Geometric Algebra

1. Geometric Algebra in 2 Dimensions

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Introduction

Present GA as a new mathematical technique

Introduce techniques through their applications

Emphasise the generality and portability of GA

Promote a cross-disciplinary view of science

- Rotations in arbitrary dimensions
- Lorentz transformations
- Lie groups
- Analytic functions
- Unifying Maxwell's equations
- Projective and conformal geometries
- Coding with GA

Resources

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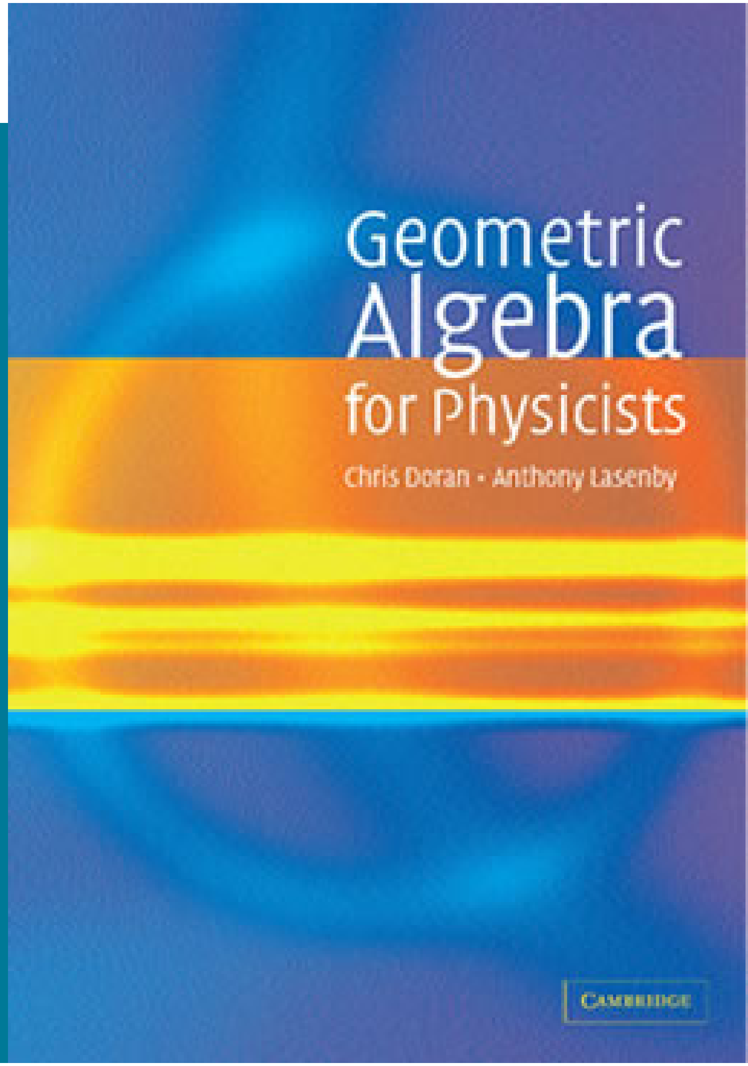
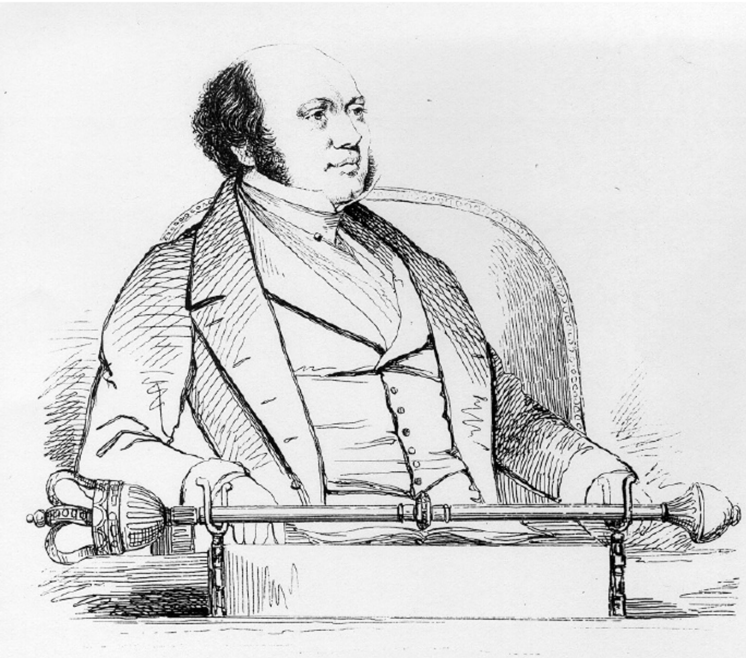
#geometricalgebra
github.com/ga



Some history

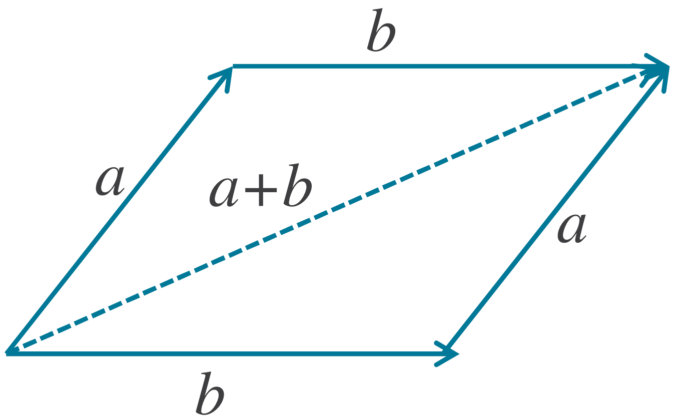


William Hamilton

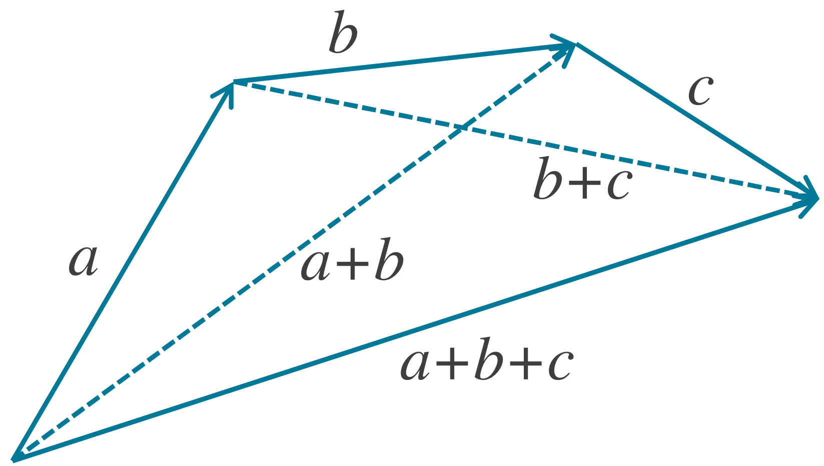
Hermann Grassmann

William Clifford

Vectors and Vector Spaces



$$a + b = b + a$$



$$a + (b + c) = (a + b) + c$$

$$\lambda(a + b) = \lambda a + \lambda b$$

$$(\lambda + \mu)a = \lambda a + \mu a$$

What is a vector?

This is not a vector: $[1.0, 2.0, 3.0]$

This is a vector: $1.0\boldsymbol{e}_1 + 2.0\boldsymbol{e}_2 + 3.0\boldsymbol{e}_3$

For this course a grade-1 tensor and a vector are different.
We will usually focus on active transformations, not passive ones

The problem

How do you multiply two vectors together?

Inner product

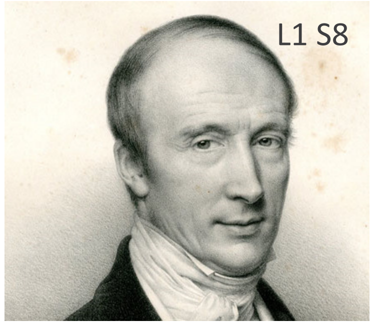
$$\boldsymbol{a}\cdot\boldsymbol{b} = ab\cos\theta$$
$$= \sum_i a_i b_i$$

Cross product

$$\boldsymbol{a}\times\boldsymbol{b} = ab\sin\theta\,\boldsymbol{n}$$

$\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{n}\}$ Right-handed set

Complex numbers

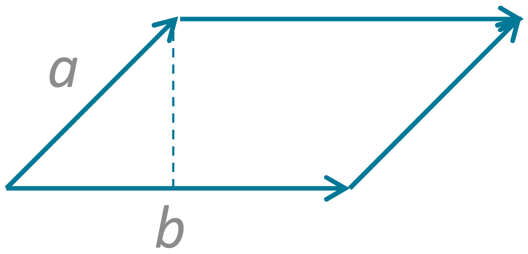


Complex numbers give us a potential product in 2D. If we form *aa** we get the length^2 of the vector.

This suggests forming *ab**.

The result contains an inner product and an area term.

$$\begin{aligned} \boldsymbol{ab}^* &= (a_1 + a_2 i)(b_1 - b_2 i) \\ &= a_1 b_1 + a_2 b_2 \\ &\quad + i(a_2 b_1 - a_1 b_2) \\ &= ab\cos\theta + iab\sin\theta \end{aligned}$$



Quaternions

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ab = -a \cdot b + a \times b$$

Generalises complex numbers, introduced the cross product and some notation still in use today.

Confusion over status of vectors, but quaternions are very powerful for describing rotations.

Quaternion properties

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$

$$\boldsymbol{ij} = -\boldsymbol{ijkk} = \boldsymbol{k}$$

$$\boldsymbol{ji} = -\boldsymbol{jiijk} = \boldsymbol{jjk} = -\boldsymbol{k}$$

Generators anti-commute

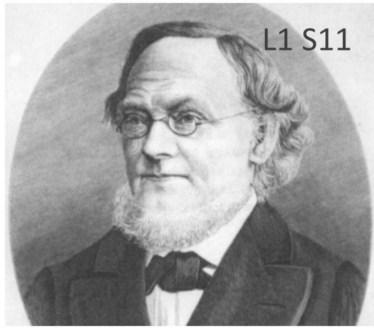
$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \quad \boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$$

$$\boldsymbol{ab} = c_0 + \boldsymbol{c}$$

$$\boldsymbol{c} = (a_2b_3 - b_2a_3)\boldsymbol{i} + (a_3b_1 - b_3a_1)\boldsymbol{j} + (a_1b_2 - b_1a_2)\boldsymbol{k}$$

The cross product

Grassmann algebra



German schoolteacher (1809-1877) who struggled for recognition in his own lifetime.

Published the *Lineale Ausdehnungslehre* in 1844.

Introduced a new, outer product that is *antisymmetric*.

$$\boldsymbol{a} \wedge \boldsymbol{b} = -\boldsymbol{b} \wedge \boldsymbol{a} \qquad \boldsymbol{a} \wedge \boldsymbol{a} = 0$$

$$\begin{aligned} \boldsymbol{a} &= a_1 \boldsymbol{e}_1 + a_2 \boldsymbol{e}_2 \\ \boldsymbol{b} &= b_1 \boldsymbol{e}_1 + b_2 \boldsymbol{e}_2 \end{aligned}$$

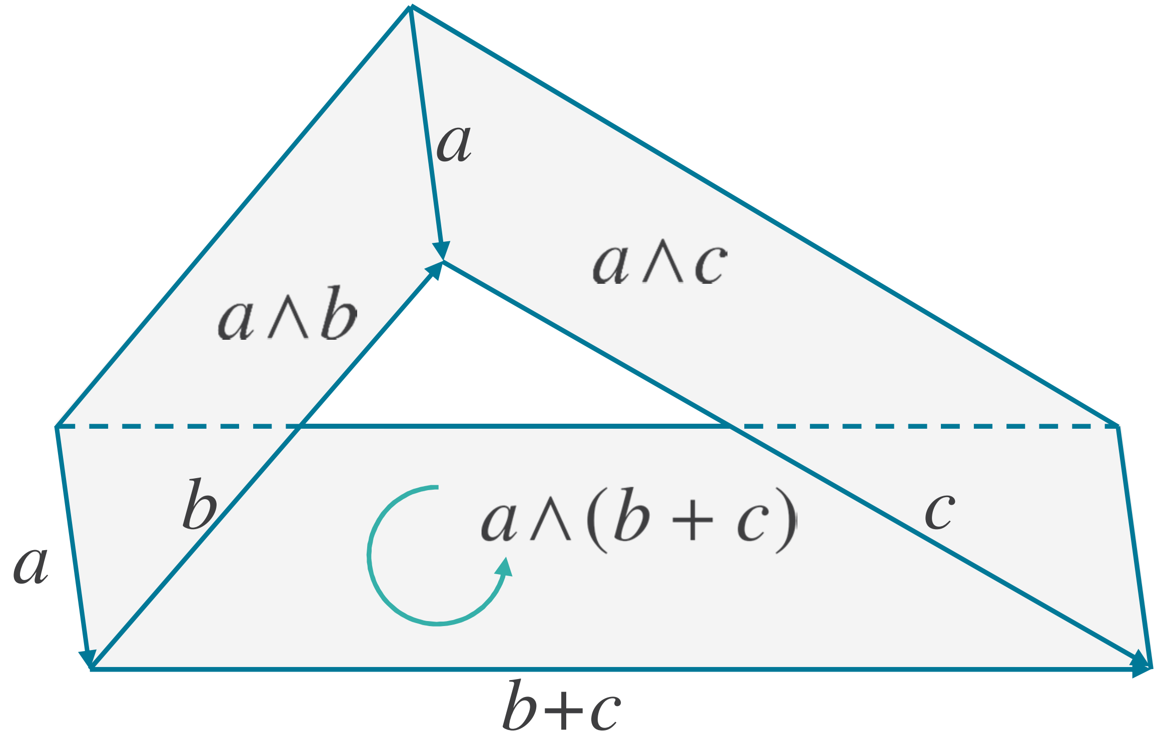
$$\begin{aligned} \boldsymbol{a} \wedge \boldsymbol{b} &= a_1 b_2 \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 + a_2 b_1 \boldsymbol{e}_2 \wedge \boldsymbol{e}_1 \\ &= (a_1 b_2 - a_2 b_1) \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \end{aligned}$$

Properties of the outer product

The result of the outer product is a new object: a BIVECTOR

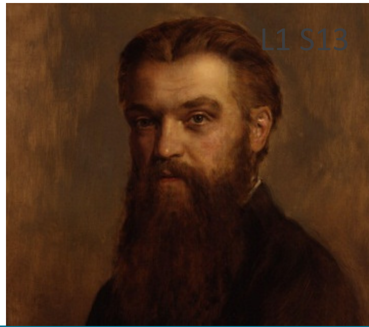
Bivectors form a linear space

We can visualise bivector addition in 3D



$$a \wedge (b + c) = a \wedge b + a \wedge c$$

Geometric algebra



$$\boldsymbol{ab} = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\wedge\boldsymbol{b}$$

W.K. Clifford (1845 – 1879) introduced the *geometric* product of two vectors.

The product of two vectors is the sum of a *scalar* and a *bivector*.

Think of the sum as like the real and imaginary parts of a complex number.

The geometric product

The geometric product is associative and distributive.

$$a(bc) = (ab)c = abc$$

$$a(b + c) = ab + ac$$

The square of any vector is a scalar. This makes the product invertible.

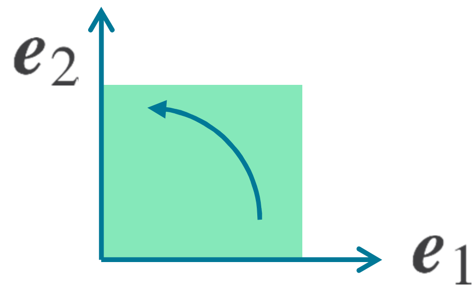
$$(a + b)^2 = a^2 + b^2 + (ab + ba)$$

Define the inner (scalar) and outer products in terms of the geometric product.

$$a \cdot b = \tfrac{1}{2}(ab + ba)$$

$$a \wedge b = \tfrac{1}{2}(ab - ba)$$

Two dimensions



2D sufficient to understand basic results. Construct an orthonormal basis.

$$e_1 \cdot e_1 = e_2 \cdot e_2 = 1$$
$$e_1 \cdot e_2 = 0$$

Parallel vectors commute.

$$\begin{aligned} e_1 e_1 &= e_1 \cdot e_1 + e_1 \wedge e_1 \\ &= 1 \end{aligned}$$

Orthogonal vectors anticommute.

$$\begin{aligned} e_1 e_2 &= e_1 \cdot e_2 + e_1 \wedge e_2 \\ &= -e_2 \wedge e_1 = -e_2 e_1 \end{aligned}$$

The bivector

The unit bivector has negative square.

Follows purely form the axioms of geometric algebra.

We have not said anything about complex numbers, or solving polynomial equations.

We have invented complex numbers!

$$
\begin{aligned}
(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2)^2 &= (\boldsymbol{e}_1 \boldsymbol{e}_2)(\boldsymbol{e}_1 \boldsymbol{e}_2) \\
&= \boldsymbol{e}_1 \boldsymbol{e}_2 \boldsymbol{e}_1 \boldsymbol{e}_2 \\
&= -\boldsymbol{e}_1 \boldsymbol{e}_1 \boldsymbol{e}_2 \boldsymbol{e}_2 \\
&= -\boldsymbol{e}_2 \boldsymbol{e}_2 \\
&= -1
\end{aligned}
$$

Unification

Complex numbers arise naturally in the geometric algebra of the plane.

Products in 2D

Call the highest grade element the pseudoscalar: $I = \boldsymbol{e}_1\boldsymbol{e}_2$

$$I\boldsymbol{e}_1 = (-\boldsymbol{e}_2\boldsymbol{e}_1)\boldsymbol{e}_1 = -\boldsymbol{e}_2\boldsymbol{e}_1\boldsymbol{e}_1 = -\boldsymbol{e}_2$$

$$I\boldsymbol{e}_2 = (\boldsymbol{e}_1\boldsymbol{e}_2)\boldsymbol{e}_2 = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_2 = \boldsymbol{e}_1$$

A 90 degree rotation clockwise (negative sense)

$$\boldsymbol{e}_1 I = \boldsymbol{e}_1(\boldsymbol{e}_1\boldsymbol{e}_2) = \boldsymbol{e}_2$$

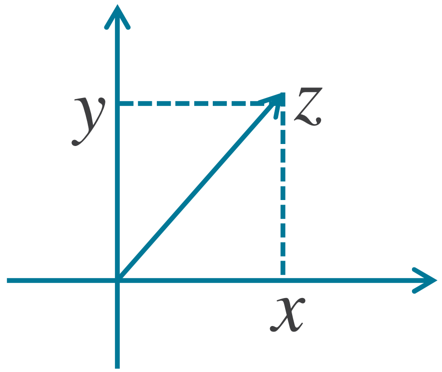
$$\boldsymbol{e}_2 I = \boldsymbol{e}_2(\boldsymbol{e}_1\boldsymbol{e}_2) = -\boldsymbol{e}_1$$

A 90 degree rotation anti-clockwise (positive sense)

Rotating through 90^{o} twice is a 180 ^{o} rotation. Equivalent to multiplication by -1 in 2D.

Complex numbers and vectors

$$z = x + y\boldsymbol{e}_1\boldsymbol{e}_2 = x + Iy$$



Want to map between complex numbers and vectors

$$\boldsymbol{x} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2$$

Answer is straightforward:

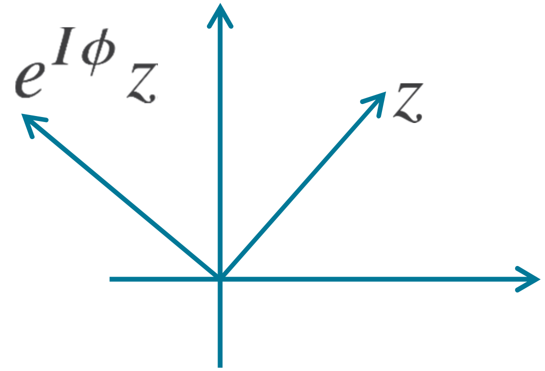
$$\boldsymbol{e}_1\boldsymbol{x} = x + y\boldsymbol{e}_1\boldsymbol{e}_2 = x + Iy = z$$

The real axis transforms between vectors and even elements in 2D GA
This only works is 2D

Rotations

Complex numbers are efficient for handling rotations in 2D

$$z \mapsto e^{I\phi} z$$



In terms of vectors:

$$\boldsymbol{x} = \boldsymbol{e}_1 z \mapsto \boldsymbol{x}' = \boldsymbol{e}_1 z'$$

$$\boldsymbol{x}' = \boldsymbol{e}_1 e^{I\phi} z = e^{-I\phi} \boldsymbol{e}_1 z = e^{-I\phi} \boldsymbol{x}$$

NB anti-commutation of *I* with vectors puts minus sign in exponent. See next lecture.

$$\boldsymbol{x}' = e^{-I\phi} \boldsymbol{x} = \boldsymbol{x} e^{I\phi}$$

First example of how rotations are handled efficiently in GA

Resources

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