Geometric Algebra

3. Applications to 3D dynamics

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Recap

Even grade = quaternions

Grade 0 1 Scalar	Grade 1 3 Vectors	Grade 2 3 Plane / bivector	Grade 3 1 Volume / trivector
1	$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$	$\{\boldsymbol{e}_1\boldsymbol{e}_2, \boldsymbol{e}_2\boldsymbol{e}_3, \boldsymbol{e}_3\boldsymbol{e}_1\}$	$\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$
$\tilde{\alpha} = \alpha$	$\tilde{a} = a$	$\tilde{B} = -B$	$\tilde{I} = -I$

Rotation $a \mapsto Ra\tilde{R}$

Rotor $R = e^{-B/2}$

$$aB = a\cdot B + a\wedge B$$

Antisymmetric Symmetric

Inner product

Should confirm that rotations do indeed leave inner products invariant

$$
\begin{aligned}
a' \cdot b' &= (Ra\tilde{R}) \cdot (Rb\tilde{R}) \\
&= \tfrac{1}{2}(Ra\tilde{R}Rb\tilde{R} + Rb\tilde{R}Ra\tilde{R}) \\
&= \tfrac{1}{2}R(ab + ba)\tilde{R} \\
&= a \cdot bR\tilde{R} \\
&= a \cdot b
\end{aligned}
$$

Can also show that rotations do indeed preserve handedness

Angular momentum

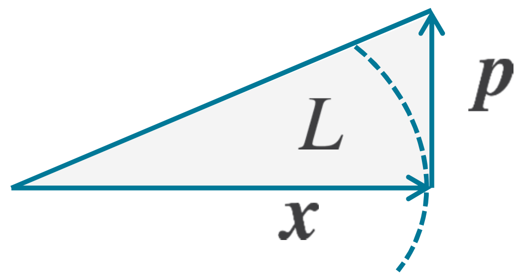
Trajectory $\boldsymbol{x}(t)$

Velocity $\boldsymbol{v} = \dot{\boldsymbol{x}}$

Momentum $\boldsymbol{p} = m\boldsymbol{v}$

Force $\boldsymbol{f}$

Angular momentum measures area swept out



Traditional definition $\boldsymbol{l} = \boldsymbol{x} \times \boldsymbol{p}$

An ‘axial’ vector instead of a ‘polar’ vector

Much better to treat angular momentum as a bivector $L = \boldsymbol{x} \wedge \boldsymbol{p}$

Torque

Differentiate the angular momentum $\dot{L} = \boldsymbol{v} \wedge (m\boldsymbol{v}) + \boldsymbol{x} \wedge (\dot{\boldsymbol{p}}) = \boldsymbol{x} \wedge \boldsymbol{f}$

Define the torque bivector $N = \boldsymbol{x} \wedge \boldsymbol{f}$

Define $\boldsymbol{x} = r\hat{\boldsymbol{x}}$

Differentiate $\dot{\boldsymbol{x}} = \dot{r}\hat{\boldsymbol{x}} + r\dot{\hat{\boldsymbol{x}}}$

So
$$
\begin{aligned}
L &= m\boldsymbol{x} \wedge (\dot{r}\hat{\boldsymbol{x}} + r\dot{\hat{\boldsymbol{x}}}) \\
&= mr\hat{\boldsymbol{x}} \wedge (\dot{r}\hat{\boldsymbol{x}} + r\dot{\hat{\boldsymbol{x}}}) \\
&= mr^2\, \hat{\boldsymbol{x}} \wedge \dot{\hat{\boldsymbol{x}}}
\end{aligned}
$$

But $\hat{\boldsymbol{x}}^2 = 1$

$$0 = \frac{d}{dt}(\hat{\boldsymbol{x}}^2) = 2\hat{\boldsymbol{x}} \cdot \dot{\hat{\boldsymbol{x}}}$$

$$\boxed{L = -mr^2 \dot{\hat{\boldsymbol{x}}}\hat{\boldsymbol{x}}}$$

Inverse-square force

$$m\ddot{\boldsymbol{x}} = -\frac{k}{r^2}\hat{\boldsymbol{x}} = -\frac{k}{r^3}\boldsymbol{x}$$

Simple to see that torque vanishes, so *L* is conserved. This is one of two conserved vectors.

$$L\dot{\boldsymbol{v}} = -\frac{k}{mr^2}L\hat{\boldsymbol{x}}$$

$$= -k\hat{\boldsymbol{x}}\dot{\hat{\boldsymbol{x}}}\hat{\boldsymbol{x}} = k\dot{\hat{\boldsymbol{x}}}$$

So $$\frac{d}{dt}(L\boldsymbol{v} - k\hat{\boldsymbol{x}}) = 0$$

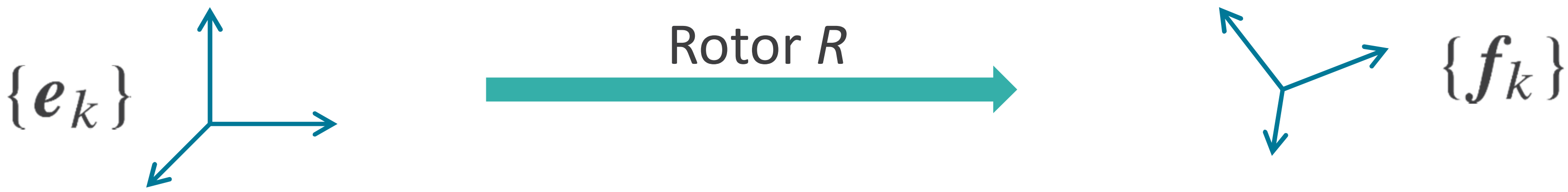
Define the eccentricity vector

$$L\boldsymbol{v} = k(\hat{\boldsymbol{x}} + \boldsymbol{e})$$

Forming scalar part of *Lvx* find

$$r = \frac{l^2}{km(1 + \boldsymbol{e}\cdot\hat{\boldsymbol{x}})}$$

Rotating frames



Frames related by a time dependent rotor $f_k(t) = R(t)e_k\tilde{R}(t)$

Traditional definition of angular velocity $\dot{f}_k = \omega \times f_k$

Replace this with a bivector

$$\dot{f}_k = \dot{R}e_k\tilde{R} + Re_k\dot{\tilde{R}} = \dot{R}\tilde{R}f_k + f_kR\dot{\tilde{R}}$$

Need to understand the rotor derivative, starting from $R\tilde{R} = 1$

Rotor derivatives

$$0 = \frac{d}{dt}(R\tilde{R}) = \dot{R}\tilde{R} + R\dot{\tilde{R}}$$

$$\dot{R}\tilde{R} = -R\dot{\tilde{R}} = -(\dot{R}\tilde{R})^{\sim}$$

An even object equal to minus it's own reverse, so must be a bivector

$$2\dot{R}\tilde{R} = -\Omega$$

$$\dot{R} = -\frac{1}{2}\Omega R$$

Lie group

Lie algebra

$$\dot{f}_k = \dot{R}\tilde{R}f_k - f_k\dot{R}\tilde{R} = f_k \cdot \Omega$$

As expected, angular momentum now a bivector

$$\Omega = I\omega$$

Constant angular velocity

$$\dot{R} = -\tfrac{1}{2}\Omega R$$

Integrates easily in the case of constant Omega $R = e^{-\Omega t/2} R_0$

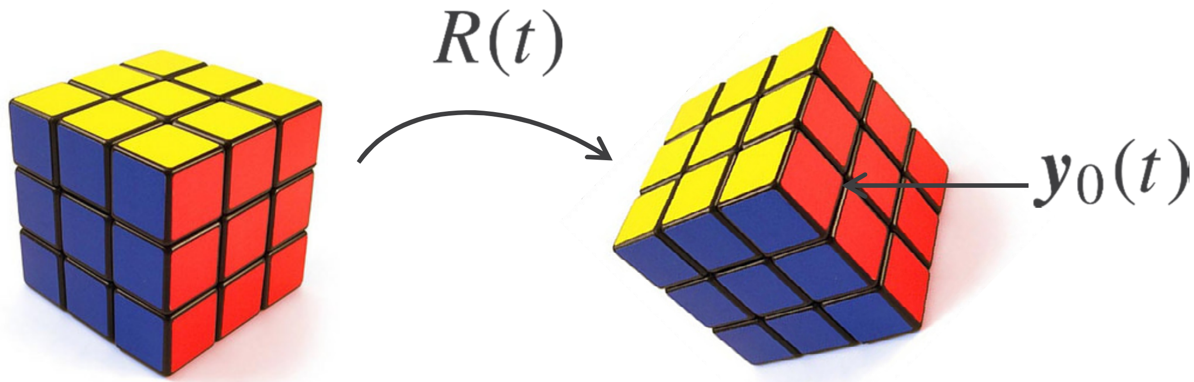
$$f_k(t) = e^{-\Omega t/2} \underline{R_0 \boldsymbol{e}_k \tilde{R}_0} e^{\Omega t/2}$$

Fixed frame at *t=0*

Example – motion around a fixed *z* axis:

$$\Omega = -\omega I \boldsymbol{e}_3 = -\omega \boldsymbol{e}_1 \boldsymbol{e}_2$$
$$R_0 = 1$$

Rigid-body dynamics



Dynamic position of the centre of mass

Fixed reference copy of the object. Origin at CoM.

Dynamic position of the actual object

$\boldsymbol{x}_i$ Constant position vector in the reference copy

$\boldsymbol{y}_i(t)$ Position of the equivalent point in space

$$\boldsymbol{y}_i(t) = R(t)\boldsymbol{x}_i\tilde{R}(t) + \boldsymbol{y}_0(t)$$

Velocity and momentum

$$\dot{R} = -\tfrac{1}{2}\Omega R = -\tfrac{1}{2}R\Omega_B$$

Spatial bivector

Body bivector

$$\Omega = R\Omega_B\tilde{R}$$

$$\begin{aligned} \boldsymbol{v}(t) &= \dot{R}\boldsymbol{x}\tilde{R} + R\boldsymbol{x}\dot{\tilde{R}} + \dot{\boldsymbol{y}}_0 \\ &= R\,\boldsymbol{x}\cdot\Omega_B\,\tilde{R} + \boldsymbol{v}_0 \end{aligned}$$

True for all points. Have dropped the index

Use continuum approximation

Centre of mass defined by

$$\int d^3x\,\rho\boldsymbol{x} = 0$$

Momentum given by

$$\begin{aligned} \int d^3x\,\rho\boldsymbol{v} &= \int d^3x\,\rho(R\,\boldsymbol{x}\cdot\Omega_B\,\tilde{R} + \boldsymbol{v}_0) \\ &= M\boldsymbol{v}_0 \end{aligned}$$

Angular momentum

Need the angular momentum of the body about its instantaneous centre of mass

$$L = \int d^3x\, \rho(\boldsymbol{y} - \boldsymbol{y}_0)\wedge \boldsymbol{v}$$

$$= \int d^3x\, \rho(R\boldsymbol{x}\tilde{R})\wedge(R\,\boldsymbol{x}\cdot\Omega_B\,\tilde{R} + \boldsymbol{v}_0)$$

$$= R\Big(\underline{\int d^3x\, \rho\boldsymbol{x}\wedge(\boldsymbol{x}\cdot\Omega_B)}\Big)\tilde{R}$$

Fixed function of the angular velocity bivector

Define the Inertia Tensor

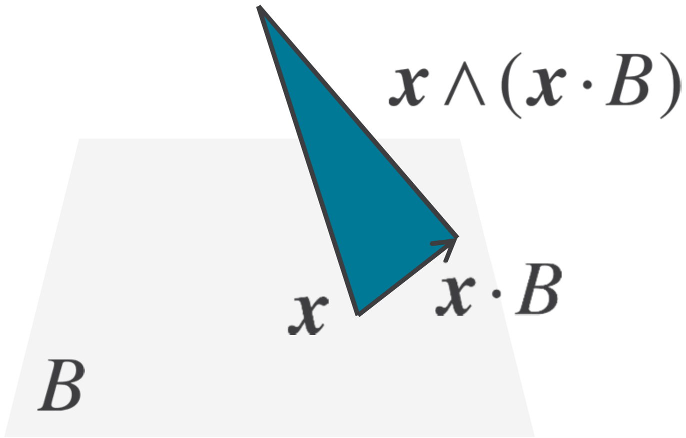
$$\mathcal{I}(B) = \int d^3x\, \rho\boldsymbol{x}\wedge(\boldsymbol{x}\cdot B)$$

This is a linear, symmetric function

$$\mathcal{I}(\lambda A + \mu B) = \lambda\mathcal{I}(A) + \mu\mathcal{I}(B)$$

$$\langle A\mathcal{I}(B)\rangle = \langle\mathcal{I}(A)B\rangle$$

The inertia tensor



Inertia tensor input is the bivector *B.*
Body rotates about centre of mass in the *B* plane.

Angular momentum of the point is $\boldsymbol{x} \wedge (\rho \boldsymbol{x} \cdot B)$

Back rotate the angular velocity to the reference copy $\Omega_B = \tilde{R}\Omega R$

Find angular momentum in the reference copy $\mathcal{I}(\Omega_B)$

Rotate the body angular momentum forward
to the spatial copy of the body $L = R\mathcal{I}(\Omega_B)\tilde{R}$

Equations of motion

$$\begin{aligned}\dot{L} &= \dot{R}\mathcal{I}(\Omega_B)\tilde{R} + R\mathcal{I}(\Omega_B)\dot{\tilde{R}} + R\mathcal{I}(\dot{\Omega}_B)\tilde{R} \\ &= R[\mathcal{I}(\dot{\Omega}_B) - \tfrac{1}{2}\Omega_B\mathcal{I}(\Omega_B) + \tfrac{1}{2}\mathcal{I}(\Omega_B)\Omega_B]\tilde{R} \\ &= R[\mathcal{I}(\dot{\Omega}_B) - \Omega_B\times\mathcal{I}(\Omega_B)]\tilde{R}\end{aligned}$$

From now on, use the cross symbol for the commutator product

$$A \times B = \tfrac{1}{2}(AB - BA)$$

The commutator of two bivectors is a third bivector

Introduce the principal axes and principal moments of inertia

$$\mathcal{I}(I\boldsymbol{e}_k) = i_k I\boldsymbol{e}_k \qquad \text{No sum}$$

Symmetric nature of inertia tensor guarantees these exist

Equations of motion

$$N = R[\mathcal{I}(\dot{\Omega}_B) - \Omega_B \times \mathcal{I}(\Omega_B)]\tilde{R}$$

$$\Omega = \sum_{k=1}^{3} \omega_k I \boldsymbol{f}_k$$

$$\Omega_B = \sum_{k=1}^{3} \omega_k I \boldsymbol{e}_k$$

$$L = \sum_{k=1}^{3} i_k \omega_k I \boldsymbol{f}_k$$

Objects expressed in terms of the principal axes

Inserting these in the above equation recover the famous Euler equations

$$i_1\dot{\omega}_1 - \omega_2\omega_3(i_2 - i_3) = N_1$$

$$i_2\dot{\omega}_2 - \omega_3\omega_1(i_3 - i_1) = N_2$$

$$i_3\dot{\omega}_3 - \omega_1\omega_2(i_1 - i_2) = N_3$$

Kinetic energy

$$T = \tfrac{1}{2}\int d^3x\, \rho(R\,\boldsymbol{x}\cdot\Omega_B\,\tilde{R})^2 = \tfrac{1}{2}\int d^3x\, \rho(\boldsymbol{x}\cdot\Omega_B)^2$$

Use this rearrangement

$$(\boldsymbol{x}\cdot\Omega_B)^2 = \langle \boldsymbol{x}\cdot\Omega_B\boldsymbol{x}\Omega_B\rangle = -\Omega_B\cdot(\boldsymbol{x}\wedge(\boldsymbol{x}\cdot\Omega_B))$$

$$T = -\tfrac{1}{2}\Omega_B\cdot\mathcal{I}(\Omega_B) = \tfrac{1}{2}\tilde{\Omega}_B\cdot\mathcal{I}(\Omega_B)$$

In terms of components

$$T = \tfrac{1}{2}\sum_{k=1}^{3} i_k\omega_k^2 = \tfrac{1}{2}\sum_{k=1}^{3}\frac{L_k^2}{i_k}$$

Symmetric top

Body with a symmetry axis aligned with the 3 direction, so $i_1 = i_2$

Action of the inertia tensor is

$$\mathcal{I}(\Omega_B) = i_1\omega_1\boldsymbol{e}_2\boldsymbol{e}_3 + i_1\omega_2\boldsymbol{e}_3\boldsymbol{e}_1 + i_3\omega_3\boldsymbol{e}_1\boldsymbol{e}_2$$
$$= i_1\Omega_B + (i_3 - i_1)\omega_3 I\boldsymbol{e}_3$$

Third Euler equation reduces to $i_3\dot{\omega}_3 = 0$

Can now write

$$\Omega = R\Omega_B\tilde{R} = \frac{1}{i_1}L + \frac{i_1 - i_3}{i_1}\omega_3 RI\boldsymbol{e}_3\tilde{R}$$

Symmetric top

$$\dot{R} = -\tfrac{1}{2}\Omega R = -\frac{1}{2i_1}(LR + R(i_1 - i_3)\omega_3 I \boldsymbol{e}_3)$$

Define the two constant bivectors

$$\Omega_l = \frac{1}{i_1}L, \qquad \Omega_r = \omega_3 \frac{i_1 - i_3}{i_1} I \boldsymbol{e}_3$$

Rotor equation is now $\dot{R} = -\tfrac{1}{2}\Omega_l R - \tfrac{1}{2}R\Omega_r$

$$R(t) = \exp(-\tfrac{1}{2}\Omega_l t) R_0 \exp(-\tfrac{1}{2}\Omega_r t)$$

Fully describes the motion
Internal rotation gives precession
Fixed rotor defines attitude at *t=0*
Final rotation defines attitude in space

Resources

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