Quadratic Lagrangians and Topology in Gauge Theory Gravity

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Abstract

We consider topological contributions to the action integral in a gauge theory formulation of gravity. Two topological invariants are found and are shown to arise from the scalar and pseudoscalar parts of a single integral. Neither of these action integrals contribute to the classical field equations. An identity is found for the invariants that is valid for non-symmetric Riemann tensors, generalising the usual GR expression for the topological invariants. The link with Yang-Mills instantons in Euclidean gravity is also explored. Ten independent quadratic terms are constructed from the Riemann tensor, and the topological invariants reduce these to eight possible independent terms for a quadratic Lagrangian. The resulting field equations for the parity non-violating terms are presented. Our derivations of these results are considerably simpler that those found in the literature.

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1 Introduction

In the construction of a gravitational field theory there is considerable freedom in the choice of Lagrangian. Einstein's theory is obtained when just the Ricci scalar is used, but there is no compelling reason to believe that this is anything other than a good approximation. Since quadratic terms will be small when the curvature is small one would expect them to have a small effect at low energies. However, they may have a considerable effect in cosmology or on singularity formation when the curvature gets larger. Quadratic terms may also be necessary to formulate a sensible quantum theory.

In this paper we consider the effects of quadratic Lagrangians when gravity is considered as a gauge theory. Topological invariants place restrictions on the number of independent quadratic terms one can place in the Lagrangian. In the gauge theory approach these invariants arise simply as boundary terms in the action integral. The Bianchi identity means that these terms do not contribute to the classical field equations, though they could become important in a quantum theory. The invariants have a natural analog in Euclidean gravity in the winding numbers of Yang-Mills instantons. These are characterised by two integers which can be expressed as integrals quadratic in the Riemann tensor.

Here we investigate instantons and quadratic Lagrangians in Gauge Theory Gravity (GTG) as recently formulated by Lasenby, Doran and Gull [1]. GTG is a modernised version of ECKS or U_4 spin-torsion theory where gravity corresponds to a combination of invariance under local Lorentz transformations and diffeomorphisms. With a Ricci Lagrangian GTG reproduces the results of General Relativity (GR) for all the standard tests, but also incorporates torsion in a natural manner. When quadratic terms are introduced into the Lagrangian the theories differ markedly. In GR one obtains fourth order equations for the metric,[2] whereas in GTG one has a pair of lower order equations. One of these determines the connection, which in general will differ from that used in GR. A reason for these differences can be seen in the way that the fields transform under scale transformations. In the GTG approach, all of the quadratic terms in the action transform homogeneously under scalings. In GR the only terms with this property are those formed from quadratic combinations of the Weyl tensor.

We start with a brief outline of GTG, employing the notation of the Spacetime Algebra (STA) [3, 4]. This algebraic system, based on the Dirac algebra, is very helpful in elucidating the structure of GTG. The simplicity of the derivations presented here is intended in part as an advertisement for the power of the STA.

We continue by constructing the topological invariants for the GTG action integral. We show that the two invariants are the scalar and pseudoscalar parts of a single quantity, and our derivation treats them in a unified way. The relationship with instanton solutions in Euclidean gravity is explored. For instantons in Yang-Mills theory, with the rotation gauge field becoming pure gauge at infinity, and the topological invariants give rise to a pair of winding numbers. The differences between the Euclidean and spacetime cases is shown to be due to the different sign of the square of the pseudoscalar.

We construct irreducible fields from the Riemann tensor and use these to form ten independent quadratic terms from the Riemann tensor. In an action integral the two topological terms can be ignored, so only eight terms are needed. We construct the field equations for the parity non-violating Lagrangian terms. Units with $\hbar = c = 8\pi G = 1$ are used throughout.

2 Gauge Theory Gravity (GTG)

In this paper we employ the Spacetime Algebra (STA), which is the geometric (or Clifford) algebra of Minkowski spacetime. For details of geometric algebra the reader is referred to [1, 5, 4]. The STA is generated by 4 orthonormal vectors, here denoted $\{\gamma_a\}, a = 0 \cdots 4$. These are equipped with a geometric (Clifford) product. This product is associative, and the symmetrised product of two vectors is a scalar:

$$\tfrac{1}{2}(\gamma_a\gamma_b + \gamma_b\gamma_a) = \gamma_a\cdot\gamma_b = \eta_{ab} = \mathrm{diag}(+\,-\,-\,-). \tag{2.1}$$

Clearly the γ_a vectors satisfy the same algebraic properties as the Dirac matrices. There is no need to introduce an explicit matrix representation for any of the derivations presented here, however, and to do so would pointlessly over-complicate matters. The antisymmetrised product of two vectors is a bivector, denoted with a wedge $\wedge$. For two vectors u and v we therefore have

$$uv = \tfrac{1}{2}(uv + vu) + \tfrac{1}{2}(uv - vu) = u\cdot v + u\wedge v. \tag{2.2}$$

These definitions extend to define an algebra with 16 elements:

$$\begin{array}{ccccc} 1 & \{\gamma_a\} & \{\gamma_a\wedge\gamma_b\} & \{I\gamma_\mu\} & I \\ \text{1 scalar} & \text{4 vectors} & \text{6 bivectors} & \text{4 trivectors} & \text{1 pseudoscalar} \\ \text{grade 0} & \text{grade 1} & \text{grade 2} & \text{grade 3} & \text{grade 4,} \end{array} \tag{2.3}$$

where the pseudoscalar I is defined by

$$I = \gamma_0\gamma_1\gamma_2\gamma_3. \tag{2.4}$$

The pseudoscalar satisfies $I^2 = -1$, and generates duality transformations, interchanging grade-r and grade-$(4-r)$ multivectors.

Gauge theory gravity, or GTG, was introduced in [1]. The notation there relied heavily on the use of geometric calculus. Here we have chosen to adopt a different notation which is closer to more familiar systems. These conventions are not as elegant as those of [1, 5], but they should help to make the results more accessible. The first of the gravitational gauge fields is a position-dependent linear function mapping vectors to vectors. In [1] this was denoted by $\overline{h}(a)$. Here we will formulate our equations in terms of the set of vectors

$$h^a = \overline{h}(\gamma^a). \tag{2.5}$$

The reciprocal vectors are denoted by h_a and satisfy[1]

$$h_a \cdot h^b = \delta^b_a. \tag{2.6}$$

The $\{h^a\}$ vectors have the property that

$$h^a \cdot h^b = g^{ab}, \qquad h_a \cdot h_b = g_{ab} \tag{2.7}$$

where g_{ab} is the metric. Clearly the h^a are closely related to a vierbein and this relationship is explained in detail in [1]. One point to note is that only one type of contraction is used in GTG, which is that of the underlying STA (2.3). Our use of Latin indices reflects the fact that in many formulae these indices can also be read as abstract vectors, which is closer to the notation of [1, 5].

The second gauge field is set of bivector-valued fields $\{\Omega_a\}$. These ensure invariance under local Lorentz transformations, which are written in the STA using the the double-sided formula

$$A \mapsto LA\tilde{L}. \tag{2.8}$$

Here A is an arbitrary multivector, L is a *rotor* — an even element satisfying $L\tilde{L} = 1$ — and the tilde denotes the operation of reversing the order of vectors in

[1]Following [1] we would write $h_a = \underline{h}^{-1}(\gamma_a)$.

any geometric product. Under a Lorentz transformation the Ω_a transforms as

$$\Omega_a \to L\Omega_a\tilde{L} - 2\nabla_a L\tilde{L}, \tag{2.9}$$

where $\nabla_a = \gamma_a\cdot\nabla$ is the derivative in the γ_a direction. It follows that Ω_a takes its values in the Lie algebra of the group of rotors, which in the STA is simply the space of bivectors. Of course the Ω_a are a form of spin connection, the difference here being that it takes its values explicitly in the bivector subalgebra of the STA.

The Ω_a are used to construct a derivative which is covariant under local spacetime rotations. Acting on an arbitrary multivector A we define

$$D_a A \equiv \nabla_a A + \Omega_a \times A \tag{2.10}$$

where $\times$ is the commutator product, $A \times B = \frac{1}{2}(AB - BA)$. The commutator of these derivatives defines the field strength,

$$R_{ab} \equiv \nabla_a\Omega_b - \nabla_b\Omega_a + \Omega_a \times \Omega_b. \tag{2.11}$$

This is also bivector-valued, and is best viewed as a linear function of a bivector argument (the argument being $\gamma_a\wedge\gamma_b$ in this case). From this field strength we define the covariant *Riemann tensor*

$$\mathcal{R}_{ab} \equiv R_{cd}\,\gamma_a\cdot h^c\,\gamma_b\cdot h^d. \tag{2.12}$$

Again, $\mathcal{R}_{ab}$ is best viewed as a linear map on the space of bivectors, and as such it has a total of 36 degrees of freedom. We employ the convention that fully covariant fields are written in calligraphic type. These covariant objects are at the heart of the GTG formalism, and distinguish this approach to one based on differential forms. Covariant objects such as $\mathcal{R}_{ab}$, or $h^a\nabla_a\alpha$ (where α is a scalar field), are elements of neither the tangent nor cotangent spaces. Instead they belong in a separate 'covariant' space in which all objects transform simply under displacements. In this space it is simple to formulate physical laws, and to isolate gauge invariant variables.

The remaining field equation is

$$h^b\wedge(D_b h^a) = \mathcal{T}^b h_b\cdot\gamma^a, \tag{2.13}$$

which defines the *torsion bivector* $\mathcal{T}^a$. This definition also ensures that $\mathcal{T}^a$ is a covariant tensor, in this case a map from vectors to bivectors. Since the torsion is not assumed to vanish, we cannot make any further assumptions about the symmetries of the Riemann tensor. Specifically the 'cyclic identity' of GR, $\mathcal{R}_{ab} \wedge \gamma^b = 0$, no longer holds.

From the Riemann tensor one forms two contractions, the Ricci tensor $\mathcal{R}_a$ and the Ricci scalar $\mathcal{R}$,

$$\mathcal{R}_a = \gamma^b \cdot \mathcal{R}_{ba} \qquad \mathcal{R} = \gamma^a \cdot \mathcal{R}_a. \tag{2.14}$$

The same symbol is used for the Riemann tensor, Ricci tensor and Ricci scalar, with the number of subscripts denoting which is intended. Both of the tensors preserve grade, so it is easy to keep track of the grade of the objects generated. The Einstein tensor is derived from the Ricci tensor in the obvious way,

$$\mathcal{G}_a = \mathcal{R}_a - \tfrac{1}{2}\mathcal{R}\gamma_a. \tag{2.15}$$

These are all of the definitions required to study the role of quadratic Lagrangians in GTG.

3 Topological invariants

We are interested in the behaviour of quadratic terms in the gravitational Lagrangian in GTG. We start by constructing the following quantity (which is motivated by instanton solutions in Euclidean gravity — see Section 4)

$$\mathcal{Z} \equiv \gamma^a \wedge \gamma^b \wedge \gamma^c \wedge \gamma^d \mathcal{R}_{cd}\mathcal{R}_{ab} = \gamma^a \wedge \gamma^b \wedge \gamma^c \wedge \gamma^d \tfrac{1}{2}(\mathcal{R}_{cd}\mathcal{R}_{ab} + \mathcal{R}_{ab}\mathcal{R}_{cd}). \tag{3.16}$$

This is a combination of scalar and pseudoscalar terms only, so transforms as a scalar under restricted Lorentz transformations. From equation (2.12) we can write

$$\mathcal{Z} = h^a \wedge h^b \wedge h^c \wedge h^d R_{cd} R_{ab} = h\, \gamma^a \wedge \gamma^b \wedge \gamma^c \wedge \gamma^d R_{cd} R_{ab} \equiv hZ \tag{3.17}$$

where h is the determinant defined by

$$h^a \wedge h^b \wedge h^c \wedge h^d \equiv h\, \gamma^a \wedge \gamma^b \wedge \gamma^c \wedge \gamma^d \tag{3.18}$$

and

$$Z \equiv \gamma^a \wedge \gamma^b \wedge \gamma^c \wedge \gamma^d R_{cd} R_{ab} \tag{3.19}$$

The quantity Z also has just scalar and pseudoscalar terms. We can therefore form an invariant integral that is independent of the h^a field as

$$S \equiv \int |d^4x| h^{-1} \mathcal{Z} = \int |d^4x| Z. \tag{3.20}$$

From the definition of the Riemann tensor we find that

$$\begin{aligned} Z &= \gamma^a \wedge \gamma^b \wedge \gamma^c \wedge \gamma^d (2\nabla_c \Omega_d + \Omega_c \Omega_d)(2\nabla_a \Omega_b + \Omega_a \Omega_b) \\ &= -4\gamma^a \wedge \gamma^b \wedge \gamma^c \wedge \nabla (\nabla_c \Omega_a \Omega_b + \tfrac{1}{3} \Omega_a \Omega_b \Omega_c) \\ &= 2\gamma^a \wedge \gamma^b \wedge \gamma^c \wedge \nabla (R_{ac} \Omega_b + \tfrac{1}{3} \Omega_a \Omega_b \Omega_c). \end{aligned} \tag{3.21}$$

The main step in this derivation is the observation that the totally antisymmetrised product of 4 bivectors vanishes identically in 4-d. This proof that Z is a total divergence is considerably simpler than that given in [6], where gamma matrices were introduced in order to generate a similar 'simple' proof in the Riemann-Cartan formulation. Here we have also treated the scalar and pseudoscalar parts in a single term, which halves the work.

The fact that the integral (3.20) reduces to a total divergence is the GTG equivalent of saying that the integral only contributes a topological term to the action. This demonstrates that there is no difficulty in dealing with many topological constructions with the (flat space) gauge theory approach to gravity. Similar observations were made in [7], where cosmic string solutions were shown to have a natural form in GTG, which nicely highlights their relationship to an Aharonov-Bohm potential.

Since the action integral reduces to a boundary term we expect that it should not contribute to the field equations. This is simple to check. There is no dependence on the $\overline{h}_a$ field, so no contribution arises when this field is varied. When the Ω_a field is varied one picks up terms proportional to

$$\gamma_a \wedge \gamma^b \wedge \gamma^c \wedge \gamma^d D_d R_{cb} = \tfrac{1}{3} \gamma_a \wedge \gamma^b \wedge \gamma^c \wedge \gamma^d (D_d R_{cb} + D_b R_{dc} + D_c R_{db}) = 0, \tag{3.22}$$

which vanishes by virtue of the Bianchi identity. Since the two topological terms do not contribute to the field equations, and can therefore be ignored in any action integral, is is useful to have expressions for these in terms of simpler combinations of the Riemann tensor and its contractions. For the scalar term (denoted $\langle \mathcal{Z} \rangle$) we find that

$$\begin{aligned}\langle\mathcal{Z}\rangle &= \gamma^a\wedge\gamma^b\wedge\gamma^c\wedge\gamma^d\,\mathcal{R}_{cd}\wedge\mathcal{R}_{ab}\\ &= (\gamma^a\wedge\gamma^b\wedge\gamma^c)\cdot[(\gamma^d\cdot\mathcal{R}_{cd})\wedge\mathcal{R}_{ab} + \mathcal{R}_{cd}\wedge(\gamma^d\cdot\mathcal{R}_{ab})]\\ &= (\gamma^a\wedge\gamma^b)\cdot[-\mathcal{R}\mathcal{R}_{ab} + 2\mathcal{R}_c\wedge(\gamma^c\cdot\mathcal{R}_{ab}) + 2\mathcal{R}_{cd}(\gamma^c\wedge\gamma^d)\cdot\mathcal{R}_{ab})]\\ &= \mathcal{R}^2 + 2\gamma^a\cdot[\gamma^b\cdot\mathcal{R}_c\,\gamma^c\cdot\mathcal{R}_{ab} - \gamma^b\cdot(\gamma^c\cdot\mathcal{R}_{ab})\mathcal{R}_c] + 2\mathcal{R}_{ba}\cdot\bar{\mathcal{R}}^{ab}\\ &= 2\mathcal{R}_{ba}\cdot\bar{\mathcal{R}}^{ab} - 4\mathcal{R}_a\cdot\bar{\mathcal{R}}^a + \mathcal{R}^2, \end{aligned} \tag{3.23}$$

where the adjoint functions are defined by

$$(\gamma_a\wedge\gamma_b)\cdot\bar{\mathcal{R}}_{cd} \equiv (\gamma_c\wedge\gamma_d)\cdot\mathcal{R}_{ab} \qquad \gamma_a\cdot\bar{\mathcal{R}}_b = \gamma_b\cdot\mathcal{R}_a. \tag{3.24}$$

For the pseudoscalar term (denoted $\langle\mathcal{Z}\rangle_4$) we similarly obtain

$$\begin{aligned}\langle\mathcal{Z}\rangle_4 &= \gamma^a\wedge\gamma^b\wedge\gamma^c\wedge\gamma^d\,\mathcal{R}_{cd}\cdot\mathcal{R}_{ab}\\ &= \gamma^a\wedge\gamma^b\wedge(\bar{\mathcal{R}}_{cd}\,(\gamma^c\wedge\gamma^d)\cdot\mathcal{R}_{ab})\\ &= -I(\gamma^c\wedge\gamma^d)\cdot\mathcal{R}_{ab}\,(I\gamma^a\wedge\gamma^b)\cdot\bar{\mathcal{R}}_{cd}\\ &= 2I\mathcal{R}^*_{cd}\cdot\mathcal{R}^{cd}\end{aligned} \tag{3.25}$$

where we have introduced the dual of the Riemann tensor defined by

$$\mathcal{R}^*_{ab} \equiv \mathcal{R}(I\gamma_a\wedge\gamma_b). \tag{3.26}$$

We therefore have

$$S = \frac{1}{32\pi^2}\int |d^4x| h^{-1}\Big(2\mathcal{R}_{ba}\cdot\bar{\mathcal{R}}^{ab} - 4\mathcal{R}_a\cdot\bar{\mathcal{R}}^a + \mathcal{R}^2 + 2I\mathcal{R}^*_{ab}\cdot\mathcal{R}^{ba}\Big). \tag{3.27}$$

This generalises the usual GR expressions for the topological invariants to the case where the Riemann tensor need not be symmetric, as in the case when there is torsion. Both of the scalar and pseudoscalar contributions can usually be ignored in the action integral. The standard GR expressions are recovered by setting $\bar{\mathcal{R}}_{ab} = \mathcal{R}_{ab}$ and $\bar{\mathcal{R}}_a = \mathcal{R}_a$.

4 Relation to Instantons

The derivation of topological terms in GTG has a Euclidean analog, which gives rise to instanton winding numbers as found in Yang-Mills theory. For this section we assume that we are working in a Euclidean space. Most of the formulae go through unchanged, except that now the pseudoscalar squares to +1. For this section we therefore denote the pseudoscalar by E. The proof that the integral (3.20) is a total divergence is unaffected, and so it can be converted to a surface integral. The Riemann is assumed to fall off sufficiently quickly that we can drop the R_{ac} term, so

$$S = -\frac{2}{3}\int |d^3x| n\wedge\gamma^a\wedge\gamma^b\wedge\gamma^c\,\Omega_a\Omega_b\Omega_c. \tag{4.28}$$

For the Riemann to tend to zero the Ω_a field must tend to pure gauge,

$$\Omega_a = -2\nabla_a L\tilde{L}, \tag{4.29}$$

where L is a (Euclidean) rotor. The integral is a topological invariant because, by construction, it is invariant under continuous transformations of the rotor L. We define

$$\chi + E\tau \equiv \frac{1}{6\pi^2}\int |d^3x| n\wedge\gamma^a\wedge\gamma^b\wedge\gamma^c\,\nabla_a L\tilde{L}\nabla_b L\tilde{L}\nabla_c L\tilde{L} = \frac{1}{32\pi^2}S. \tag{4.30}$$

The numbers τ and χ are instanton numbers for the solution, here given by the scalar and pseudoscalar parts of one equation. The common origin of the invariants is clear, as is the fact that one is a scalar and one a pseudoscalar. There are two integer invariants because the 4-d Euclidean rotor group is $Spin(4)$ and the homotopy groups obey

$$\pi_3(Spin(4)) = \pi_3(SU(2)\times SU(2)) = \pi_3(SU(2))\times\pi_3(SU(2)) = Z\times Z. \tag{4.31}$$

Exhibiting the common origin of these invariants in Euclidean and Lorentzian signatures shows that the gauge-theory approach has applications beyond just gravitation theory. The derivations also highlight the differences between the two signatures. Most of these result from the different sign of the square of the pseudoscalar. In Euclidean 4-d space the pseudoscalar E squares to +1 and is used to separate the bivectors into self-dual and anti-self-dual components,

$$B^{\pm} = \tfrac{1}{2}(B \pm BE), \qquad EB^{\pm} = \pm B^{\pm}. \tag{4.32}$$

These give rise to the two separate instanton numbers, one for each of the $SU(2)$ subgroups. In spacetime, however, the pseudoscalar has negative square and instead gives rise to a natural complex structure. The structure frequently re-emerges in gravitation theory. The fact that the complex structures encountered in GR are geometric in origin is often forgotten when one attempts a Euclideanised treatment of gravity.

5 Quadratic Lagrangians

We now use the preceding results to construct a set of independent Lagrangian terms for GTG which are quadratic in the field strength (Riemann) tensor $\mathcal{R}_{ab}$. None of these terms contain derivatives of the h^a, so all transform simply rescaling of the h^a vectors. Local changes of scale are determined by

$$h^a \mapsto e^{-\alpha} h^a, \quad \Omega_a \mapsto \Omega_a, \tag{5.33}$$

where α is a function of position. The field strength transforms as

$$\mathcal{R}_{ab} \mapsto e^{-2\alpha} \mathcal{R}_{ab}, \tag{5.34}$$

so all quadratic terms formed from $\mathcal{R}_{ab}$ pick up a factor of $\exp(-4\alpha)$ under scale changes. It follows that all quadratic combinations contribute a term to the action integral that is invariant under local rescalings. This situation is quite different to GR, where only combinations of the Weyl tensor are invariant. As a result the field equations from quadratic GTG (and ECKS theory) are very different to those obtained in GR.

To construct the independent terms for a quadratic Lagrangian we need to construct the irreducible parts of the Riemann tensor. To do this we write

$$\mathcal{R}_{ab} = \mathcal{W}_{ab} + \mathcal{P}_{ab} + \mathcal{Q}_{ab} \tag{5.35}$$

where

$$\gamma^a \mathcal{W}_{ab} = 0 \qquad \gamma^a \mathcal{P}_{ab} = \gamma^a \wedge \mathcal{P}_{ab} \qquad \gamma^a \cdot \mathcal{Q}_{ab} = \mathcal{R}_b. \tag{5.36}$$

In the language of Clifford analysis, this is a form of monogenic decomposition of $\mathcal{R}_{ab}$ [8, 9]. To achieve this decomposition we start by defining [1]

$$\mathcal{Q}_{ab} = \tfrac{1}{2}(\mathcal{R}_a \wedge \gamma_b + \gamma_a \wedge \mathcal{R}_b) - \tfrac{1}{6}\gamma_a \wedge \gamma_b \mathcal{R}, \tag{5.37}$$

which satisfies $\gamma^a \cdot \mathcal{Q}_{ab} = \mathcal{R}_b$. We next take the protraction of (5.35) with γ^a to obtain

$$\gamma^a \wedge \mathcal{R}_{ab} - \tfrac{1}{2}\gamma^a \wedge \mathcal{R}_a \wedge \gamma_b = \gamma^a \wedge \mathcal{P}_{ab}. \tag{5.38}$$

We now define the vector-valued function

$$\mathcal{V}_b \equiv -I\gamma^a \wedge \mathcal{R}_{ab} = \gamma^a \cdot (I\mathcal{R}_{ab}). \tag{5.39}$$

The symmetric part of $\mathcal{V}_b$ is

$$\begin{aligned}\mathcal{V}_b^+ &= \tfrac{1}{2}(\mathcal{V}_b + \gamma^a\, \mathcal{V}_a \cdot \gamma_b) \\ &= -I\tfrac{1}{2}(\gamma^a \wedge \mathcal{R}_{ab} + \gamma^a\, \gamma_b \wedge \gamma^c \wedge \mathcal{R}_{ca}) \\ &= -I(\gamma^a \wedge \mathcal{R}_{ab} - \tfrac{1}{2}\gamma^a \wedge \mathcal{R}_a \wedge \gamma_b)\end{aligned} \tag{5.40}$$

so we have

$$\gamma^a \wedge \mathcal{P}_{ab} = I\mathcal{V}_b^+. \tag{5.41}$$

It follows that

$$\mathcal{P}_{ab} = -\tfrac{1}{2}I(\mathcal{V}_a^+ \wedge \gamma_b + \gamma_a \wedge \mathcal{V}_b^+) + \tfrac{1}{6}I\gamma_a \wedge \gamma_b \mathcal{V} \tag{5.42}$$

where

$$\mathcal{V} = \gamma^a \cdot \mathcal{V}_a. \tag{5.43}$$

This construction of $\mathcal{P}_{ab}$ ensures that the tensor has zero contraction, as required.

Splitting the Ricci tensor into symmetric and antisymmetric parts we can finally write the Riemann tensor as

$$\begin{aligned}\mathcal{R}_{ab} =& \mathcal{W}_{ab} + \tfrac{1}{2}(\mathcal{R}_a^+ \wedge \gamma_b + \gamma_a \wedge R_b^+) - \tfrac{1}{6}\gamma_a \wedge \gamma_b \mathcal{R} \\ &+ \tfrac{1}{2}(\mathcal{R}_a^- \wedge \gamma_b + \gamma_a \wedge R_b^-) - \tfrac{1}{2}I(\mathcal{V}_a^+ \wedge \gamma_b + \gamma_a \wedge \mathcal{V}_b^+) + \tfrac{1}{6}I\gamma_a \wedge \gamma_b \mathcal{V}\end{aligned} \tag{5.44}$$

where $+$ and $-$ superscripts denote the symmetric and antisymmetric parts of a tensor respectively. This decomposition splits the Riemann tensor into a Weyl term ($\mathcal{W}_{ab}$) with 10 degrees of freedom, two symmetric tensors ($\mathcal{R}_a^+$ and $\mathcal{V}_a^+$) with 10 degrees of freedom each, and an anti-symmetric tensor ($\mathcal{R}_a^-$) with 6 degrees of freedom. These account for all 36 degrees of freedom in $\mathcal{R}_{ab}$. The first three terms in the decomposition are the usual ones for a symmetric Riemann tensor and would be present in GR. The remaining terms come from the antisymmetric parts of $\mathcal{R}_{ab}$ and only arise in the presence of spin or quadratic terms in the Lagrangian. One could proceed simply now to construct traceless tensors from $\mathcal{V}_a^+$ and $\mathcal{R}_a^+$ to complete the decomposition into irreducible parts.

We can write the antisymmetric part of $\mathcal{R}_a$ as

$$\mathcal{R}_a^- = a \cdot \mathcal{A} \tag{5.45}$$

where $\mathcal{A} = \frac{1}{2}\gamma^a \wedge \mathcal{R}_a$ is a bivector. Using this definition we can write down 10 independent scalar terms which are quadratic in the Riemann tensor:

$$\begin{aligned} \{\, \mathcal{W}^{ab}\cdot\mathcal{W}_{ab}, \quad \mathcal{W}^{ab}\cdot(i\mathcal{W}_{ab}), \quad \mathcal{R}^{+a}\cdot\mathcal{R}^{+}_{a}, \quad \mathcal{R}^2, \\ \mathcal{A}\cdot\mathcal{A}, \quad \mathcal{A}\cdot(i\mathcal{A}), \quad \mathcal{V}^{+a}\cdot\mathcal{V}^{+}_{a}, \quad \mathcal{V}^{+a}\cdot\mathcal{R}^{+}_{a}, \quad \mathcal{V}^2, \quad \mathcal{R}\mathcal{V}\,\} \end{aligned} \tag{5.46}$$

Six of these are invariant under parity and four are parity violating. The two topological invariants can be used to remove two terms, so there are only eight possible independent quadratic terms for the gravitational Lagrangian. The classical field equations arising from an equivalent set of terms is calculated in [10] where the Einstein-Cartan formalism is used. The theory is physically the same as GTG.

For calculational purposes it is easier to use the six parity invariant terms

$$\{\, \mathcal{R}^{ab}\cdot\mathcal{R}_{ba}, \quad \mathcal{R}^{a}\cdot\mathcal{R}_{a}, \quad \bar{\mathcal{R}}^{a}\cdot\mathcal{R}_{a}, \quad \mathcal{R}^2, \quad \mathcal{V}^{a}\cdot\mathcal{V}_{a}, \quad \mathcal{V}^2\,\} \tag{5.47}$$

and the four parity violating terms

$$\{\, \mathcal{R}^{ab}\cdot(i\mathcal{R}_{ba}), \quad \mathcal{R}^{a}\cdot\mathcal{V}_{a}, \quad \bar{\mathcal{R}}^{a}\cdot\mathcal{V}_{a}, \quad \mathcal{R}\mathcal{V}\,\} \tag{5.48}$$

which are linear combinations of the irreducible components. The topological invariants can be used to remove one term from each set. If we consider just the parity invariant terms and use the topological invariant to remove $\bar{\mathcal{R}}^a\cdot\mathcal{R}_a$ we can calculate the field equations from

$$\mathcal{L}_{R^2} = \tfrac{1}{4}\epsilon_1\mathcal{R}^2 + \tfrac{1}{2}\epsilon_2\mathcal{R}^a\cdot\mathcal{R}_a + \tfrac{1}{4}\epsilon_3\mathcal{R}^{ab}\cdot\mathcal{R}_{ba} + \epsilon_4\tfrac{1}{4}\mathcal{V}^2 + \epsilon_5\tfrac{1}{2}\mathcal{V}^a\cdot\mathcal{V}_a \tag{5.49}$$

The field equations for the h^a give a modified Einstein tensor of the form

$$\mathcal{G}'_a = \mathcal{G}_a + \epsilon_1\mathcal{G}_{1a} + \epsilon_2\mathcal{G}_{2a} + \epsilon_3\mathcal{G}_{3a} + \epsilon_4\mathcal{G}_{4a} + \epsilon_5\mathcal{G}_{5a} \tag{5.50}$$

where

$$\mathcal{G}_{1a} = \mathcal{R}(\mathcal{R}_a - \tfrac{1}{4}\gamma_a\mathcal{R}) \tag{5.51}$$

$$\mathcal{G}_{2a} = \gamma_b\,\mathcal{R}^b\cdot\mathcal{R}_a + \mathcal{R}_{ab}\cdot\mathcal{R}^b - \tfrac{1}{2}\gamma_a\,\mathcal{R}^b\cdot\mathcal{R}_b \tag{5.52}$$

$$\mathcal{G}_{3a} = \gamma_b\,\mathcal{R}^{bc}\cdot\mathcal{R}_{ca} - \tfrac{1}{4}\gamma_a\,\mathcal{R}^{bc}\cdot\mathcal{R}_{cb} \tag{5.53}$$

$$\mathcal{G}_{4a} = \mathcal{V}(\mathcal{V}_a - \tfrac{1}{4}\gamma_a\mathcal{V}) \tag{5.54}$$

$$\mathcal{G}_{5a} = \gamma_b\,\mathcal{V}^b\cdot\mathcal{V}_a + (I\mathcal{R}_{ab})\cdot\mathcal{V}^b - \tfrac{1}{2}\gamma_a\,\mathcal{V}^b\cdot\mathcal{V}_b. \tag{5.55}$$

These tensors all have zero contraction, as expected from scale invariance.

The field equations for Ω_a give the generalized torsion equation of the form

$$\mathcal{N}_a = \mathcal{S}_a \tag{5.56}$$

where $\mathcal{N}_a$ is the (generalized) torsion tensor and $\mathcal{S}_a$ is the matter spin tensor. Both of these are bivector-valued functions of their vector argument. To simplify the expressions for the various contributions to $\mathcal{N}_a$ it is useful to introduce the full covariant derivative $\mathcal{D}_a$, defined by

$$\mathcal{D}_a = h^b \cdot \gamma_a \, D_b. \tag{5.57}$$

It is also convenient to define employ the over-dot notation for the covariant derivative of tensors,

$$\dot{\mathcal{D}}_a \dot{T}_b = \mathcal{D}_a T_b - T_c \, \gamma^c \cdot (\mathcal{D}_a \gamma_b), \tag{5.58}$$

which has the property of commuting with contractions. This definition extends in the obvious manner for tensors with higher numbers of indices. The contributions to $\mathcal{N}_a$ from the five terms in the action integral are concisely written as

$$\mathcal{N}_{1a} = -\mathcal{R} \, \gamma^b \cdot (\gamma^a \wedge \mathcal{T}_b) + \gamma_a \wedge \gamma^b \, \mathcal{D}_b \mathcal{R} \tag{5.59}$$

$$\mathcal{N}_{2a} = \Big((\gamma^b \wedge \gamma^c) \cdot (\gamma_a \wedge \mathcal{T}_c) \Big) \wedge \mathcal{R}_b + \gamma_a \wedge (\dot{\mathcal{D}}_b \dot{\mathcal{R}}^b) - \gamma^b \wedge (\dot{\mathcal{D}}_b \dot{\mathcal{R}}_a) \tag{5.60}$$

$$\mathcal{N}_{3a} = \dot{\mathcal{D}}_b \dot{\mathcal{R}}_a{}^b + (\gamma^b \wedge \gamma^c) \cdot \mathcal{T}_c \, \mathcal{R}_{ab} - \tfrac{1}{2} \mathcal{R}_{bc} \, (\gamma^c \wedge \gamma^b) \cdot \mathcal{T}_a \tag{5.61}$$

$$\mathcal{N}_{4a} = I \, \gamma^b \cdot (\gamma_a \wedge \mathcal{T}_b) \mathcal{V} - I \, \gamma_a \wedge \gamma^b \, \mathcal{D}_b \mathcal{V} \tag{5.62}$$

$$\mathcal{N}_{5a} = I \Big((\gamma^c \wedge \gamma^b) \cdot (\gamma_a \wedge \mathcal{T}_c) \Big) \wedge \mathcal{V}_b + I \, \gamma^b \wedge (\dot{\mathcal{D}}_b \dot{\mathcal{V}}_a) - I \, \gamma_a \wedge (\dot{\mathcal{D}}_b \dot{\mathcal{V}}^b). \tag{5.63}$$

More elegant expressions can be obtained if one uses the full, index-free notation and conventions of geometric algebra [1].

6 Conclusions

We have shown that in gauge theory gravity topological terms are simply dealt with and reduce to boundary integrals which do not alter the (classical) field equations. These topological terms have a natural analog in the winding numbers for instanton solutions Euclidean gravity. The differences between the two constructions are due to the opposite signs of the squares of the pseudoscalars. This difference is nicely highlighted by working with the scalar and pseudoscalar invariants in a unified way. In the Euclidean setup the pseudoscalar drives duality transformations, which reduce the $Spin(4)$ group to two $SU(2)$ subgroups. In Minkowski spacetime, however, the pseudoscalar has negative square, and is responsible for the frequently made observation that there is a natural complex structure associated with the gravitational field equations [11].

We constructed ten possible terms for a quadratic Lagrangian, which the topological invariants then restrict to eight independent terms. The field equations for these have been derived elsewhere, but the derivations and formulae presented here are considerably simpler than in previous approaches. A detailed account of how to translate between the results of other approaches and flat space gauge viewpoint adopted here will be provided elsewhere.

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