Geometric Algebra

6. Geometric Calculus

Dr Chris Doran
ARM Research

The vector derivative

Define a vector operator that returns the derivative in any given direction by

$$a \cdot \nabla F(x) = \lim_{\epsilon \mapsto 0} \frac{F(x + \epsilon a) - F(x)}{\epsilon}$$

Define a set of Euclidean coordinates

$$x^k = e^k \cdot x$$

$$\nabla = \sum_k e^k \frac{\partial}{\partial x^k} = e^k \partial_k$$

This operator has the algebraic properties of a vector in a geometric algebra, combined with the properties of a differential operator.

Basic Results

Extend the definition of the dot and wedge product

$$\nabla \cdot A_r = \langle \nabla A_r \rangle_{r-1},$$
$$\nabla \wedge A_r = \langle \nabla A_r \rangle_{r+1}$$

The exterior derivative defined by the wedge product is the *d* operator of differential forms

The exterior derivative applied twice gives zero

$$\begin{aligned}\nabla \wedge (\nabla \wedge A) &= e^{i} \wedge \partial_i (e^{j} \wedge \partial_j A) \\ &= e^{i} \wedge e^{j} \wedge (\partial_i \partial_j A) \\ &= 0\end{aligned}$$

Same for inner derivative

Leibniz rule

$$\nabla(AB) = \nabla AB + \dot{\nabla} A \dot{B}$$

Where

$$\dot{\nabla} A \dot{B} = e^k A \, \partial_k B$$

Overdot notation very useful in practice

Two dimensions

$$\boldsymbol{r} = xe_1 + ye_2 \qquad \boldsymbol{\nabla} = e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y}$$

Now suppose we define a 'complex' function $\psi = u + Iv$

$$\boldsymbol{\nabla}\psi = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) e_1 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) e_2$$

These are precisely the terms that vanish for an analytic function – the Cauchy-Riemann equations

$$\boldsymbol{\nabla}\psi = 0$$

Unification

The Cauchy-Riemann equations arise naturally from the vector derivative in two dimensions.

Analytic functions

Any function that can be that can be written as a function of *z* is analytic: $f(z)$ $\quad z = x + Iy = e_1\boldsymbol{r}$

$$\boldsymbol{\nabla} z = \boldsymbol{\nabla}(x + I) = e_1 + e_2 I = e_1 - e_1 = 0$$

$$\boldsymbol{\nabla}(z - z_0)^n = n\boldsymbol{\nabla}(x + Iy)(z - z_0)^{n-1} = 0$$

- The CR equations are the same as saying a function is independent of *z**.
- In 2D this guarantees we are left with a function of *z* only
- Generates of solution of the more general equation $\boldsymbol{\nabla}\psi = 0$

Three dimensions

Maxwell equations in vacuum around sources and currents, in natural units

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$$

$$-\boldsymbol{\nabla}\times\boldsymbol{E} = \frac{\partial}{\partial t}\boldsymbol{B} \qquad \boldsymbol{\nabla}\times\boldsymbol{B} = \frac{\partial}{\partial t}\boldsymbol{E} + \boldsymbol{J}$$

Remove the curl term via

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -I\,\boldsymbol{\nabla}\wedge\boldsymbol{E}$$

Find

$$\boldsymbol{\nabla}\boldsymbol{E} = \boldsymbol{\nabla}\cdot\boldsymbol{E} + \boldsymbol{\nabla}\wedge\boldsymbol{E} = \rho - \partial_t(I\boldsymbol{B})$$

$$\boldsymbol{\nabla}(I\boldsymbol{B}) = I(\boldsymbol{\nabla}\cdot\boldsymbol{B} + \boldsymbol{\nabla}\wedge\boldsymbol{B}) = -\partial_t\boldsymbol{E} - \boldsymbol{J}$$

$$\boldsymbol{\nabla}(\boldsymbol{E} + I\boldsymbol{B}) = -\partial_t(\boldsymbol{E} + I\boldsymbol{B}) + \rho - \boldsymbol{J}$$

All 4 of Maxwell’s equations in 1!

Spacetime

The key differential operator in spacetime physics

$$\nabla = \gamma^\mu \frac{\partial}{\partial x^\mu} = \gamma_0 \frac{\partial}{\partial t} + \gamma^i \frac{\partial}{\partial x^i}$$

Form the relative split

$$\nabla \gamma_0 = \partial_t + \gamma^i \gamma_0 \partial_i = \partial_t - \boldsymbol{\nabla}$$

Or

$$\gamma_0 \nabla = \partial_t + \boldsymbol{\nabla}$$

So

$$\gamma_0 \nabla x \gamma_0 = (\partial_t + \boldsymbol{\nabla})(t + \boldsymbol{x})$$
$$= 4$$

Recall Faraday bivector

$$F = \boldsymbol{E} + I\boldsymbol{B}$$

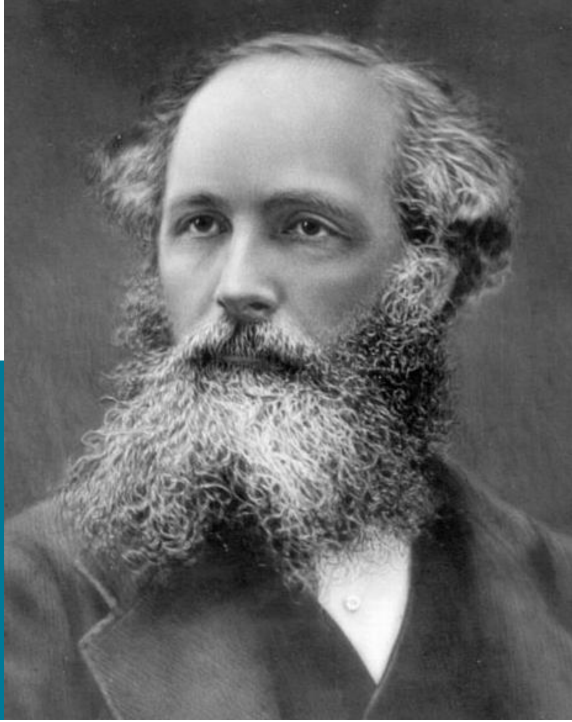
$$\gamma_0 \nabla F = \rho - \boldsymbol{J}$$
$$= \gamma_0 \cdot J - J \wedge \gamma_0$$

So finally

$$\boxed{\nabla F = J}$$

Unification

$$\nabla F = J$$

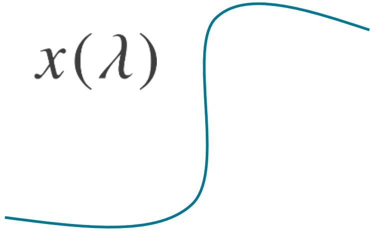


The most compact formulation of the Maxwell equations. Unifies all four equations in one.

More than some symbolic trickery. The vector derivative is an invertible operator.

Directed integration

Start with a simple line integral along a curve



$$\int F(x)\frac{dx}{d\lambda}\,d\lambda = \int F\,dx = \lim_{n\mapsto\infty}\sum_{i=1}^{n}\bar{F}^i\Delta x^i$$

$$\Delta x^i = x_i - x_{i-1} \qquad \bar{F}^i = \tfrac{1}{2}\left(F(x_{i-1}) + F(x_i)\right)$$

Key concept here is the vector-valued measure

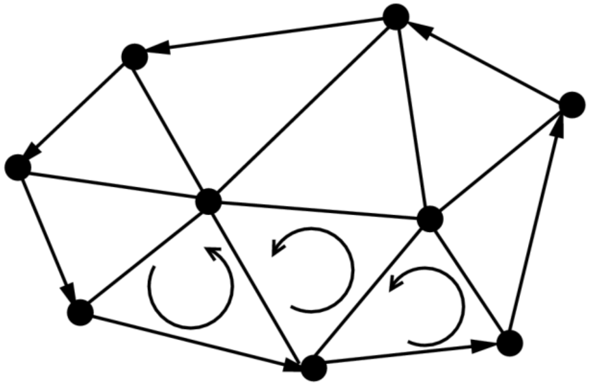
$$dx = \frac{\partial x(\lambda)}{\partial\lambda}d\lambda$$

More general form of line integral is

$$\int F(x)\frac{dx}{d\lambda}G(x)\,d\lambda = \int F(x)\,dx\,G(x)$$

Surface integrals

Now consider a 2D surface embedded in a larger space



$$dX = \frac{\partial x}{\partial x^1} \wedge \frac{\partial x}{\partial x^2}\, dx^1\, dx^2 = e_1 \wedge e_2\, dx^1\, dx^2$$

Directed surface element

This extends naturally to higher dimensional surfaces

$$dX = e_1 \wedge \cdots \wedge e_n\, d\lambda^1 \cdots d\lambda^n$$

- The surface element is a blade
- It enters integrals via the geometric product

Fundamental theorem

Overdots show where the vector derivative acts

Left-sided version

$$\oint_{\partial V} F\, dS = \int_V \dot{F}\dot{\nabla}\, dX$$

Right-sided version

$$\oint_{\partial V} dS\, G = \int_V \dot{\nabla}\, dX\, \dot{G}$$

General result

$$\oint_{\partial V} L(dS) = \int_V \dot{L}(\dot{\nabla} dX)$$

L is a multilinear function

Divergence theorem

Set $L(A) = \langle J A I^{-1} \rangle$

Vector — Grade *n-1* — Constant Grade *n*

L is a scalar-valued linear function of *A*

$$\int_V \langle \dot{J} \dot{\nabla} \underbrace{dX I^{-1}}_{\text{Scalar measure}} \rangle = \int_V \nabla \cdot J \, |dX| = \oint_{\partial V} \langle J dS I^{-1} \rangle$$

$$n|dS| = dS\, I^{-1}$$

The divergence theorem

$$\int_V \nabla \cdot J \, |dX| = \oint_{\partial V} n \cdot J \, |dS|$$

Cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a} dz$$

One of the most famous results in 19^{th} century mathematics

Knowledge of an analytic function around a curve is enough to learn the value of the function at each interior point

We want to understand this in terms of Geometric Algebra

And extend it to arbitrary dimensions!

Cauchy integral formula

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

$$z = \boldsymbol{e}_1\boldsymbol{r} \qquad dz = \frac{\partial x}{\partial \lambda}d\lambda = \boldsymbol{e}_1\frac{\partial \boldsymbol{r}}{\partial \lambda}$$

$$f(z) = f(\boldsymbol{r}) \qquad \boldsymbol{\nabla} f = 0$$

$$\frac{1}{z-a} = \frac{z^* - a^*}{(z-a)(z-a)^*} = \frac{(\boldsymbol{r}-\boldsymbol{a})\boldsymbol{e}_1}{(\boldsymbol{r}-\boldsymbol{a})^2}$$

$$\frac{f(z)}{z-a}dz = \frac{\boldsymbol{r}-\boldsymbol{a}}{(\boldsymbol{r}-\boldsymbol{a})^2}d\boldsymbol{r}$$

Translate the various terms into their GA equivalents

Find the dependence on the real axis drops out of the integrand

Cauchy integral formula

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

$$f(\boldsymbol{a}) = \oint f(\boldsymbol{r})\frac{\boldsymbol{r}-\boldsymbol{a}}{2\pi(\boldsymbol{r}-\boldsymbol{a})^2}d\boldsymbol{r}\, I^{-1}$$

$$= \int \left(f(\boldsymbol{r})\frac{\boldsymbol{r}-\boldsymbol{a}}{2\pi(\boldsymbol{r}-\boldsymbol{a})^2}\right)\overleftarrow{\boldsymbol{\nabla}}\, dX\, I^{-1}$$

Applying the fundamental theorem of geometric calculus

$$dX\, I^{-1} = dA$$ Scalar measure

$$\boldsymbol{\nabla} f = 0$$ Function is analytic

$$\boldsymbol{\nabla}\frac{\boldsymbol{r}-\boldsymbol{a}}{2\pi(\boldsymbol{r}-\boldsymbol{a})^2} = \delta(\boldsymbol{r}-\boldsymbol{a})$$

The Green’s function for the vector derivative in the plane

Cauchy integral formula

$$\oint f(\boldsymbol{r})\frac{\boldsymbol{r}-\boldsymbol{a}}{2\pi(\boldsymbol{r}-\boldsymbol{a})^2}d\boldsymbol{r}\, I^{-1} = \int f(\boldsymbol{a})\delta(\boldsymbol{r}-\boldsymbol{a})dA = f(\boldsymbol{a})$$

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

1. *dz* encodes the tangent vector
2. Complex numbers give a geometric product
3. The integrand includes the Green's function in 2D
4. The I comes from the directed volume element

Generalisation

$$\nabla\psi = 0$$

$$\psi(y) = \frac{1}{IS_n}\oint_{\partial V}\frac{x-y}{|x-y|^n}\,dS\,\psi(x)$$

This extends Cauchy's integral formula to arbitrary dimensions

Unification

The Cauchy integral formula, the divergence theorem, Stoke’s theorem, Green’s theorem etc. are all special cases of the fundamental theorem of geometric calculus

Resources

geometry.mrao.cam.ac.uk
chris.doran@arm.com
cjld1@cam.ac.uk
@chrisjldoran
#geometricalgebra
github.com/ga