Geometric Algebra

7. Conformal Geometric Algebra

Dr Chris Doran
ARM Research

Motivation

- Projective geometry showed that there is considerable value in treating points as vectors
- Key to this is a homogeneous viewpoint where scaling does not change the geometric meaning attached to an object
- We would also like to have a direct interpretation for the inner product of two vectors
- This would be the distance between points
- Can we satisfy all of these demands in one algebra?

Inner product and distance

Suppose *X* and *Y* represent points
Would like

$$X \cdot Y \propto d_{xy}^2$$

Quadratic on grounds of units

Immediate consequence:

$$X \cdot X = X^2 = d_{xx}^2 = 0$$

Represent points with null vectors
Borrow this idea from relativity

$$(\gamma_0 + \gamma_1)^2 = 1 - 1 = 0$$

Key idea was missed in 19th century

Also need to consider homogeneity
Idea from projective geometry is to introduce a point at infinity:

$$n, \quad n^2 = 0$$

Inner product and distance

Natural Euclidean definition is $$\left(\frac{X}{X\cdot n} - \frac{Y}{Y\cdot n}\right)^2 = d_{xy}^2$$

But both *X* and *Y* are null, so $$\frac{-2X\cdot Y}{X\cdot n\, Y\cdot n} = d_{xy}^2 = (\boldsymbol{x} - \boldsymbol{y})^2$$

As an obvious check, look at the distance to the point at infinity

$$\frac{-2X\cdot n}{X\cdot n\, n\cdot n} = \infty$$

We have a concept of distance in a homogeneous representation

Need to see if this matches our Euclidean concept of distance.

Origin and coordinates

Pick out a preferred point to represent the origin $C, \quad C^2 = 0$

Look at the displacement vector $$\frac{X}{X\cdot n} - \frac{C}{C\cdot n} = \frac{(X\wedge C)\cdot n}{X\cdot n\, C\cdot n}$$

Would like a basis vector containing this, but orthogonal to *C*

Add back in some amount of *n*

$$C\cdot\left(\frac{(X\wedge C)\cdot n}{X\cdot n\, C\cdot n} + \lambda n\right) = 0$$

Get this as our basis vector:

$$\frac{1}{X\cdot n\, C\cdot n}\left((X\wedge C)\cdot n - X\cdot C\, n\right)$$

$$-(C\wedge X)\cdot n - C\cdot X\, n$$

$$= -\langle CXn\rangle_1$$

Origin and coordinates

Now have
$$\frac{X}{X\cdot n} = \frac{C}{C\cdot n} - \frac{\langle CXn\rangle_1}{X\cdot n\, C\cdot n} + \frac{C\cdot X}{C\cdot n\, X\cdot n} n$$

Write as
$$\frac{-X}{X\cdot n} = \frac{-C}{C\cdot n} + x + \frac{x^2}{2} n$$

$X\cdot n$ is negative

Euclidean vector from origin

Historical convention is to write

$$\bar{n} = \frac{2C}{C\cdot n} \qquad n\cdot\bar{n} = 2$$

$$e = \tfrac{1}{2}(n+\bar{n}) \qquad \bar{e} = \tfrac{1}{2}(n-\bar{n})$$

$$e^2 = 1, \qquad \bar{e}^2 = -1$$

$$n = e + \bar{e}, \qquad \bar{n} = e - \bar{e}$$

Is this Euclidean geometry?

Look at the inner product of two Euclidean vectors

$$x \cdot y = \frac{\langle CXn \rangle_1}{X \cdot n \, C \cdot n} \cdot \frac{\langle CYn \rangle_1}{Y \cdot n \, C \cdot n} = \tfrac{1}{2}(x^2 + y^2 - (x-y)^2) \quad \checkmark$$

$$\begin{aligned}
\langle CXn \rangle_1 \cdot \langle CYn \rangle_1 &= \tfrac{1}{2}\langle (CXn + nCX)CYn \rangle \\
&= \tfrac{1}{2}\langle CXnCYn \rangle \\
&= C \cdot X \langle nCYn \rangle - \tfrac{1}{2}\langle XCnCYn \rangle \\
&= -C \cdot n \langle XCYn \rangle \\
&= -C \cdot n (X \cdot C \, Y \cdot n - X \cdot Y \, C \cdot n \\
&\quad + X \cdot n \, Y \cdot c)
\end{aligned}$$

Checks out as we require

The inner product is the standard Euclidean inner product

Can introduce an orthonormal basis

Summary of idea

Represent the Euclidean point *x* by null vectors

$$X = -\bar{n} + 2x + x^2 n$$

Distance is given by the inner product

$$\frac{-2X\cdot Y}{X\cdot n\, Y\cdot n} = (\boldsymbol{x} - \boldsymbol{y})^2$$

Normalised form has $X\cdot n = -2$

Basis vectors are $\{e, \bar{e}, e_i\}$ $\quad x = x_i e_i$

Null vectors $n = e + \bar{e}, \quad \bar{n} = e - \bar{e}$

1D conformal GA

Basis algebra is

$$1 \qquad \{e, \bar{e}, e_1\} \qquad \{e\bar{e}, ee_1, \bar{e}e_1\} \qquad e\bar{e}e_1$$

NB pseudoscalar squares to +1

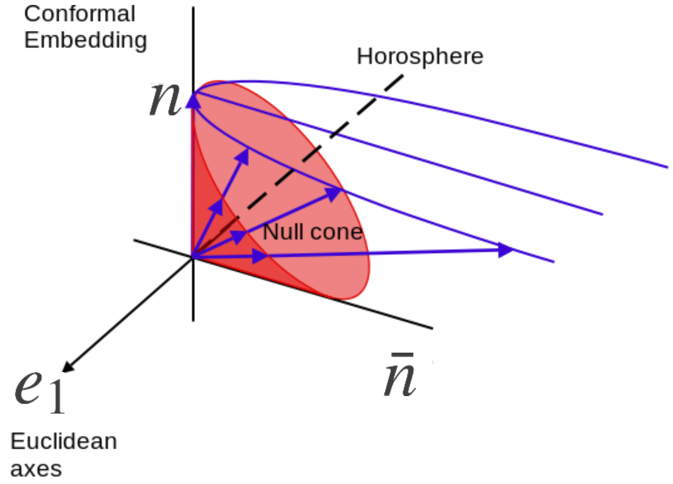
Simple example in 1D

$$X = -\bar{n} + 2xe_1 + x^2 n$$

$$Y = -\bar{n} + 2ye_1 + y^2 n$$

$$X \cdot Y = 4xy - 2x^2 - 2y^2$$

$$= -2(x-y)^2$$



Transformations

Any rotor that leaves *n* invariant must leave distance invariant

$$\frac{X'\cdot Y'}{X'\cdot n\, Y'\cdot n} = \frac{(RX\tilde{R})\cdot(RY\tilde{R})}{(RX\tilde{R})\cdot n\,(RY\tilde{R})\cdot n} = \frac{X\cdot Y}{X\cdot(\tilde{R}nR)\,Y\cdot(\tilde{R}nR)} = \frac{X\cdot Y}{X\cdot n\,Y\cdot n}$$

Rotations around the origin work simply

$$x' = Rx\tilde{R}, \qquad R = e^{-\theta e_1 e_2/2}$$

$$X' = -\bar{n} + 2x' + x'^2 n$$

$$= RX\tilde{R}$$

Remaining generators that commute with *n* are of the form

$$B = an, \qquad a\cdot n = 0$$

$$Bn = nB = 0$$

$$B^2 = 0$$

Null generators

$$T_a = e^{na/2} = 1 + \tfrac{1}{2}na$$

Taylor series terminates after two terms

$$T_a n \tilde{T}_a = n + \tfrac{1}{2}nan + \tfrac{1}{2}nan + \tfrac{1}{4}nanan = n$$

$$T_a \bar{n} \tilde{T}_a = \bar{n} - 2a - a^2 n$$

Since $a \cdot n = a \cdot \bar{n} = 0$

$$T_a x \tilde{T}_a = x + n(a \cdot x)$$

$$T_a X \tilde{T}_a = x^2 n + 2(x + a \cdot x\, n) - (\bar{n} - 2a - a^2 n)$$

$$= (x+a)^2 n + 2(x+a) - \bar{n}$$

Conformal representation of the translated point

Dilations

Suppose we want to dilate about the origin $x \mapsto x' = e^{-\alpha} x$

$$X' = e^{-\alpha} \underline{(x^2 e^{-\alpha} n + 2x + e^{\alpha} \bar{n})}$$

↑

Generate this part via a rotor, then use homogeneity

$$n \mapsto e^{-\alpha} n, \qquad \bar{n} \mapsto e^{\alpha} \bar{n}$$

Define $N = e\bar{e} = \frac{1}{2} \bar{n} \wedge n$

$$D_\alpha = e^{\alpha N/2}$$

$$D_\alpha n \tilde{D}_\alpha = e^{-\alpha} n$$

$$D_\alpha \bar{n} \tilde{D}_\alpha = e^{\alpha} \bar{n}$$

Rotor to perform a dilation

To dilate about an arbitrary point replace origin with conformal representation of the point

$$D_\alpha = \exp\left(\frac{\alpha}{2} \frac{A \wedge n}{A \cdot n}\right)$$

Unification

In conformal geometric algebra we can use rotors to perform translations and dilations, as well as rotations

Results proved at one point can be translated and rotated to any point

Geometric primitives

Find that bivectors don't represent lines. They represent point pairs.

Look at $x = \lambda a + (1-\lambda)b$

$$X(\lambda) = (\lambda^2 a^2 + 2\lambda(1-\lambda)a\cdot b + (1-\lambda)^2 b^2)n + 2\lambda a + 2(1-\lambda)b - \bar{n}$$

$$= \lambda A + (1-\lambda)B + \tfrac{1}{2}\lambda(1-\lambda)A\cdot B\, n$$

Point *a* Point *b* Point at infinity

Points along the line satisfy $\underline{(A\wedge B\wedge n)}\wedge X = 0, \qquad X^2 = 0$

This is the line

Lines as trivectors

Suppose we took any three points, do we still get a line?

$$A_1 = -\bar{n} + 2e_1 + n$$

$$A_2 = -\bar{n} + 2e_2 + n$$

$$A_3 = -\bar{n} - 2e_1 + n$$

$$A_1 \wedge A_2 \wedge A_3 = 16 e_1 e_2 \bar{e}$$

Need null vectors in this space

Up to scale find

$$X = \cos\theta e_1 + \sin\theta e_2 + \bar{e} = -\tfrac{1}{2}\bar{n} + \cos\theta e_1 + \sin\theta e_2 + \tfrac{1}{2}n$$

The outer product of 3 points represents the circle through all 3 points.

Lines are special cases of circles where the circle include the point at infinity

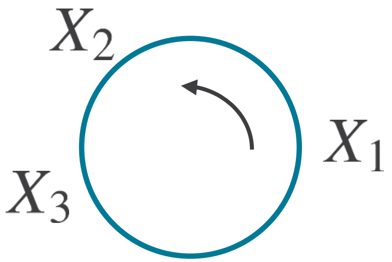
Circles

$$C = X_1 \wedge X_2 \wedge X_3$$

Everything in the conformal GA is oriented

Objects can be rescaled, but you mustn't change their sign!

Important for intersection tests



$$\rho^2 = -\frac{C^2}{(C \wedge n)^2}$$

Radius from magnitude.

Metric quantities in homogenous framework

If the three points lie in a line then $C \wedge n = 0$

Lines are circles with infinite radius

All related to inversive geometry

4-vectors

4 points define a sphere or a plane $P = A_1 \wedge A_2 \wedge A_3 \wedge A_4$

If the points are co-planar find $X = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \delta n$

So *P* is a plane iff $P \wedge n = 0$

Unit sphere is $S = e_1 e_2 e_3 \bar{e}$

Radius of the sphere is

$$\rho^2 = \frac{P^2}{(P \wedge n)^2}$$

Note if *L* is a line and *A* is a point, the plane formed by the line and the point is $P = L \wedge A$

5D representation of 3D space

Object	Grade	Dimension	Interpretation
Scalar	0	1	Scalar values
Vector	1	5	Points (null), dual to spheres and planes.
Bivector	2	10	Point pairs, generators of Euclidean transformations, dilations.
Trivectors	3	10	Lines and circles
4-vectors	4	5	Planes and spheres
Pseudoscalar	5	1	Volume factor, duality generators

Angles and inversion

Angle between two lines that meet at a point or point pair

$$\cos\theta = \frac{L_1 \cdot L_2}{|L_1||L_2|}$$

Works for straight lines and circles!

All rotors leave angles invariant – generate the conformal group

Reflect the conformal vector in *e*

$$-eXe = n + 2x - x^2\bar{n}$$
$$= x^2\left(-\bar{n} + \frac{2x}{x^2} + \frac{1}{x^2}n\right)$$

The is the result of inverting space in the origin.

Can translate to invert about any point – conformal transformations

Reflection

1-2 plane is represented by $P = e_1 e_2 N, \quad P^2 = -1$

$$Pe_1 = -e_1 P$$

In the plane

$$Pe_3 = e_3 P$$

Out of the plane

So if *L* is a line through the origin $L = aN$

The reflected line is $L' = -PLP$

But we can translate this result around and the formula does not change

$$L' = -PLP$$

Reflects any line in any plane, without finding the point of intersection

Intersection

Use same idea of the meet operator

Duality still provided by the appropriate pseudoscalar (technically needs the join)

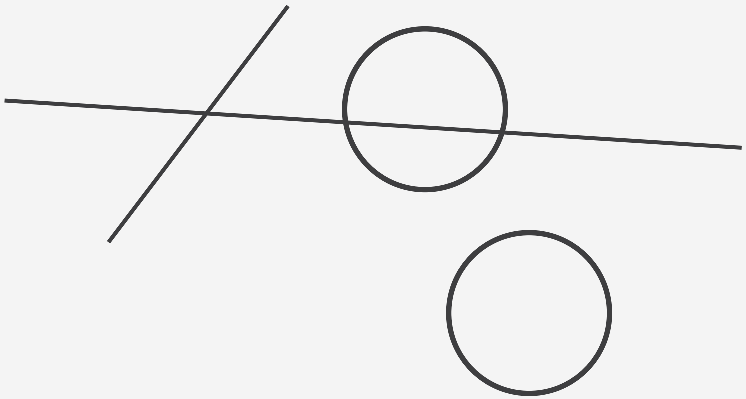
$$X \wedge M_1 = X \wedge M_2 = 0$$

$$X \wedge (M_1^* \wedge M_2^*)^* = 0$$

Example – 2 lines in a plane

$$B = (L_1^* \wedge L_2^*)^* = I(L_1 \times L_2)$$

$$B^2 = \begin{cases} > 0 & \text{2 points of intersection} \\ 0 & \text{1 point of intersection} \\ < 0 & \text{0 points of intersection} \end{cases}$$

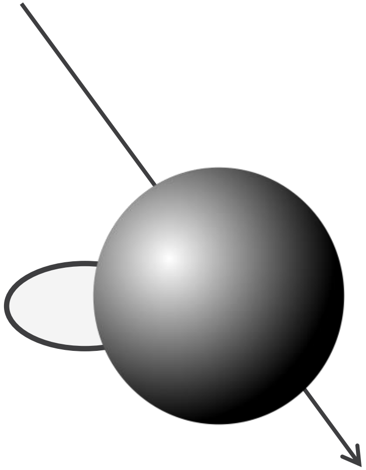


Intersection

Circle / line and sphere / plane

$$B = (P^* \wedge L^*)^* = (IP)\cdot L = I\langle PL\rangle_3$$

$$B^2 = \begin{cases} > 0 & \text{2 points of intersection} \\ 0 & \text{1 point of intersection} \\ < 0 & \text{0 points of intersection} \end{cases}$$



All cases covered in a single application of the geometric product

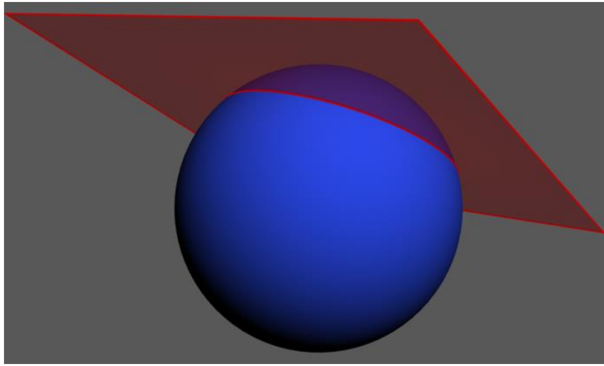
Orientation tracks which point intersects on way in and way out

In line / plane case, one of the points is at infinity $n\wedge L = n\wedge P = 0$

Intersection

Plane / sphere and a plane / sphere intersect in a line or circle

$$L = I\langle S_1 S_2 \rangle_2$$



Norm of *L* determines whether or not it exists.

If we normalise a plane *P* and sphere *S* to -1 can also test for intersection

$$P \cdot S = \begin{cases} > 1 & \text{Sphere above plane} \\ -1 \cdots 1 & \text{Sphere and plane intersect} \\ < -1 & \text{Sphere below plane} \end{cases}$$

Resources

geometry.mrao.cam.ac.uk
chris.doran@arm.com
cjld1@cam.ac.uk
@chrisjldoran
#geometricalgebra
github.com/ga