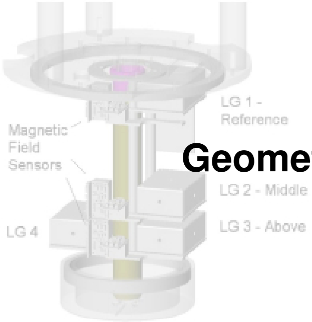
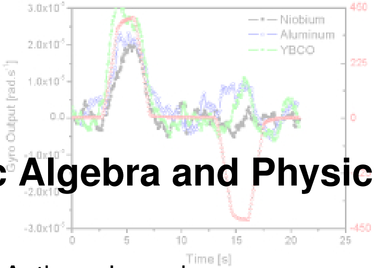


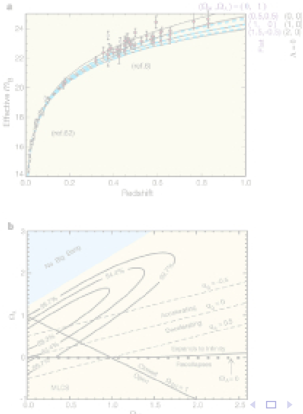




# Geometric Algebra and Physics

Anthony Lasenby

Astrophysics Group,
Cavendish Laboratory,
Cambridge, UK

Oxford, 14 April 2008

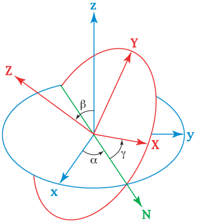


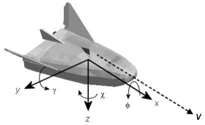
# Overview

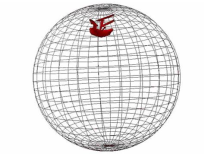
- I am basically a cosmologist, interested in the Cosmic Microwave Background, and early universe
- Why am I here talking about 'Geometric Algebra'?
- Came across it several years ago — an extremely useful approach to the mathematics of physics, that allows one to use a common language in a huge variety of contexts
- E.g. complex variables, vectors, quaternions, matrix theory, differential forms, tensor calculus, spinors, twistors, all subsumed under a common approach
- Therefore results in great efficiency — can quickly get into new areas
- Also tends to suggest new geometrical (therefore physically clear, and coordinate-independent) ways of looking at things
- Will try today to introduce a few aspects of it in more detail — principally applications to electromagnetism and quantum mechanics
- For further info and pointers to where else it's useful, look at `http://www.mrao.cam.ac.uk/~clifford`

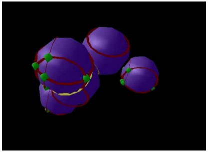
# What is GA?

- In 2d gives a geometric origin for complex numbers
- In 3d, gives geometrical explanation of quaternions and their properties
- Advantages of quaternions in e.g. spacecraft navigation and computer graphics already well known (more efficient than Euler angles and rotation matrices, which they replace)
- GA effectively extends them to the relativistic domain
- And via 'conformal geometric algebra' gives a whole new language for doing geometry on the computer (being exploited currently in computer graphics)

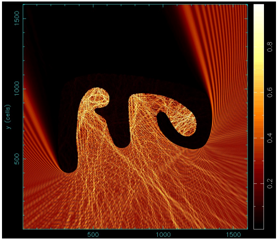


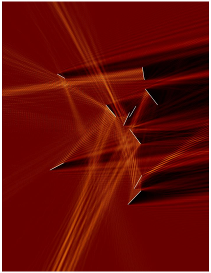





# Applications of GA



- Works extremely well with electromagnetism
- All four Maxwell equations combine to one: $\nabla F = J$, in which the $\nabla$ is invertible
- Leads to novel methods for treating EM scattering
- GA also leads to a different approach to quantum information theory (which can now be studied relativistically)

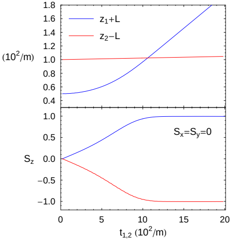
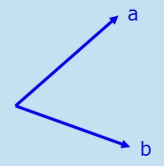



**Figure 5.16:** Single two-path and spin components for the spin entangled state (5.52) after a local pulse in region $A$. This two-path is a member of the $|\uparrow\downarrow\rangle$ sub-wavepacket. The spin $S_z$ of particle 1 (blue) goes from 0 to 1 while the spin of particle 2 (red) goes from 0 to -1. This occurs despite there being no motion on the part of particle 2. The effect of the localized pulse in $A$ has a nonlocal effect on particle 2 in region $B$.

# Geometric Algebra

- Know that for complex numbers there is a unit imaginary $i$
- Main property is that $i^2 = -1$
- How can this be? (any ordinary number squared is positive)
- Troubled some very good mathematicians for many years
- Usually these days an object with these properties just defined to exist, and complex numbers are defined as $x + iy$ ($x$ and $y$ ordinary numbers)
- But consider following: Suppose have two directions in space

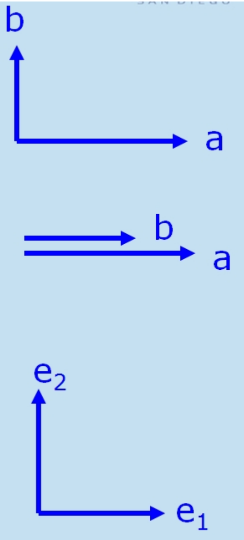



$a$ and $b$ (these are called vectors as usual)
- And suppose we had a language in which we could use vectors as words and string together meaningful phrases and sentences with them So e.g. $ab$ or $bab$ or $abab$ would be meaningful phrases

# Geometric Algebra (contd.)

Now introduce two rules:

- If $a$ and $b$ perpendicular, then $ab = -ba$
- If $a$ and $b$ parallel (same sense) then $ab = |a||b|$ (product of lengths)
- Just this does an amazing amount of mathematics!
- E.g. suppose have two unit vectors at right angles
- Rules say $e_1^2 = e_1 e_1 = 1$ , $e_2^2 = e_2 e_2 = 1$ and $e_1 e_2 = -e_2 e_1$



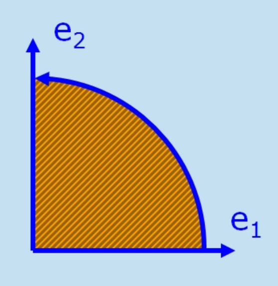
# Geometric Algebra (contd.)

Try $(e_1 e_2)^2$

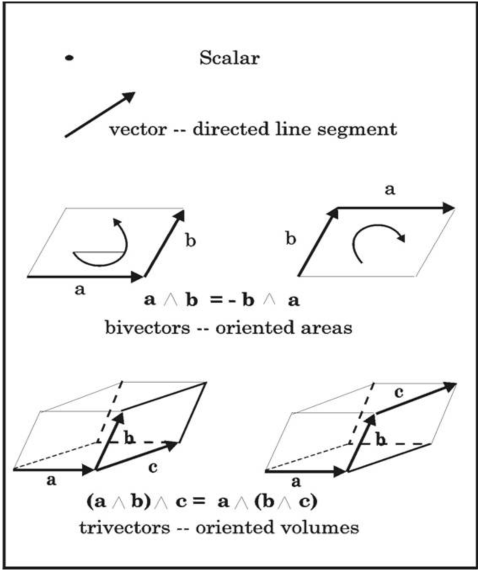
- This is

$$e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -1$$

- We have found a geometrical object $(e_1 e_2)$ which squares to minus 1 !
- Can now see complex numbers are objects of the form $x + (e_1 e_2) y$
- What is $(e_1 e_2)$ ? — we call it a bivector
- Can think of it as an oriented plane segment swept out in going from $e_1$ to $e_2$



# An algebra of geometric objects



# Geometric Algebra

Consider a vector space with the usual inner product;

$$\boldsymbol{a}\cdot\boldsymbol{b}$$

Also have an outer or wedge product which produces a new quantity called a bivector

$$\boldsymbol{a}\wedge\boldsymbol{b}$$

Combine these into a single geometric product:

$$\boldsymbol{ab} = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b}$$

Unlike the inner and outer products, this product is **INVERTIBLE**
Note taking the geometric product as primary, we have

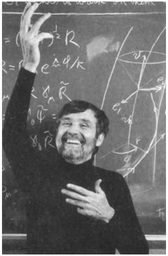
$$\boldsymbol{a}\cdot\boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab} + \boldsymbol{ba})$$

and

$$\boldsymbol{a}\wedge\boldsymbol{b} = \frac{1}{2}(\boldsymbol{ab} - \boldsymbol{ba})$$

This is basis for axiomatic development.

# Some History



- Hamilton (1840s): 3D rotations via quaternions
- Grassmann (1870s): exterior (wedge) product; oriented objects
- Clifford (1870s): combining products to form geometric product
- Hestenes (1960–): Formalism taking clifford algebra to geometric algebra (Clifford's own name).

# 3D Geometric Algebras cont...

In 3D we have three orthonormal basis vectors: $e_1, \; e_2, \; , e_3$

$$e_1^2 = 1, \quad e_2^2 = 1, \quad e_3^2 = 1, \quad e_1 \cdot e_2 = e_2 \cdot e_3 = e_3 \cdot e_1 = 0$$

$$e_1 e_2 e_3 = e_1 \wedge e_2 \wedge e_3 \equiv I$$

Again, look at the properties of this trivector, $I$:

$$I^2 = (e_1 e_2 e_3)(e_1 e_2 e_3) = e_1 e_1 e_2 e_3 e_2 e_3 = -(e_1 e_1)(e_2 e_2)(e_3 e_3) = -1$$

So, we have another real geometric object which squares to -1 !
Indeed there are many such objects which square to $-1$ ; this means that we seldom have need for complex numbers....
Call the highest grade object in the space the pseudoscalar – unique up to scale

# Reflections

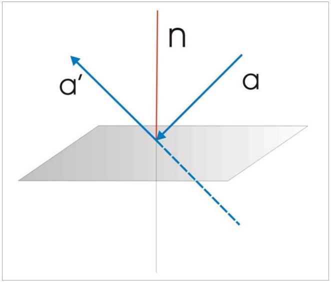
Reflections are very easy to implement in GA and will be of crucial importance later. Consider reflecting a vector $a$ in a plane with unit normal $n$ , the reflected vector $a$ is given by:

$$a' = -nan$$

This can easily be seen via the following expansion of $-nan$

$$-nan = a - 2(n \cdot a)n$$

[write $-nan$ as $-(na)n$ and expanding $na$ as $n \cdot a + n \wedge a$ and then writing $2(n \wedge a) = na - an$].



# Rotations

For many applications rotations are also an extremely important aspect of GA first consider rotations in 3D:
Recall that two reflections form a rotation:

$$a \mapsto -m(-nan)m = mnanm$$

We therefore define our rotor $R$ to be

$$R = mn \quad \text{and rotations are given by} \quad a \mapsto Ra\tilde{R}$$

Note that this is a geometric product!
The operation of reversion is the reversing of the order of products, eg

$$\tilde{R} = nm \quad \text{and therefore} \quad R\tilde{R} = 1$$

Works in spaces of any dimension or signature. Works for all grades of multivectors

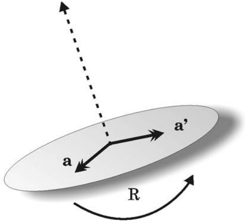
$$A \mapsto RA\tilde{R}$$

# Rotations cont...

A rotor, $R$, is therefore an element of the algebra and can also be written as the exponential of a bivector.

$$R = e^{-B}, \quad B = In\theta/2$$

$$R = \cos\frac{\theta}{2} - In\sin\frac{\theta}{2}$$




The bivector $B$ gives us the plane of rotation (cf Lie groups and quaternions). A rotor is a scalar plus bivector.
Comparing with quaternions

$$q = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \qquad \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$

$$\boldsymbol{i} = Ie_1, \quad \boldsymbol{j} = -Ie_2 \quad \boldsymbol{k} = Ie_3$$

# An Algebra for Spacetime I

Aim — to construct the geometric algebra of spacetime. Invariant interval is

$$s^2 = c^2 t^2 - \mathrm{x}^2 - \mathrm{y}^2 - \mathrm{z}^2$$

Work in natural units, $c = 1$.
Need four vectors $\{e_0, e_i\}, i = 1 \dots 3$ with properties

$$e_0{}^2 = 1, \qquad e_i{}^2 = -1$$
$$e_0 \cdot e_i = 0, \qquad e_i \cdot e_j = -\delta_{ij}$$

Summarised by

$$e_\mu \cdot e_\nu = \mathrm{diag}(+ - - -)\,, \quad \mu, \nu = 0 \dots 3$$

Bivectors
$4 \times 3/2 = 6$ bivectors in algebra. Two types

1. Those containing $e_0$, *e.g.* $\{e_i \wedge e_0\}$,
2. Those not containing $e_0$, *e.g.* $\{e_i \wedge e_j\}$.

# An Algebra for Spacetime II

For any pair of vectors $a$ and $b$, with $a \cdot b = 0$, have

$$(a \wedge b)^2 = abab = -abba = -a^2 b^2$$

The two types have different squares

$$(e_i \wedge e_j)^2 = -{e_i}^2 {e_j}^2 = -1$$

Spacelike Euclidean bivectors, generate rotations in a plane.

$$(e_i \wedge e_0)^2 = -{e_i}^2 {e_0}^2 = 1$$

Timelike bivectors. Generate hyperbolic geometry:

$$\begin{aligned} e^{\alpha e_1 e_0} &= 1 + \alpha e_1 e_0 + \alpha^2/2! + \alpha^3/3!\, e_1 e_0 + \cdots \\ &= \cosh\alpha + \sinh\alpha\, e_1 e_0 \end{aligned}$$

Crucial to treatment of Lorentz transformations.
Put $R = e^{\alpha/2 e_1 e_0}$, then $R$ is a rotor carrying out Lorentz boosts with velocity parameter $\alpha$ in the $x$-direction.

- Generalise this to $R = e^B$ where $B$ is any bivector in the Spacetime Algebra.
- This rotor provides general Lorentz transformations.
- Given any object $M$ in the algebra, we rotate it with $M' = RM\tilde{R}$
- Very simple!

## The Pseudoscalar

Define the pseudoscalar $I$

$$I = e_0 e_1 e_2 e_3$$

Since $I$ is grade 4, it has

$$\tilde{I} = e_3 e_2 e_1 e_0 = I$$

Compute the square of $I$ :

$$I^2 = I\tilde{I} = (e_0 e_1 e_2 e_3)(e_3 e_2 e_1 e_0) = -1$$

# An Algebra for Spacetime II

Multiply bivector by $I$, get grade $4-2=2$ — another bivector. Provides map between bivectors with positive and negative square:

$$Ie_1e_0 = e_1e_0I = e_1e_0e_0e_1e_2e_3 = -e_2e_3$$

Have four vectors, and four trivectors in algebra. Interchanged by duality

$$e_1e_2e_3 = e_0e_0e_1e_2e_3 = e_0I = -Ie_0$$

NB $I$ anticommutes with vectors and trivectors. (In space of even dimensions). $I$ always commutes with even-grade.

# An Algebra for Spacetime

Now have available the basic tool for the relativistic physics — the STA

| $1$ | $\{\gamma_\mu\}$ | $\{\gamma_\mu\wedge\gamma_\nu\}$ | $\{I\gamma_\mu\}$ | $I=\gamma_0\gamma_1\gamma_2\gamma_3$ |
|---|---|---|---|---|
| 1 | 4 | 6 | 4 | 1 |
| scalar | vectors | bivectors | trivectors | pseudoscalar |

The spacetime algebra or STA. Use the new name $\{\gamma_\mu\}$ for preferred orthonormal frame. Also define

$$\sigma_i = \gamma_i\gamma_0$$

Not used $i$ for the pseudoscalar. The $\{\gamma_\mu\}$ satisfy

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}$$

This is the Dirac matrix algebra! (with identity matrix on right). A matrix representation of the STA. Explains notation, but $\{\gamma_\mu\}$ are vectors, not a set of matrices in 'isospace'.

# The Even Subalgebra

Each inertial frame defines a set of relative vectors. These are spacetime areas swept out while moving along the velocity vector of the frame.
Therefore model these as spacetime bivectors. Take timelike vector $\gamma_0$, relative vectors $\sigma_i = \gamma_i\gamma_0$. Satisfy

$$\begin{aligned}\sigma_i\cdot\sigma_j &= \tfrac{1}{2}(\gamma_i\gamma_0\gamma_j\gamma_0 + \gamma_j\gamma_0\gamma_i\gamma_0) \\ &= \tfrac{1}{2}(-\gamma_i\gamma_j - \gamma_j\gamma_i) = \delta_{ij}\end{aligned}$$

Generators for a 3-d algebra!
This is GA of the 3-d relative space in rest frame of $\gamma_0$. Volume element

$$\sigma_1\sigma_2\sigma_3 = (\gamma_1\gamma_0)(\gamma_2\gamma_0)(\gamma_3\gamma_0) = -\gamma_1\gamma_0\gamma_2\gamma_3 = I$$

so 3-d subalgebra shares same pseudoscalar as spacetime. Note have

$$\tfrac{1}{2}(\sigma_i\sigma_j - \sigma_j\sigma_i) = \epsilon_{ijk} I\sigma_k$$

which is the algebra of the Pauli spin matrices.

# The Even Subalgebra II

Both relative vectors and relative bivectors are spacetime bivectors. Projected onto the even subalgebra of the STA.

$$1 \cdots\cdots \{\gamma_\mu\} \cdots\cdots \{\boldsymbol{\sigma}_i, I\boldsymbol{\sigma}_i\} \cdots\cdots \{I\gamma_\mu\} \cdots\cdots I \qquad 4-d$$

$$1 \qquad \{\boldsymbol{\sigma}_i\} \qquad \{I\boldsymbol{\sigma}_i\} \qquad I \qquad 3-d$$

The 6 spacetime bivectors split into relative vectors and relative bivectors. This split is observer dependent. A very useful technique.

# Electromagnetism

- The spactime vector derivative is

$$\nabla = \gamma^\mu \frac{\partial}{\partial x^\mu} = \gamma^\mu \partial_\mu$$

  which splits as

$$\nabla \gamma_0 = \partial_t - \boldsymbol{\sigma}_i \partial_i = \partial_t - \boldsymbol{\nabla}$$

  where $\boldsymbol{\nabla}$ is the relative (3-d) vector derivative
- Maxwell's equations:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

$$\boldsymbol{\nabla} \wedge \boldsymbol{E} = \partial_t(I\boldsymbol{B}) \qquad \boldsymbol{\nabla} \wedge \boldsymbol{B} = I(\boldsymbol{J} + \partial_t \boldsymbol{E})$$

- Using the geometric product, reduce to

$$\boldsymbol{\nabla}(\boldsymbol{E} + I\boldsymbol{B}) + \partial_t(\boldsymbol{E} + I\boldsymbol{B}) = \rho - \boldsymbol{J}$$

# Electromagnetism

- Defining the Lorentz-covariant field strength $F = \boldsymbol{E} + I\boldsymbol{B}$ and current $J = (\rho + \boldsymbol{J})\gamma_0$, we obtain the single, covariant equation

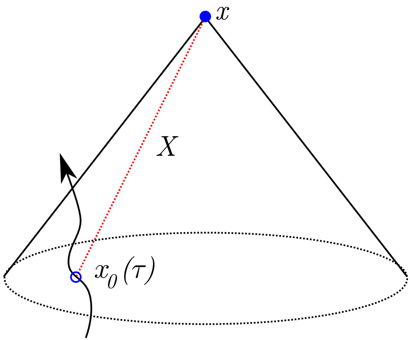
$$\nabla F = J$$

- The advantage here is not merely notational - just as the geometric product is invertible, unlike the separate dot and wedge product, the geometric product with the vector derivative is invertible (via Green's functions) where the separate divergence and curl operators are not.
- Since $\nabla \wedge F = 0$, we can introduce a vector potential $A$ such that $F = \nabla \wedge A$
- If we impose $\nabla \cdot A = 0$, so that $F = \nabla A$, then $A$ obeys the wave equation

$$\nabla F = \nabla^2 A = J$$

# Point Charge Fields

- Start with wave equation, $\nabla^2 A = J$
- Since radiation doesn't travel backwards in time, we have the electromagnetic influence propagating along the future light-cone of the charge.




- An observer at $x$ receives an influence from the intersection of their past light-cone with the charge's worldline, $x_0$, so the separation vector down the light-cone $X = x - x_0$ is null.
- In the rest frame of the charge, the potential is pure $1/r$ electrostatic, so

$$A = \frac{q}{4\pi}\frac{v}{r} = \frac{q}{4\pi}\frac{v}{X \cdot v}$$

(the Liénard-Wiechert potential)

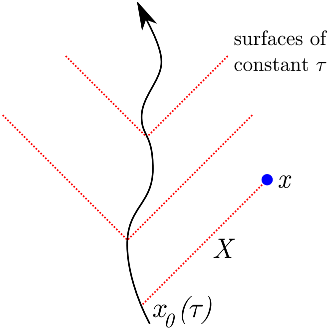
# Point Charge Fields

- Now we want to find $F = \nabla A$
- We need a few differential identities:
- Since $X^2 = 0$,

$$
\begin{aligned}
0 = \mathring{\nabla}(\mathring{X} \cdot X) &= \mathring{\nabla}(\mathring{x} \cdot X) - \mathring{\nabla}(\mathring{x_0}(\tau) \cdot X) \\
&= X - \gamma^\mu (X \cdot \partial_\mu x_0(\tau)) \\
&= X - \gamma^\mu (X \cdot (\partial_\mu \tau) \partial_\tau x_0) \\
&= X - (\nabla \tau)(X \cdot v)
\end{aligned}
$$

$$
\Rightarrow \quad \nabla \tau = \frac{X}{X \cdot v}
$$

where we treat $\tau$ as a scalar field, with its value at $x_0(\tau)$ being extended over the charge's forward light-cone



# Point Charge fields

- To differentiate $X$, we need

$$\nabla x_0(\tau) = \gamma^\mu \partial_\mu x_0(\tau) = \gamma^\mu (\partial_\mu \tau) \partial_\tau x_0 = (\nabla \tau) v$$

- Since $A \propto v/(X \cdot v)$ we also want

$$\begin{aligned}
\nabla(X \cdot v) &= \mathring{\nabla}(\mathring{X} \cdot v) + \mathring{\nabla}(X \cdot \mathring{v}) \\
&= \mathring{\nabla}(\mathring{x} \cdot v) - \mathring{\nabla}(x_0(\mathring{\tau}) \cdot v) + \mathring{\nabla}(X \cdot \mathring{v}) \\
&= v - \nabla\tau(v \cdot v) + \nabla\tau(X \cdot \dot{v}) \\
&= v - \frac{X}{X \cdot v} + \frac{X(X \cdot \dot{v})}{X \cdot v}
\end{aligned}$$

where the over-circles denote the term being differentiated

# Point Charge Fields

- Now:

$$
\begin{aligned}
F = \nabla A = \frac{q}{4\pi}\left(\frac{\nabla v(\tau)}{X\cdot v} - \frac{1}{(X\cdot v)^2}(\nabla(X\cdot v))v\right) \\
= \frac{q}{4\pi}\left(\frac{(\nabla\tau)\dot{v}}{X\cdot v} - \frac{1}{(X\cdot v)^3}((X\cdot v)v - X + X(X\cdot\dot{v}))v\right) \\
= \frac{q}{4\pi}\left(\frac{X\dot{v}}{(X\cdot v)^2} - \frac{1}{(X\cdot v)^2} - \frac{(X(X\cdot\dot{v}) - X)}{(X\cdot v)^3}v\right) \\
= \frac{q}{4\pi}\left(\frac{X\wedge\dot{v}}{(X\cdot v)^2} + \frac{X\wedge v - (X\cdot\dot{v})X\wedge v}{(X\cdot v)^3}\right)
\end{aligned}
$$

since $F$ is a pure bivector

- Using

$$
(X\cdot v)X\wedge\dot{v} - (X\cdot\dot{v})X\wedge v = -X(X\cdot(\dot{v}\wedge v)) = \frac{1}{2}X(\dot{v}\wedge v)X
$$

get (with $\Omega_v = \dot{v}\wedge v$)

$$
F = \frac{q}{4\pi}\frac{X\wedge v + \frac{1}{2}X\Omega_v X}{(X\cdot v)^3}
$$

# Point Charge Fields

$$F = \frac{q}{4\pi}\frac{X \wedge v + \frac{1}{2}X\Omega_v X}{(X \cdot v)^3}$$

- Equation displays clean split into Coulomb field in rest frame of charge, and radiation term

$$F_{rad} = \frac{q}{4\pi}\frac{\frac{1}{2}X\Omega_v X}{(X \cdot v)^3}$$

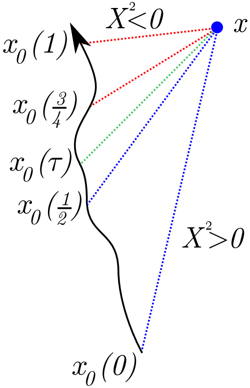
proportional to rest-frame acceleration projected down the null vector $X$.

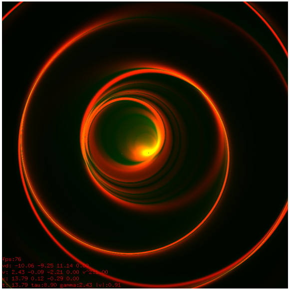
- $X \cdot v$ is distance in rest-frame of charge, so $F_{rad}$ goes as $1/\mathrm{distance}$, and energy-momentum tensor $T(a) = -\frac{1}{2}FaF$ drops off as $1/\mathrm{distance}^2$. Thus the surface integral of $T$ doesn't vanish at infinity - energy-momentum is carried away from the charge by radiation.

# Point Charge Fields

For a numerical solution:

- Store particle's history (position, velocity, acceleration)
- To calculate the fields at $x$, find the null vector $X$ by bisection search (or similar)
- Retrieve the particle velocity, acceleration at the corresponding $\tau$ - above formulae give us $A$ and $F$



# Quantum Theory

- The algebraic structure of wave mechanics arises naturally from the geometric algebra of spacetime
- Allows us to reformulate standard QM in more geometrical way
- Also suggests new lines of interpretation ...

# Non-relativistic spin

- For a spin-$\frac{1}{2}$ particle, the operator returning the spin along the $\boldsymbol{\sigma}_k$ axis (where $\{\boldsymbol{\sigma}_k\}$ is an orthonormal frame for 3-space) is $\hat{s}_k = \frac{1}{2}\hbar\hat{\sigma}_k$, where $\hat{\sigma}_k$ are the Pauli matrices:

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- These have commutation relations $\hat{\sigma}_i\hat{\sigma}_j = \delta_{ij}\mathbf{I} + i\epsilon_{ijk}\hat{\sigma}_k$ (where $\mathbf{I}$ is the identity matrix)
- But working entirely with the vectors $\boldsymbol{\sigma}_k$, we have

$$\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j = \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j + \boldsymbol{\sigma}_i \wedge \boldsymbol{\sigma}_j = \delta_{ij} + I\epsilon_{ijk}\boldsymbol{\sigma}_k$$

  where $I = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$ is the pseudoscalar.
- Pauli matrices are just a representation of the geometric algebra of 3-space! (with the pseudoscalar acting as the unit imaginary)

# Non-relativistic spin

- Operators (such as Pauli matrices) act on the wavefunction, which is (in the non-relativistic case) is a Pauli spinor, with two complex coefficients:

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

- Natural question - can we represent this 4-DoF object as a multivector, acted on by the $\boldsymbol{\sigma}_k$ vectors? Simplest choice is

$$|\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \quad \leftrightarrow \quad \psi = a^0 + a^k I \boldsymbol{\sigma}_k$$

so a Pauli spinor corresponds to a weighted spatial rotor! The spin up and spin down states correspond to

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightarrow 1 \quad , \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftrightarrow -I\boldsymbol{\sigma}_2$$

# Non-relativistic spin

- Actions of Pauli operators are

$$\hat{\sigma}_k|\psi\rangle \leftrightarrow \boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3$$

where $\boldsymbol{\sigma}_3$ acts to keep result in the even subalgebra.

- Multiplication by unit imaginary is equivalent to multiplication by bivector $I\boldsymbol{\sigma}_3$

$$i|\psi\rangle = \psi I\boldsymbol{\sigma}_3$$

- Hermitian adjoint corresponds to 3-d reversion
- Quantum inner product is given by

$$\langle\psi|\phi\rangle \leftrightarrow \langle\psi^\dagger\phi\rangle - \langle\psi^\dagger\phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3$$

which projects out the 1 and $I\boldsymbol{\sigma}_3$ components of $\psi^\dagger\phi$

# Non-relativistic spin

- Expectation value of spin in $k$-direction is

$$\langle\psi|\hat{\sigma}_k|\psi\rangle \leftrightarrow \langle\psi^\dagger \boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3\rangle - \langle\psi^\dagger \boldsymbol{\sigma}_k \psi I\rangle I\boldsymbol{\sigma}_3$$
$$= \langle\boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3 \psi^\dagger\rangle$$

since $\psi^\dagger \boldsymbol{\sigma}_k \psi$ is a vector.

- Defining the spin vector,

$$\boldsymbol{s} = \frac{1}{2}\hbar\psi\boldsymbol{\sigma}_3\psi^\dagger$$

this reduces to

$$\langle\psi|\hat{s}_k|\psi\rangle \leftrightarrow \frac{1}{2}\hbar\langle\boldsymbol{\sigma}_k\psi\boldsymbol{\sigma}_3\psi^\dagger\rangle = \boldsymbol{\sigma}_k \cdot \boldsymbol{s}$$

So "forming the expectation value of the $s_k$ operator" reduces to projecting out the $\boldsymbol{\sigma}_k$ component of the vector $\boldsymbol{s}$

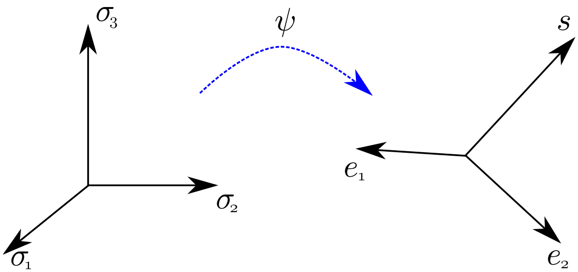
- What exactly is the status of $\boldsymbol{s}$? Often hear things like:

  > *It turns out that the spin vector is not very useful in actual quantum mechanical calculations, because it cannot be measured directly – $s_x$, $s_y$ and $s_z$ cannot possess simultaneous definite values, because of a quantum uncertainty relation between them.*

  [Wikipedia]

# Non-relativistic spin

- Remembering that Pauli spinors are weighted rotors, we can write $\psi = \rho^{1/2} R$, so that $\boldsymbol{s} = \frac{1}{2}\hbar\rho R \boldsymbol{\sigma}_3 R^\dagger$
- Picking out $\boldsymbol{\sigma}_3$ is a result of chosing the $\hat{\sigma}_3$ matrix to be diagonal, and doesn't break the rotational symmetry of the theory. In rigid-body dynamics, we often choose an arbitrary reference configuration, and formulate the dynamics in terms of the transformation needed to rotate this configuration to the physical one.
- The situation here is analogous - we could have chosen any constant vector, and made $\psi$ so that it transformed this into $\boldsymbol{s}$



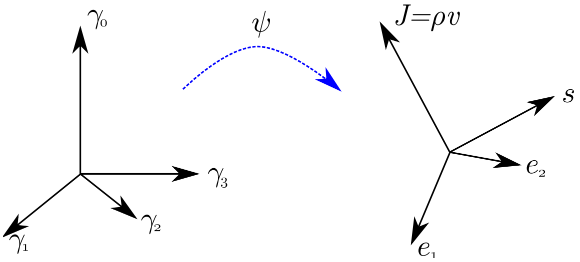
# Dirac Theory

- In the relativistic theory of spin-$\frac{1}{2}$ particles, things are similar
- Instead of the wavefunction being a weighted spatial rotor, it's now a full Lorentz spinor:

$$\psi = \rho^{1/2} e^{I\beta/2} R$$

with the addition of a slightly mysterious $\beta$ term related to antiparticle states.

- Five observables in all, including the current, $J = \psi\gamma_0\psi = \rho R\gamma_0\tilde{R}$, and the spin vector $s = \psi\gamma_3\psi = \rho R\gamma_3\tilde{R}$



# Dirac Theory

- The wavefunction obeys the Dirac equation:

$$\nabla \psi I \boldsymbol{\sigma}_3 - eA\psi = m\psi\gamma_0$$

- This implies that the current $J$ is conserved,

$$\nabla \cdot J = 0$$

with the implication that a fermion cannot be created or destroyed (pair annihilation / production are multiparticle processes, not covered by the Dirac equation)

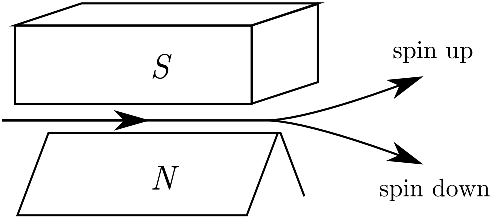
- The timelike component of J is positive definite, and is interpreted as a probability density: a normalised wavefunction has

$$\int d^3x \, J_0 = 1$$

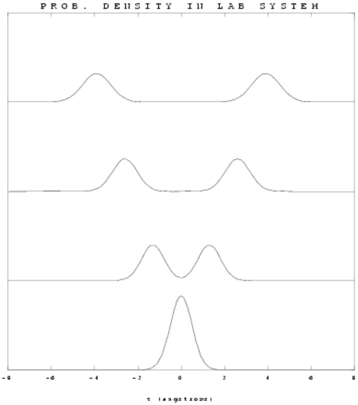
- Conservation of $J$ implies that the probability density “flows” along non-intersecting streamlines - useful for visualisation.

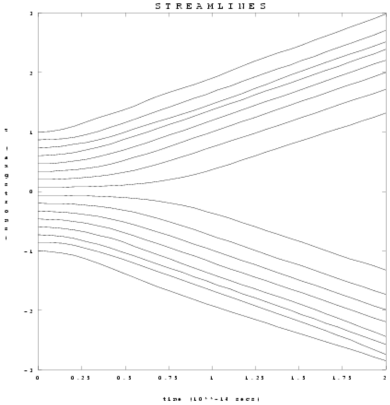
# Dirac Theory

- A sample application - Stern-Gerlach apparatus
- Apply a delta-function magnetic field gradient to simulate the apparatus, and numerically calculate the effect of this shock on a wave-packet, with spin initially orthogonal to the magnetic field
- Result : the wave-packet splits into two parts, spins aligned/anti-aligned with the magnetic shock, with streamlines bifurcating depending
- Instead of viewing the device as ‘measuring’ the spin in z direction, and obtaining one of two eigenstates, the apparatus acts as a spin polariser, forcing the spins to align with the magnetic shock

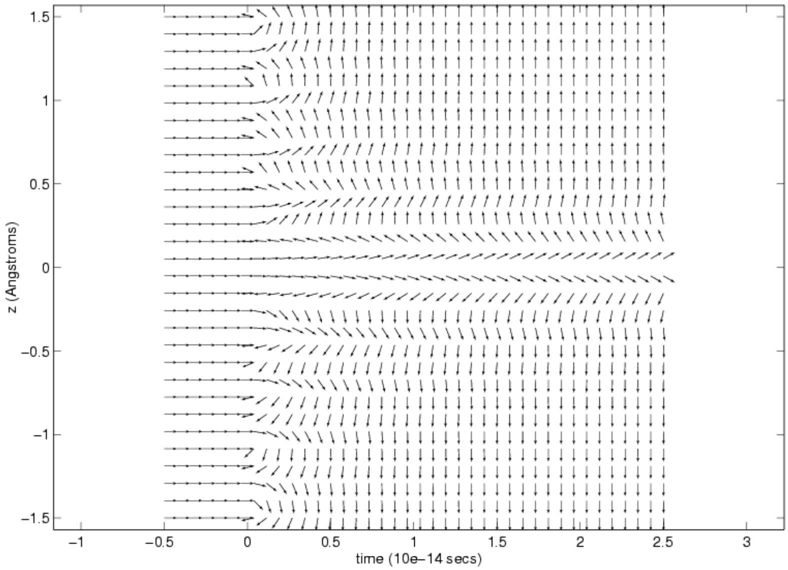


# Dirac Theory



# Dirac Theory



# Dirac Theory — Tunnelling through barriers

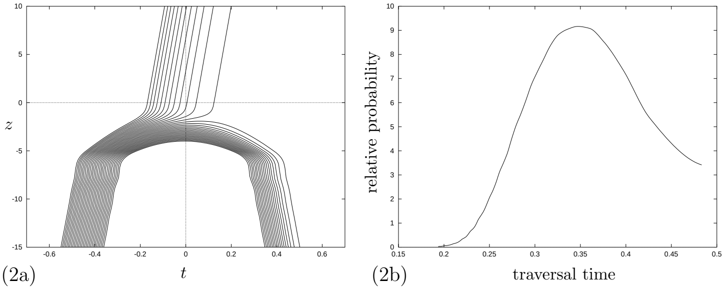
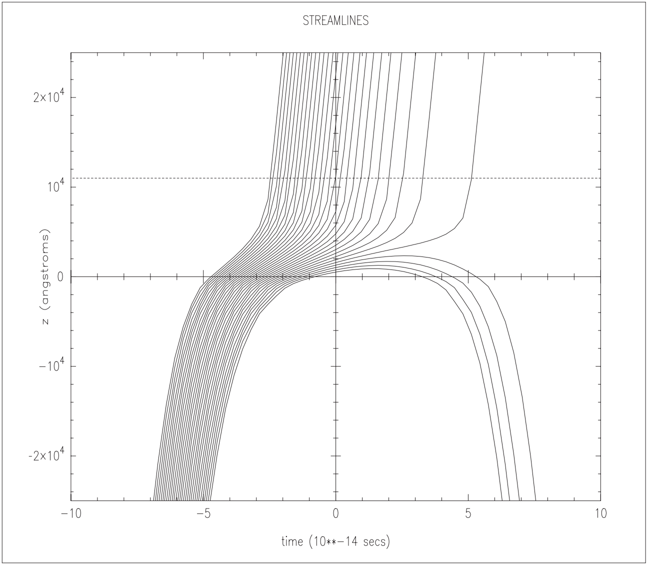



Figure 2: *Particle streamlines and time spent in the barrier. Figure 2a shows the streamlines for the front of the wavepacket, indicating that only the streamlines from the front of the packet cross the barrier. Each streamline slows down as it passes through the barrier. Figure 2b is a histogram of time the streamlines spend in the barrier. Distance is measured in* Å *and time in* $10^{-14}$s.

# Photon Tunnelling - no Superluminality!



# What is Gauge Theory Gravity?

- This is a version of gravity that aims to be as much like our best descriptions of the other 3 forces of nature:
  - The strong force (which binds nuclei)
  - The weak force (e.g. responsible for radiactivity)
  - electromagnetism
- These are all described in terms of Yang-Mills type gauge theories (unified in quantum chromodynamics) in a flat spacetime background
- In the same way, Gauge Theory Gravity (GTG) is expressed in a flat spacetime
- Has two gauge fields
- One corresponds to invariance under arbitrary remappings of spacetime onto itself
- The other corresponds to invariance under local rotations at a point
- Standard GR cannot even see changes of the latter type, since metric is invariant under such changes

# What is Gauge Theory Gravity? (contd.)

- Advantages of GTG include being clear about what the physical predictions of the theory are (since a gauge theory)
- Conceptually simpler that standard GR (since works in a flat space background)
- Linked with this, all the tools of flat spacetime Geometric Algebra are available (rotors, integral theorems, etc.)
- Locally, theory reproduces predictions of an extension of GR known as Einstein-Cartan theory (incorporates quantum spin)
- Differs on global issues such as nature of horizons, and topology
- A very big advantage, is that since it is as much like other forces and gauge theories as possible, can start to do quantum calculations is similar ways as in these
- With colleagues have carried out the first calculations of this kind:

# Gravitational atoms, bound states and scattering

- The gravitational equivalent of the Hydrogen atom — electrons forming bound states with a black hole
- Can get out whole spectrum of states – black hole spectroscopy! (Lasenby *et al.*, Physical Review D, **72**, 105014 (2005))
- Also can look at scatterring - we have produced the cross section for the first interesting Feynman diagram of an electron interacting with a black hole (gravitational Mott scattering) (Doran + Lasenby, PRD, **74**, 064005 (2006))
- Bremsstrahlung is next to carry out - may solve longstanding problem of radiation (or not) of freely falling electrons

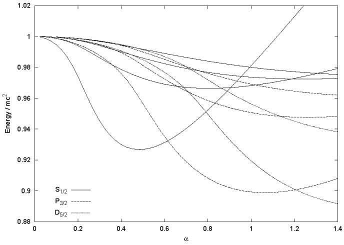



Figure 2: The energy spectra of the $S_{1/2}$, $P_{3/2}$ and $D_{5/2}$ states. At around $\alpha = 0.6$, the $2P$ state becomes the ground state, and beyond $\alpha = 1.2$ it is replaced by the $3D$ state.