



Geometric Algebra

2. Geometric Algebra in 3 Dimensions

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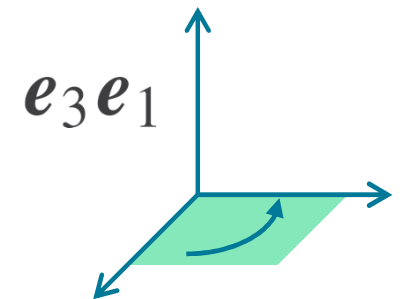
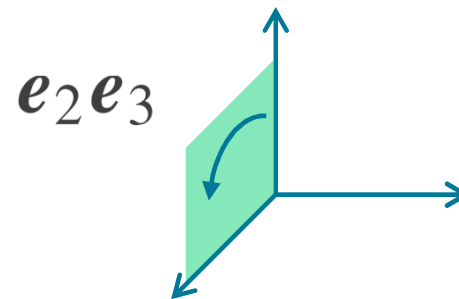
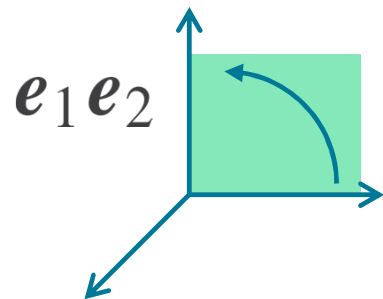
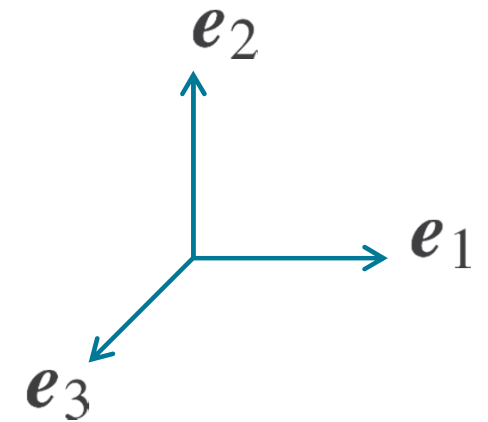
Three dimensions

Introduce a third vector.

These all anticommute.

$$\{e_1, e_2, e_3\}$$

$$e_1 e_3 = -e_3 e_1 \quad \dots$$



Bivector products

The product of a vector and a bivector can contain two different terms.

The product of two perpendicular bivectors results in a third bivector.

Now define i , j and k . We have discovered the quaternion algebra buried in 3 (not 4) dimensions

$$\mathbf{e}_1(\mathbf{e}_1\mathbf{e}_2) = \mathbf{e}_2$$

$$\mathbf{e}_1(\mathbf{e}_2\mathbf{e}_3) = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$$

$$\begin{aligned} (\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_2\mathbf{e}_3) &= \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_3 \\ &= \mathbf{e}_1\mathbf{e}_3 = -\mathbf{e}_3\mathbf{e}_1 \end{aligned}$$

$$i = -\mathbf{e}_2\mathbf{e}_3, \quad j = -\mathbf{e}_3\mathbf{e}_1, \quad k = -\mathbf{e}_1\mathbf{e}_2$$

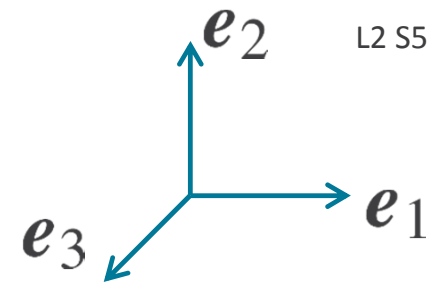
$$i^2 = j^2 = k^2 = ijk = -1$$



Unification

Quaternions arise naturally in the geometric algebra of space.

The directed volume element



$$I = e_1 e_2 e_3$$

Has negative square

$$\begin{aligned} I^2 &= e_1 e_2 e_3 e_1 e_2 e_3 \\ &= e_1 e_2 e_1 e_2 = -1 \end{aligned}$$

Commutates with vectors

$$\begin{aligned} e_1 I &= e_1 e_1 e_2 e_3 \\ &= -e_1 e_2 e_1 e_3 \\ &= e_1 e_2 e_3 e_1 = I e_1 \end{aligned}$$

Swaps lines and planes

$$\begin{aligned} I e_1 &= e_2 e_3 \\ I e_2 e_3 &= -e_1 \end{aligned}$$

3D Basis



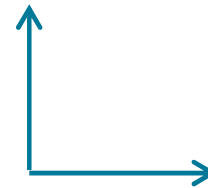
Grade 0
1 Scalar

1



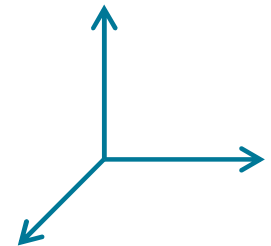
Grade 1
3 Vectors

$\{e_i\}$



Grade 2
3 Plane / bivector

$\{e_i \wedge e_j\}$



Grade 3
1 Volume / trivector

$e_1 \wedge e_2 \wedge e_3$

A linear space of dimension 8

Note the appearance of the binomial coefficients - this is general
General elements of this space are called multivectors

Products in 3D

$$ab = a \cdot b + a \wedge b \qquad a = \sum_{i=1}^3 a_i \mathbf{e}_i \qquad b = \sum_{i=1}^3 b_i \mathbf{e}_i$$

$$\begin{aligned} a \wedge b = & (a_2 b_3 - b_3 a_2) \mathbf{e}_2 \wedge \mathbf{e}_3 + (a_3 b_1 - a_1 b_3) \mathbf{e}_3 \wedge \mathbf{e}_1 \\ & + (a_1 b_2 - a_2 b_1) \mathbf{e}_1 \wedge \mathbf{e}_2 \end{aligned}$$

We recover the cross product from duality:
Can only do this in 3D

$$a \times b = -I a \wedge b$$



Unification

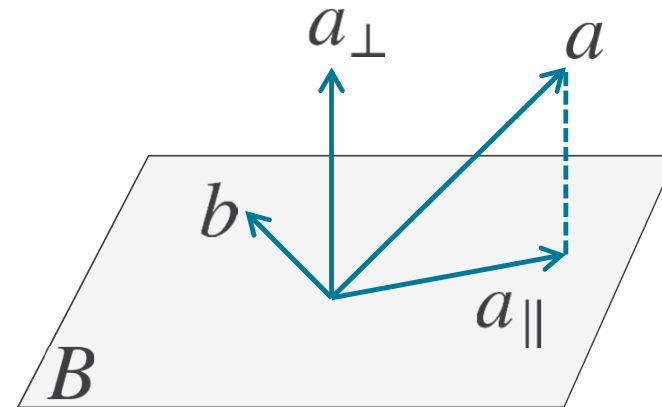
The cross product is a disguised form of the outer product in three dimensions.

Vectors and bivectors

Decompose vector
into terms into and
normal to the plane

$$\mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp}$$

$$\mathbf{B} = \mathbf{a}_{\parallel} \wedge \mathbf{b}$$



$$\mathbf{a}_{\parallel} \mathbf{B} = \mathbf{a}_{\parallel} (\mathbf{a}_{\parallel} \wedge \mathbf{b}) = \mathbf{a}_{\parallel} (\mathbf{a}_{\parallel} \mathbf{b}) = (\mathbf{a}_{\parallel})^2 \mathbf{b}$$

A **vector** lying in the
plane

$$\mathbf{a}_{\perp} \mathbf{B} = \mathbf{a}_{\perp} (\mathbf{a}_{\parallel} \wedge \mathbf{b}) = \mathbf{a}_{\perp} \mathbf{a}_{\parallel} \mathbf{b}$$

Product of three orthogonal vectors,
so a **trivector**

Vectors and bivectors

Write the combined product: $aB = a \cdot B + a \wedge B$



Lowest grade Highest grade

$$a \cdot B = a_{\parallel}^2 b = -(a_{\parallel} b) a_{\parallel} = -B \cdot a$$

Inner product is antisymmetric, so **define**

$$a \cdot B = \frac{1}{2}(aB - Ba)$$

This always returns a vector

With a bit of work, prove that

$$a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b$$

A very useful result. Generalises the vector triple product.

Vectors and bivectors

Symmetric component of product gives a **trivector**:

$$\mathbf{a} \wedge \mathbf{B} = \frac{1}{2}(\mathbf{a}\mathbf{B} + \mathbf{B}\mathbf{a}) = \mathbf{B} \wedge \mathbf{a}$$

Can defined the outer product of three vectors

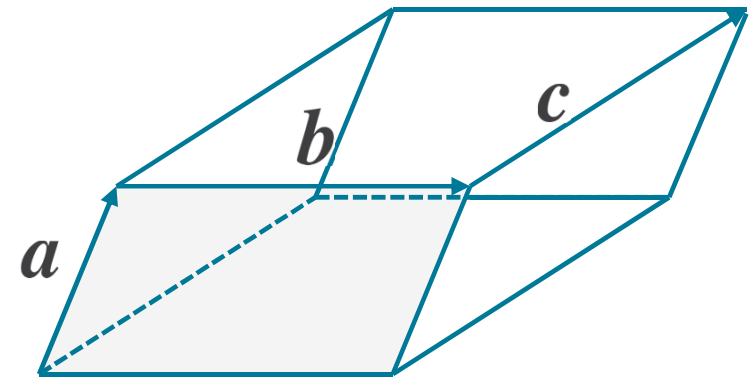
$$\begin{aligned}\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) &= \langle \mathbf{a}(\mathbf{b} \wedge \mathbf{c}) \rangle_3 \\ &= \langle \mathbf{a}(\mathbf{bc} - \mathbf{b} \cdot \mathbf{c}) \rangle_3\end{aligned}$$

Vector part does not contribute

$$\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = \langle \mathbf{a}(\mathbf{bc}) \rangle_3 = \langle \mathbf{abc} \rangle_3$$

The outer product is associative

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$$

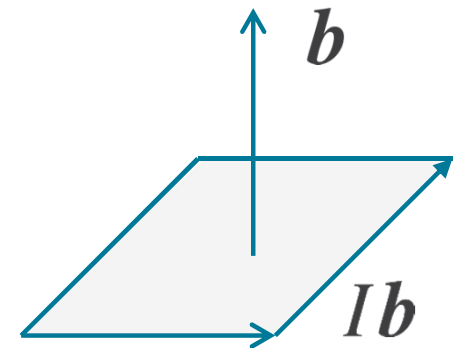


Duality

Seen that the pseudoscalar
interchanges planes and vectors in 3D

$$\mathbf{e}_1 \mathbf{e}_2 = I \mathbf{e}_3$$

$$I \mathbf{e}_1 \mathbf{e}_2 = -\mathbf{e}_3$$



Can use this in 3D to understand
product of a vector and a bivector

$$B = I\mathbf{b}$$

$$\mathbf{a}B = \mathbf{a}(I\mathbf{b}) = I\mathbf{a}\mathbf{b} = I(\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b})$$

Symmetric part is a trivector

$$\mathbf{a} \wedge B = I(\mathbf{a} \cdot \mathbf{b}) = \frac{1}{2}(\mathbf{a}B + B\mathbf{a})$$

Antisymmetric part is a vector

$$\mathbf{a} \cdot B = I(\mathbf{a} \wedge \mathbf{b}) = \frac{1}{2}(\mathbf{a}B - B\mathbf{a})$$

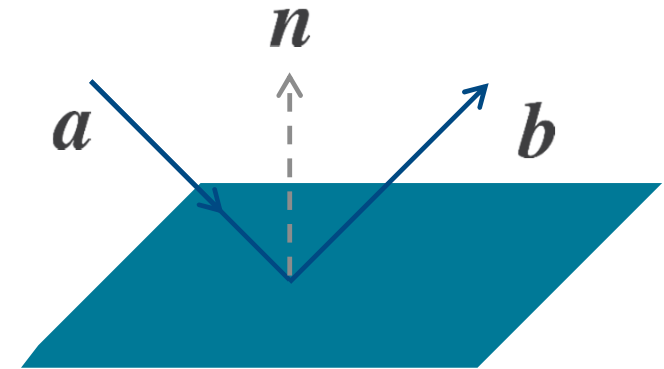
Reflections

See the power of the geometric product when looking at operations.

Decompose a into components into and out of the plane.

Form the reflected vector

Now re-express in terms of the geometric product.



$$a_{\perp} = (a \cdot n)n$$

$$a_{\parallel} = a - (a \cdot n)n$$

$$b = a_{\parallel} - a_{\perp}$$

$$b = a - 2(a \cdot n)n$$

$$= a - (an + na)n = -nan$$

Rotations

Two reflections generate a rotation.

Define a *rotor* R . This is formed from a geometric product!

Rotations now formed by

This works for higher grade objects as well. Will prove this later.

$$a \mapsto -m(-nan)m \\ = mnamm$$

$$R = mn$$

$$a \mapsto Ra\tilde{R}$$

$$A \mapsto RA\tilde{R}$$

Rotors in 3D

$$R = mn$$

Rotors are even grade, so built out of a scalar and the three bivectors.

These are the terms that map directly to quaternions.

Rotors are normalised.

$$R\tilde{R} = mn\tilde{n}\tilde{m} = 1$$

Reduces the degrees of freedom from 4 to 3.

This is precisely the definition of a unit quaternion.

Rotors are elements of a 4-dimensional space normalised to 1.

They live on a 3-sphere.

This is the GROUP MANIFOLD.

Exponential form

$$R = \boldsymbol{m} \boldsymbol{n} = \cos \theta + \boldsymbol{m} \wedge \boldsymbol{n}$$

Use the following
useful, general result.

$$\begin{aligned} (a \wedge b)^2 &= (ab - a \cdot b)(a \cdot b - ba) \\ &= -a^2 b^2 + a \cdot b(ab + ba - a \cdot b) \\ &= (a \cdot b)^2 - a^2 b^2 \\ &= -a^2 b^2 \sin^2 \theta \end{aligned}$$

Polar decomposition $R = \cos \theta + \sin \theta \hat{B} = e^{\theta \hat{B}}$

Exponential form

Sequence of two reflections gives a rotation through twice the angle between the vectors

$$\begin{aligned}
 m \cdot m' &= \langle m(mnmmn) \rangle \\
 &= \langle mnmmn \rangle \\
 &= \cos^2 \theta - \sin^2 \theta = \cos 2\theta
 \end{aligned}$$

Useful result when vector a lies in the plane B

$$\begin{aligned}
 e^{\theta \hat{B}} a &= (\cos \theta + \sin \theta \hat{B}) a \\
 &= a(\cos \theta - \sin \theta \hat{B}) = a e^{-\theta \hat{B}}
 \end{aligned}$$

Also need to check orientation

$$e^{\theta e_1 e_2 / 2} e_1 e^{-\theta e_1 e_2 / 2} = e^{\theta e_1 e_2} e_1 = \cos \theta e_1 - \sin \theta e_2$$

Rotors in 3D

The rotor for a rotation through $|B|$ with handedness of B : $R = \exp(-B/2)$
 In terms of an axis: $R = \exp(-\theta I \mathbf{n} / 2)$

Decompose a vector
 into terms in and out of the plane $e^{-B/2}(a_{\parallel} + a_{\perp})e^{B/2} = a_{\parallel}e^B + a_{\perp}$

Can work in terms of Euler angles, but best avoided:

$$R = e^{-\mathbf{e}_1\mathbf{e}_2\phi/2} e^{-\mathbf{e}_2\mathbf{e}_3\theta/2} e^{-\mathbf{e}_1\mathbf{e}_2\psi/2}$$

Unification

Every rotor can be written as $R = \pm \exp(-B/2)$

Rotations of any object, of any grade, in any space of any signature can be written as $A \mapsto RA\tilde{R}$

Unification

Every finite Lie group can be realised as a group of rotors.

Every Lie algebra can be realised as a set of bivectors.

Resources

geometry.mrao.cam.ac.uk
chris.doran@arm.com
cjld1@cam.ac.uk
[@chrisjldoran](https://twitter.com/chrisjldoran)
[#geometricalgebra](https://twitter.com/geometricalgebra)
github.com/ga

