mu e_r \gamma_0 g_\mu &= -e_r g^\mu \gamma_0 g_\mu + 2g^\mu \cdot e_r \, \gamma_0 g_\mu \\ &= e_r \gamma_0 g^\mu g_\mu - 2e_r (g^\mu \cdot \gamma_0) g_\mu + 2\gamma_0 e_r \\ &= 4e_r\gamma_0 - 2e_r\gamma_0 + 2\gamma_0 e_r = 0. \end{aligned} \tag{4.12}$$

Combining these we find that

$$\begin{aligned} g^\mu \cdot (g_\mu \wedge g_\nu + 3\boldsymbol{\sigma}_r g_\mu \wedge g_\nu \boldsymbol{\sigma}_r) &= 3g_\nu + 3g^\mu \boldsymbol{\sigma}_r (g_\mu g_\nu - g_\mu \cdot g_\nu)\boldsymbol{\sigma}_r \\ &= 3g_\nu - 3g^\mu g_\mu \cdot g_\nu \boldsymbol{\sigma}_r \boldsymbol{\sigma}_r \\ &= 0 \end{aligned} \tag{4.13}$$

which confirms that we have a vacuum solution. The fact that $\mathsf{R}_{\mu\nu} \neq 0$ also confirms that our solution is non-trivial — there are genuine physical effects associated with the solution. The factor of GM/r^3 controlling the magnitude of the field strength is to be expected, since the Riemann tensor measures the strength of *tidal forces*.

4.2 Point Particle Trajectories

For radial motion we have $x(\tau) = t(\tau)\gamma_0 + r(\tau)e_r$, so

$$v = \dot{t} g_0 + \dot{r}\frac{x^i}{r} g_i = \dot{t}\gamma_0 + (\dot{r} + \sqrt{(2M/r)}\dot{t})e_r. \tag{4.14}$$

The constraint $v^2 = 1$ also allows us to write

$$v = \cosh(\alpha)\gamma_0 + \sinh(\alpha)e_r = \mathrm{e}^{\alpha\boldsymbol{\sigma}_r}\gamma_0, \tag{4.15}$$

so

$$\dot{v} = \dot{\alpha}\boldsymbol{\sigma}_r \cdot v. \tag{4.16}$$

The geodesic equation (3.10) includes the term $\dot{x}^\mu \Omega_\mu$. For the connection of equation (4.8) this simplifies to

$$\dot{x}^\mu \Omega_\mu = \frac{-1}{2r}\left(\frac{2M}{r}\right)^{1/2} (2v \wedge \gamma_0 + 3v \cdot e_r \, \boldsymbol{\sigma}_r). \tag{4.17}$$

For radial motion this reduces to

$$\dot{x}^\mu \Omega_\mu = \sinh(\alpha)\frac{1}{2r}\left(\frac{2M}{r}\right)^{1/2} \boldsymbol{\sigma}_r, \tag{4.18}$$

so the free-fall equations reduce to the set

$$\begin{aligned} \dot{t} &= \cosh(\alpha) \\ \dot{r} &= \sinh(\alpha) - \sqrt{(2M/r)}\cosh(\alpha) \\ \dot{\alpha} &= -\sinh(\alpha)\sqrt{(2M/r)}/(2r). \end{aligned} \tag{4.19}$$

We arrive at a set of three first-order equations, which are sufficient to specify a unique trajectory, given initial values of position (r and t) and velocity ($\tanh\alpha$).

One immediate solution to the radial equations is to set $\alpha = 0$. This solution has

$$\dot{r} = -\sqrt{(2M/r)}, \tag{4.20}$$

which represents a particle in free fall from rest at infinity. The solution also has $\dot{t} = 1$, so the time coordinate t now has a physical interpretation: t is the proper time as measured by observers in free fall from rest at infinity. This particular choice of time coordinate has many attractive features. Furthermore, the velocity for these freely falling observers is simply $v = \gamma_0$! This is one of the most convenient features of this gauge.

Taking the second derivative of the $\dot{r}$ equation, we find that

$$\ddot{r} = -\frac{GM}{r^2} \tag{4.21}$$

so the Newtonian force law is still present. The differences with Newtonian physics now lie in the meaning of the variables. The variable r is now a *local* observable, fixed by the magnitude of the tidal force. Similarly, the derivatives in $\ddot{r}$ are taken with respect to the local particle proper time, rather than a global Newtonian time. This transition from global to local variables is in keeping with the gauging process. Often the trick is to find a gauge and a set of global coordinates such that the values of the coordinates coincide with local, physical observables.

The equation for $\dot{r}$ contains a further surprise. On writing

$$\dot{r}/\cosh(\alpha) = \tanh(\alpha) - \sqrt{(2M/r)} \tag{4.22}$$

we see that if $2M/r > 1$ then $\dot{r}$ is necessarily negative. There is no way for the particle to escape. The place where this happens, $r = 2M = 2GM/c^2$, is the radius where the escape velocity $\sqrt{2GM/r}$ is greater than the speed of light. This is called the Schwarzschild radius, though the possibility of bodies becoming so dense that light could not escape was first suggested by John Michell (~ 1782).

4.3 Photon Paths

Radial photon paths are even easier to compute than for massive point particles. Photons satisfy the same geodesic equation (3.10) except that now we have

$$v^2 = 0. \tag{4.23}$$

For radial paths we still have

$$v = \dot{t}\gamma_0 + (\dot{r} + \sqrt{(2M/r)}\dot{t})e_r, \tag{4.24}$$

so the null condition $v^2 = 0$ forces

$$\dot{t} = \pm(\dot{r} + \sqrt{(2M/r)}\dot{t}). \tag{4.25}$$

The different signs correspond to ingoing and outgoing photons. The equations can be summarised as

$$\frac{dr}{dt} = -\sqrt{(2M/r)} + \begin{cases} +1 & \text{outgoing} \\ -1 & \text{ingoing,} \end{cases} \tag{4.26}$$

and these are straightforward to integrate.

A summary of the possible radial trajectories is contained in Figure 4.1, which includes both particle and photon tracks. An immediate feature is that for $r < 2M$ all photon paths point inwards. This confirms that not even light can escape from inside the horizon of a black hole.

4.4 The Dirac Equation

As a final application, we return to the Dirac equation and include the new gauge fields. We require the frame vectors in the form of (4.4). With this frame we have

$$g^\mu \partial_\mu = \nabla - \gamma_0\sqrt{(2GM/r)}(x^i/r)\partial_{x^i} = \nabla - \gamma_0\sqrt{(2GM/r)}\partial_r. \tag{4.27}$$

The connection term in the Dirac equation evaluates to

$$\tfrac{1}{2}g^\mu\Omega_\mu = \frac{-1}{4r}\left(\frac{2M}{r}\right)^{1/2} g^\mu\left(2g_\mu\wedge\gamma_0 + 3g_\mu\cdot e_r\,\boldsymbol{\sigma}_r\right) = \frac{-3}{4r}\left(\frac{2M}{r}\right)^{1/2}\gamma_0. \tag{4.28}$$

The Dirac equation therefore reduces to the simple equation

$$\nabla\psi I\boldsymbol{\sigma}_3 - \gamma_0\left(\frac{2M}{r}\right)^{1/2}\left(\partial_r\psi + 3/(4r)\psi\right)I\boldsymbol{\sigma}_3 = m\psi\gamma_0. \tag{4.29}$$

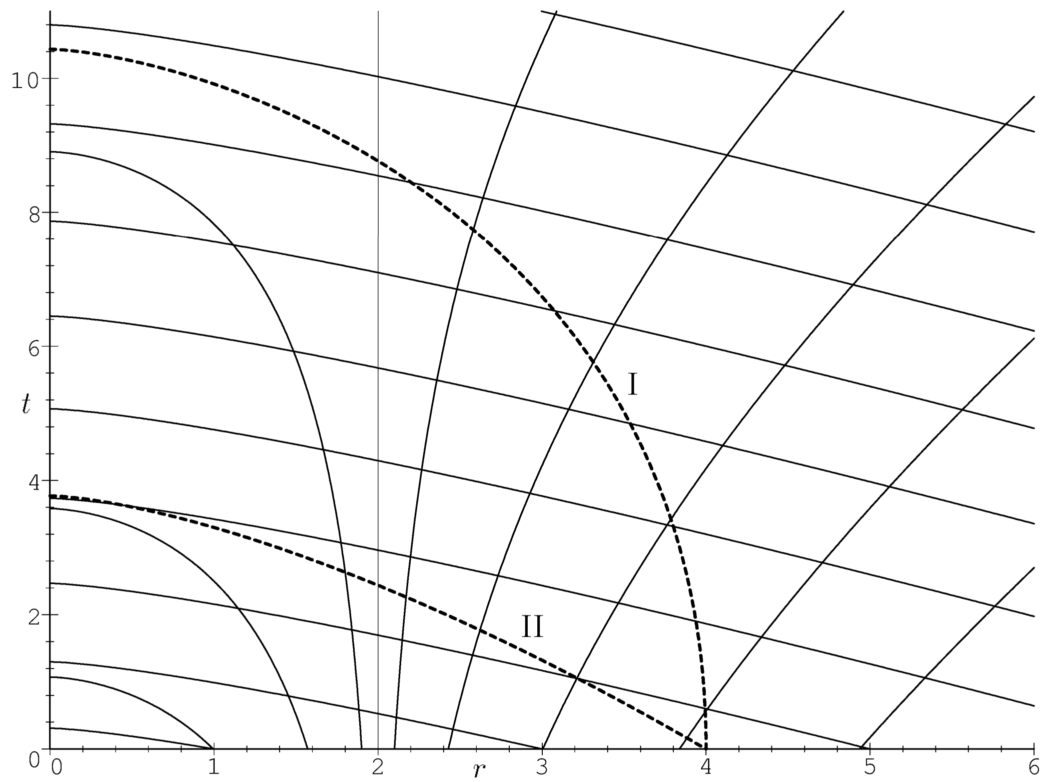

Figure 4.1: *Matter and photon trajectories in a black hole background.* The solid lines are photon trajectories, $GM = 1$ and the horizon lies at r=2. Outside the horizon it is possible to send photons out to infinity, and hence communicate with the rest of the universe. As one approaches the horizon, these photons are strongly redshifted and take a long time to escape. Once inside the horizon, all photon paths end on the singularity. The broken lines represent two possible trajectories for infalling matter. Trajectory I is for a particle released from rest at $r = 4$. Trajectory II is for a particle released from rest at $r = \infty$.

Premultiplying by γ_0 converts this equation to Hamiltonian form, and shows that all of the gravitational effects are contained in the single interaction term

$$\hat{H}_I = i\hbar \left(\frac{2M}{r}\right)^{1/2} \big(\partial_r + 3/(4r)\big). \tag{4.30}$$

This Hamiltonian has many remarkable features, some of which are explored on the final example sheet. Despite the apparent complexity of studying relativistic quantum mechanics in a black hole background, one can reduce the entire problem to studying the properties of a single Hamiltonian operator!