Geometric Algebra

9. Unification

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Euclidean geometry

Represent the Euclidean point *x* by null vectors

$$X = -\bar{n} + 2x + x^2 n$$

Distance is given by the inner product

$$\frac{-2X \cdot Y}{X \cdot n \, Y \cdot n} = (\boldsymbol{x} - \boldsymbol{y})^2$$

$$\frac{-X}{X \cdot n} = -\tfrac{1}{2}\bar{n} + x + \tfrac{1}{2}x^2 n$$

Read off the Euclidean vector

Depends on the concept of the origin

Spherical geometry

Suppose instead we form

$$\frac{-X}{X\cdot\bar{e}} = \hat{x} + \bar{e}$$

($\hat{x}$:) Unit vector in an n+1 dimensional space

Instead of plotting points in Euclidean space, we can plot them on a sphere

No need to pick out a preferred origin any more

$$\frac{-X\cdot Y}{X\cdot\bar{e}\,Y\cdot\bar{e}} = -(\hat{x}\cdot\hat{y} - 1)$$

$$= 2\sin^2(\theta/2)$$

Spherical geometry

Spherical distance

$$d(\hat{x}, \hat{y}) = 2\sin^{-1}\left(\frac{-X\cdot Y}{2X\cdot\bar{e}\,Y\cdot\bar{e}}\right)^{1/2}$$

Same pattern as Euclidean case

'Straight' lines are now

$$X\wedge Y\wedge\bar{e} = \hat{x}\wedge\hat{y}\bar{e}$$

The $\bar{e}$ term now becomes essentially redundant and drops out of calculations

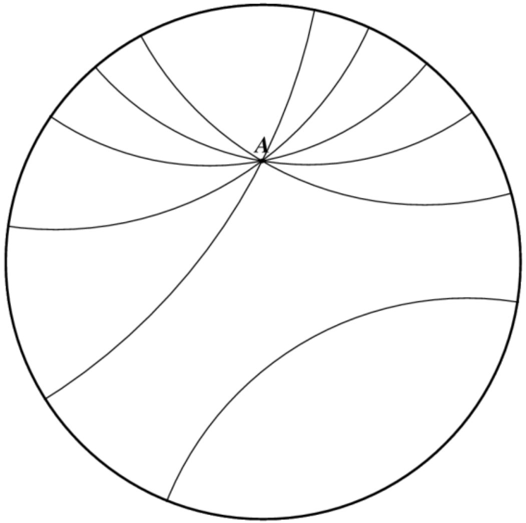
Invariance group are the set of rotors satisfying $R\bar{e}\tilde{R} = \bar{e}$

Generators satisfy

$$B\cdot\bar{e} = 0$$

Left with standard rotors in a Euclidean space. Just rotate the unit sphere

non-Euclidean geometry



Historically arrived at by replacing the parallel postulate

'Straight' lines become d-lines. Intersect the unit circle at 90º

Model this in our conformal framework

Unit circle $e_1 e_2 \bar{e} = Ie$

d-lines $L \wedge e = 0$

d-line between *X* and *Y* is

$$L = X \wedge Y \wedge e$$

$$L^2 > 0$$

Translation along a d-line generated by

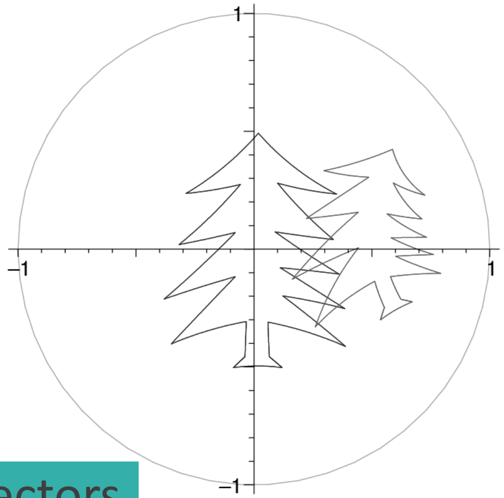
$$B = Le \quad B^2 > 0$$

Rotor generates hyperbolic transformations

non-Euclidean geometry

$$Y = e^{\alpha \hat{B}/2} X e^{-\alpha \hat{B}/2} \qquad \hat{B} = \frac{B}{|B|}$$

Generator of translation along the d-line.
Use this to define distance.



Write $X = \hat{x} + e, \quad Y = \hat{y} + e$

Unit time-like vectors

$$\hat{x} \cdot \hat{y} = \cosh(\alpha)$$

Boost factor from special relativity

$$\cosh(\alpha) = 1 - \frac{X \cdot Y}{X \cdot e \, Y \cdot e}$$

$$\sinh^2(\alpha/2) = -\frac{X \cdot Y}{2X \cdot e \, Y \cdot e}$$

$$d(x, y) = 2 \sinh^{-1} \left(-\frac{X \cdot Y}{2X \cdot e \, Y \cdot e} \right)^{1/2}$$

Distance in non-Euclidean geometry

non-Euclidean distance

$$d(x, y) = 2\sinh^{-1}\left(\frac{|x-y|^2}{(1-x^2)(1-y^2)}\right)^{1/2}$$

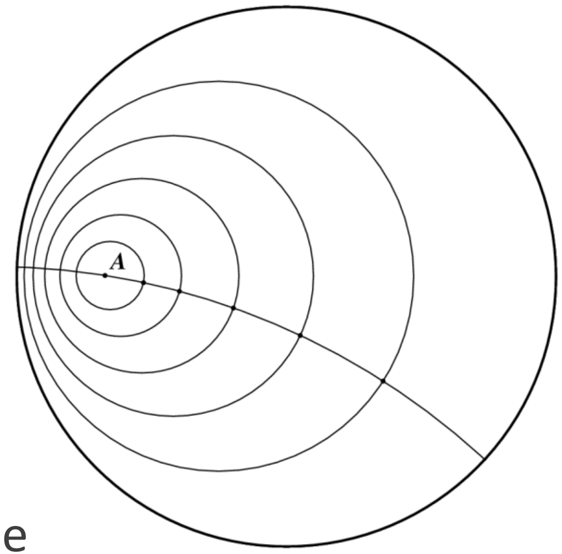
Distance expands as you get near to the boundary

Circle represents a set of points at infinity

This is the Poincare disk view of non-Euclidean geometry

non-Euclidean circles

$$-\frac{X\cdot C}{2X\cdot e\, C\cdot e} = \text{constant} = \alpha^2$$



$$X\cdot(C + 2\alpha^2 C\cdot e\, e) = 0$$

$$s = IS \qquad X\wedge S = 0$$

Formula unchanged from the Euclidean case

Still have $S = X_1\wedge X_2\wedge X_3$

Non-Euclidean circle

Definition of the centre is not so obvious. Euclidean centre is

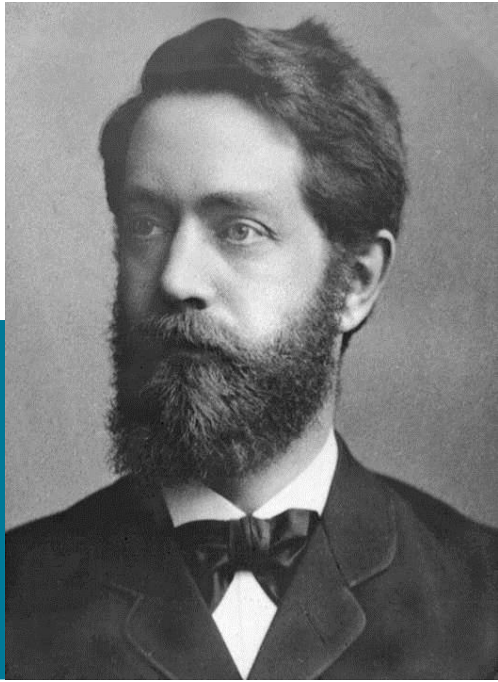
$$C = SnS$$

Reverse the logic above and define $C = s + \lambda e$

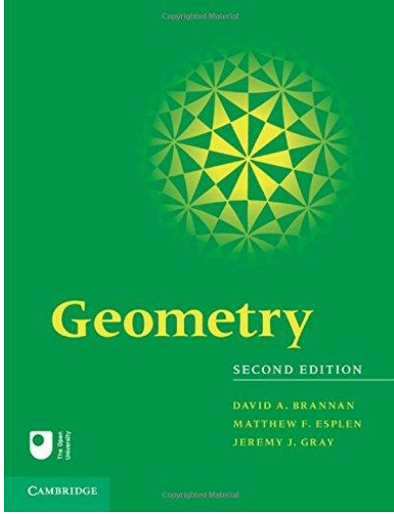
$$C^2 = 0 \implies \lambda$$

Unification

Conformal GA unifies Euclidean, projective, spherical, and hyperbolic geometries in a single compact framework.



Geometries and Klein



Understand geometries in terms of the underlying transformation groups

Euclidean

$$\boldsymbol{x} \mapsto \boldsymbol{U}\boldsymbol{x} + \boldsymbol{a}$$

Affine

$$\boldsymbol{x} \mapsto \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$$

Projective

$$[\boldsymbol{x}] \mapsto [\boldsymbol{A}\boldsymbol{x}]$$

Conformal

$$X \mapsto RX\tilde{R}$$

Mobius /Inversive

$$z \mapsto (az + b)/(cz + d)$$

Spherical

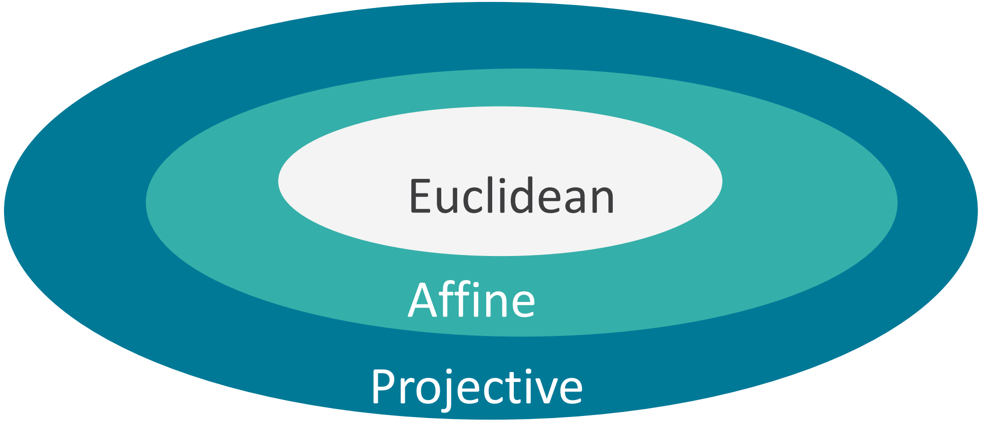
$$\hat{\boldsymbol{x}} \mapsto R\hat{\boldsymbol{x}}\tilde{R}$$

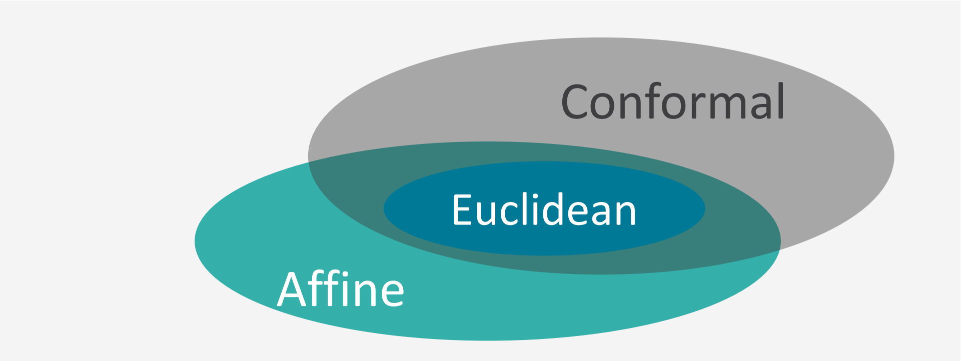
non-Euclidean

$$Re\tilde{R} = e$$

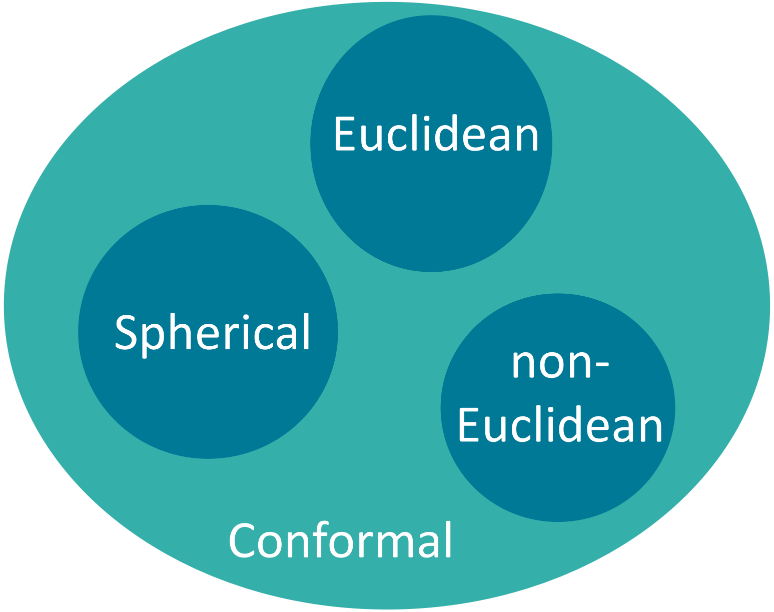
Geometries and Klein

Projective viewpoint





Conformal viewpoint



Groups

Have seen that we can perform dilations with rotors

Every linear transformation is rotation + dilation + rotation via SVD $A = U\Lambda V$

Trick is to double size of space

$$\{e_i, f_i\}, \quad e_i \cdot e_j = \delta_{ij}, \quad f_i \cdot f_j = -\delta_{ij}, \quad e_i \cdot f_j = 0$$

Null basis $n_i = e_i + f_i, \quad \bar{n}_i = e_i - f_i$

Define bivector

$$K = \sum_i e_i f_i \qquad (a \cdot K) \cdot K = a$$

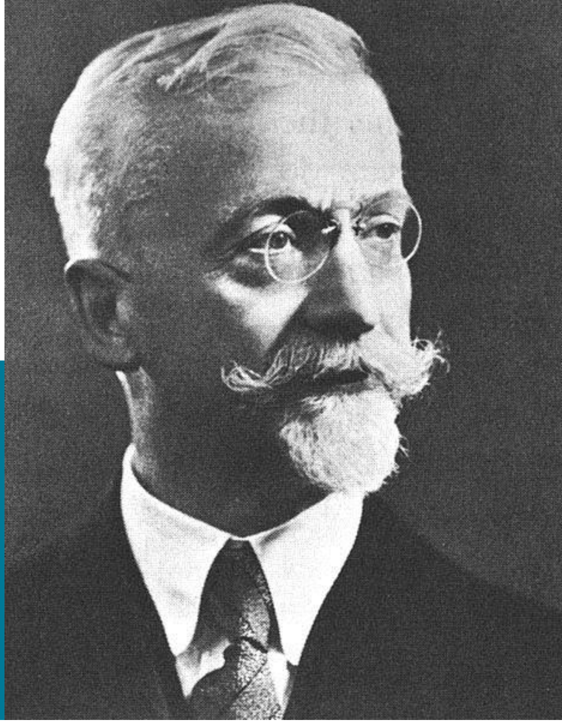
Construct group from constraint

$$RK\tilde{R} = K$$

Keeps null spaces separate. Within null space give general linear group.

Unification

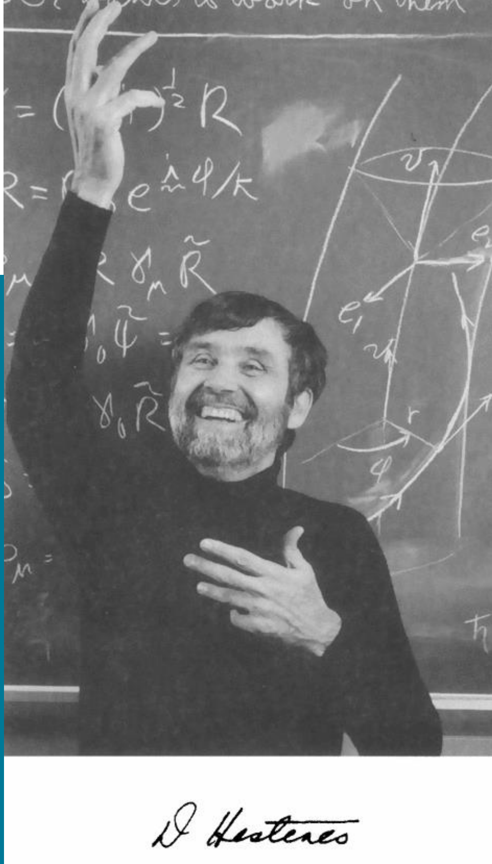
Every matrix group can be realised as a rotor group in some suitable space. There is often more than one way to do this.



Design of mathematics

Coordinate geometry
Complex analysis
Vector calculus
Tensor analysis
Matrix algebra
Lie groups
Lie algebras
Spinors
Gauge theory

Grassmann algebra
Differential forms
Berezin calculus
Twistors
Quaternions
Octonions
Pauli operators
Dirac theory
Gravity…



Spinors and twistors

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$$

Spin matrices act on 2-component wavefunctions

These are spinors

Very similar to qubits

$$|\psi\rangle \mapsto \rho R$$

Roger Penrose has put forward a philosophy that spinors are more fundamental than spacetime

Start with 2-spinors and build everything up from there

Twistors

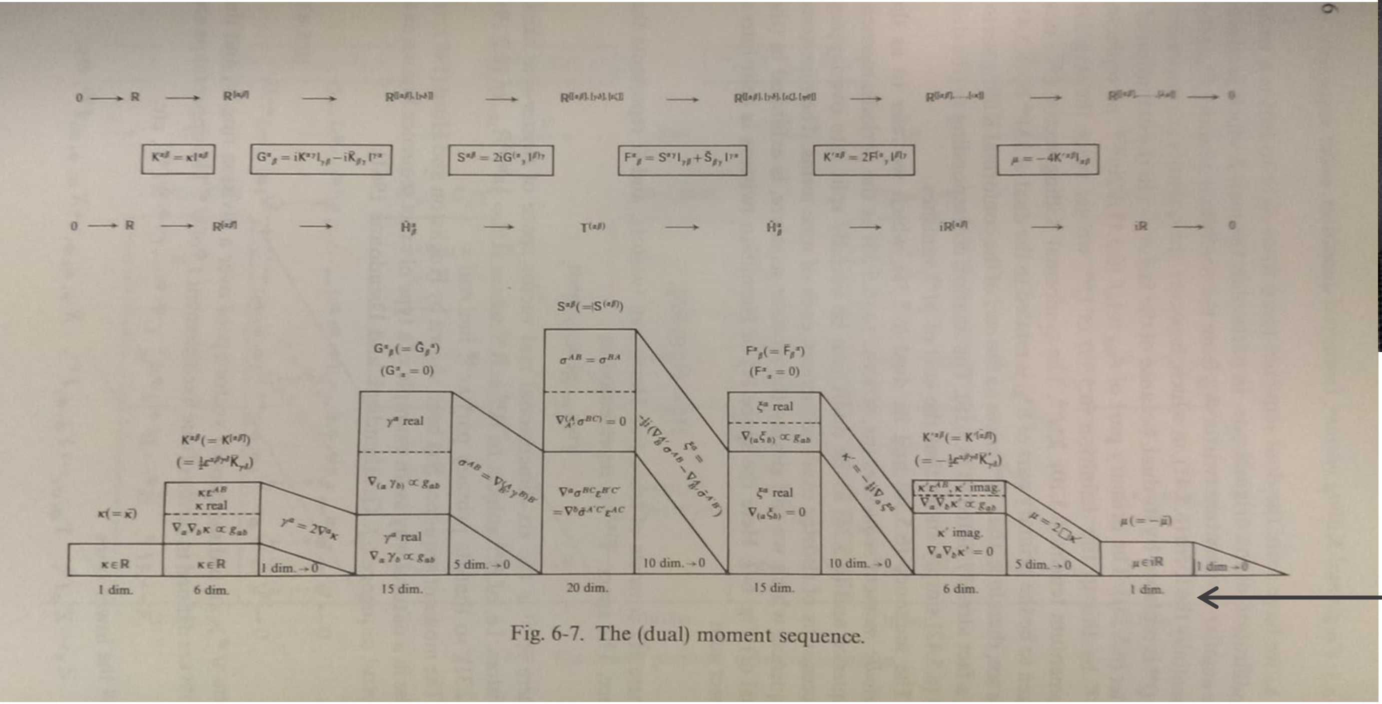
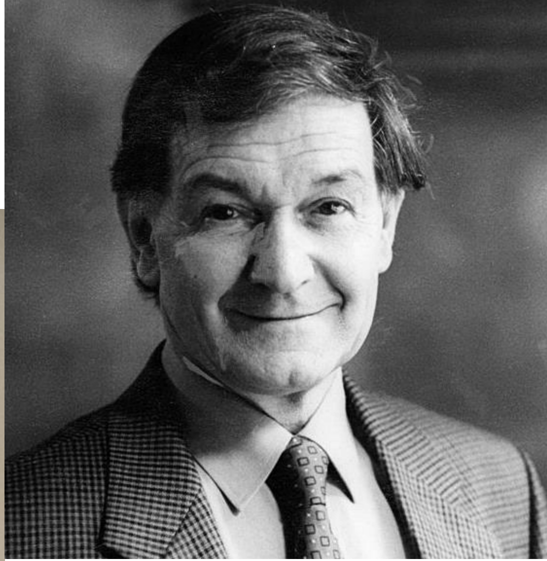


Fig. 6-7. The (dual) moment sequence.



Look at dimensionality of objects in twistor space

Conformal GA of spacetime!

Forms and exterior calculus

Working with just the exterior product, exterior differential and duality recovers the language of forms

Motivation is that this is the 'non-metric' part of the geometric product

Interesting development to track is the subject of discrete exterior calculus

This has a discrete exterior product

$$\langle \alpha^k \wedge \beta^l, \sigma^{k+l} \rangle = \frac{1}{(k+l)!} \sum_{\tau \in S_{k+l+1}} \text{sign}(\tau) \frac{|\sigma^{k+l} \cap \star v_{\tau(k)}|}{|\sigma^{k+l}|} \alpha \smile \beta(\tau(\sigma^{k+l})),$$

This is associative! Hard to prove.

Challenge – can you do better?

Resources

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