Geometric Algebra

4. Algebraic Foundations and 4D

Dr Chris Doran
ARM Research

Axioms

Elements of a geometric algebra are called multivectors

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A(BC) = (AB)C$$

$$A(B + C) = AB + AC$$

$$(B + C)A = BA + CA$$

Space is linear over the scalars. All simple and natural

Multivectors can be classified by grade

$$A = \langle A \rangle_0 + \langle A \rangle_1 + \langle A \rangle_2 + \cdots$$

Grade-0 terms are real scalars

$$\langle A \rangle_0 = \langle A \rangle \in \mathcal{R}$$

Grading is a projection operation

$$\langle A + B \rangle_r = \langle A \rangle_r + \langle B \rangle_r$$

$$\langle \langle A \rangle_r \rangle_r = \langle A \rangle_r$$

$$\langle \lambda A \rangle_r = \lambda \langle A \rangle_r$$

Axioms

The grade-1 elements of a geometric algebra are called vectors

$$a^2 \in \mathcal{R}$$

$$ab + ba = (a+b)^2 - a^2 - b^2$$

So we define

$$a \cdot b = \tfrac{1}{2}(ab + ba)$$

$$a \wedge b = \tfrac{1}{2}(ab - ba)$$

The antisymmetric produce of *r* vectors results in a grade-*r* blade

Call this the outer product

$$a_1 \wedge a_2 \wedge \cdots \wedge a_r = \frac{1}{r!} \sum (-1)^{\epsilon} a_{k_1} a_{k_2} \cdots a_{k_r}$$

Sum over all permutations with epsilon +1 for even and -1 for odd

Simplifying result

Given a set of linearly-independent vectors $\{a_1, \ldots a_r\}$

We can find a set of anti-commuting vectors such that

$$a_1 \wedge a_2 \wedge \cdots \wedge a_r = e_1 e_2 \cdots e_r$$

$\mathsf{M}_{ij} = a_i \cdot a_j$ Symmetric matrix

$$\mathsf{R}_{ik}\mathsf{M}_{kl}\mathsf{R}^{\mathsf{t}}_{lj} = \mathsf{R}_{ik}\mathsf{R}_{jl}\mathsf{M}_{kl} = \Lambda_{ij}$$

Define

$$e_i = \mathsf{R}_{ij} a_j$$

These vectors all anti-commute

$$\begin{aligned} e_i \cdot e_j &= (\mathsf{R}_{ik} a_k) \cdot (\mathsf{R}_{jl} a_l) \\ &= \mathsf{R}_{ik}\mathsf{R}_{jl}\mathsf{M}_{kl} \\ &= \Lambda_{ij} \end{aligned}$$

The magnitude of the product is also correct

Decomposing products

Make repeated use of $ab = 2a\cdot b - ba$

$$
\begin{aligned}
a(b\wedge c) &= \tfrac{1}{2}a(bc - cb) \\
&= a\cdot b\, c - a\cdot c\, b + \tfrac{1}{2}(cab - bac) \\
&= 2a\cdot b\, c - 2a\cdot c\, b + \tfrac{1}{2}(bc - cb)a \\
&= 2a\cdot b\, c - 2a\cdot c\, b + (b\wedge c)a
\end{aligned}
$$

Define the inner product of a vector and a bivector $a\cdot B = \tfrac{1}{2}(aB - Ba)$

$$a\cdot(b\wedge c) = a\cdot b\, c - a\cdot c\, b$$

General result

$$aa_1a_2\cdots a_r = 2a\cdot a_1\, a_2\cdots a_r - a_1aa_2\cdots a_r$$

Grade r-1

$$= 2\sum_{k=1}^{r}(-1)^{k+1}a\cdot a_k\, a_1a_2\cdots\check{a}_k\cdots a_r + (-1)^r a_1a_2\cdots a_r a$$

Over-check means this term is missing

Define the inner product of a vector and a grade-r term

$$a\cdot A_r = \frac{1}{2}\left(aA_r - (-1)^r A_r a\right) = \langle aA_r\rangle_{r-1}$$

Remaining term is the outer product

$$a\wedge A_r = \frac{1}{2}\left(aA_r + (-1)^r A_r a\right) = \langle aA_r\rangle_{r+1}$$

Can prove this is the same as earlier definition of the outer product

General product

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \cdots + \langle A_r B_s \rangle_{r+s}$$

$$A_r \cdot B_s = \langle A_r B_s \rangle_{|r-s|}$$

$$A_r \wedge B_s = \langle A_r B_s \rangle_{r+s}$$

Extend dot and wedge symbols for homogenous multivectors

The definition of the outer product is consistent with the earlier definition (requires some proof). This version allows a quick proof of associativity:

$$(A_r \wedge B_s) \wedge C_t = \langle A_r B_s \rangle_{r+s} \wedge C_t = \langle (A_r B_s) C_t \rangle_{r+s+t}$$

$$= \langle A_r B_s C_t \rangle_{r+s+t} = A_r \wedge B_s \wedge C_t$$

Reverse, scalar product and commutator

The reverse, sometimes written with a dagger

$$(ab\cdots c)^{\sim} = c\cdots ba$$

$$\tilde{A}_r = (-1)^{r(r-1)/2} A_r$$

$+ \; + \; - \; - \; + \; + \; - \; \cdots \leftarrow$ Useful sequence

Write the scalar product as

$$\langle AB \rangle = \sum_r \langle A_r B_r \rangle$$

Scalar product is symmetric

$$\langle AB \rangle = \langle BA \rangle$$

$$\langle ABC \rangle = \langle CAB \rangle$$

Occasionally use the commutator product

$$A \times B = \frac{1}{2}(AB - BA)$$

Useful property is that the commutator with a bivector *B* preserves grade

$$B \times A_r = \langle B \times A_r \rangle_r$$

Rotations

$$a \mapsto a' = Ra\tilde{R} \qquad R\tilde{R} = 1$$

Combination of rotations

$$a \mapsto R_2(R_1 a \tilde{R}_1)\tilde{R}_2$$
$$= R_2 R_1 a (R_2 R_1)\tilde{}$$

So the product rotor is

$$R = R_2 R_1$$

Rotors form a group

$$R\tilde{R} = R_2 R_1 \tilde{R}_1 \tilde{R}_2 = 1$$

Suppose we now rotate a blade

$$A_r = a_1 a_2 \cdots a_r$$
$$A_r \mapsto Ra_1\tilde{R}Ra_2\tilde{R} \cdots Ra_r\tilde{R}$$
$$= Ra_1 a_2 \cdots a_r \tilde{R}$$

So the blade rotates as

$$A_r \mapsto RA_r\tilde{R}$$

Fermions?

Take a rotated vector through a further rotation $R_\theta = \exp(-\theta B/2)$

The rotor transformation law is $R \mapsto R_\theta R$

Now take the rotor on an excursion through 360 degrees. The angle goes through 2π, but we find the rotor comes back to minus itself.

$$R' = \exp(-\pi B)R = -R$$

This is the defining property of a fermion!

Unification

One of the defining properties of spin-half particles drops out naturally from the properties of rotors.

Linear algebra

Linear function *f*

$$f(\lambda a + \mu b) = \lambda f(a) + \mu f(b)$$

Extend *f* to multivectors

$$f(a \wedge b \cdots) = f(a) \wedge f(b) \cdots$$

This a grade-preserving linear function

The pseudoscalar is unique up to scale so we can define

$$f(I) = \det(f) I$$

Form the product function *fg*

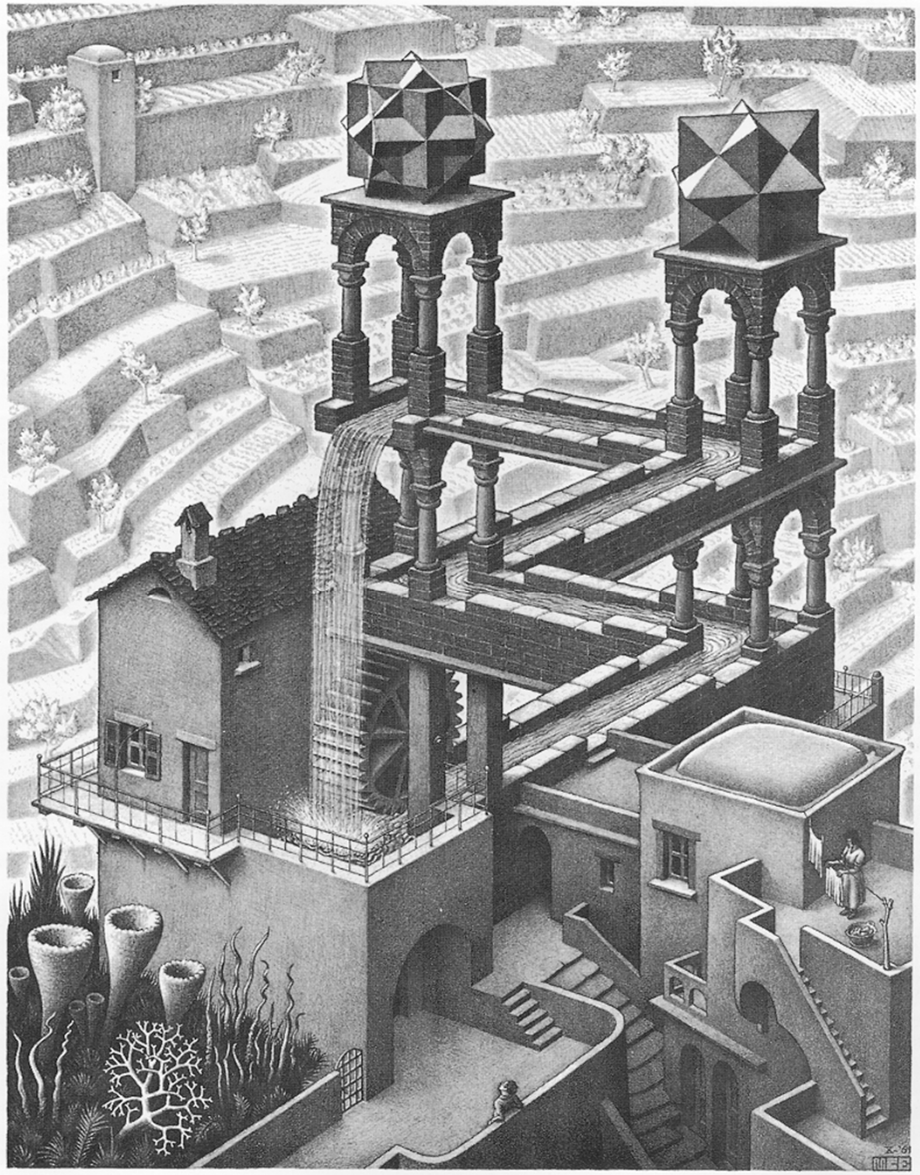
$$\begin{aligned} fg(a \wedge b \cdots) &= fg(a) \wedge fg(b) \cdots \\ &= f(g(a) \wedge g(b) \ldots) \\ &= f(g(a \wedge b \cdots)) \end{aligned}$$

Quickly prove the fundamental result

$$\begin{aligned} \det(fg)I &= fg(I) = f(g(I) \\ &= \det(g) f(I) = \det(f)\det(g) I \end{aligned}$$

Projective geometry

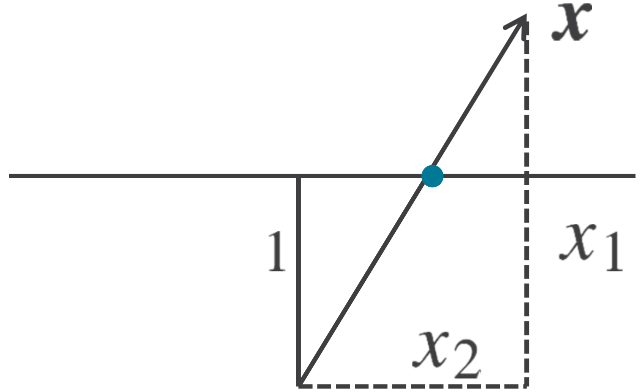
- Use projective geometry to emphasise expressions in GA have multiple interpretations
- Closer to Grassmann's original view
- Our first application of 4D GA
- Core to many graphics algorithms, though rarely taught



Projective line

Point *x* represented by homogeneous coordinates (x_1, x_2)

$$x = \frac{x_1}{x_2}$$



A point as a vector in a GA

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2$$

Outer product of two points represents a line

$$\boldsymbol{x}\wedge\boldsymbol{y} = (x_1y_2 - x_2y_1)\boldsymbol{e}_1\wedge\boldsymbol{e}_2$$
$$= x_2y_2(x-y)I$$

Scale factors

Distance between the points

This representation of points is homogeneous

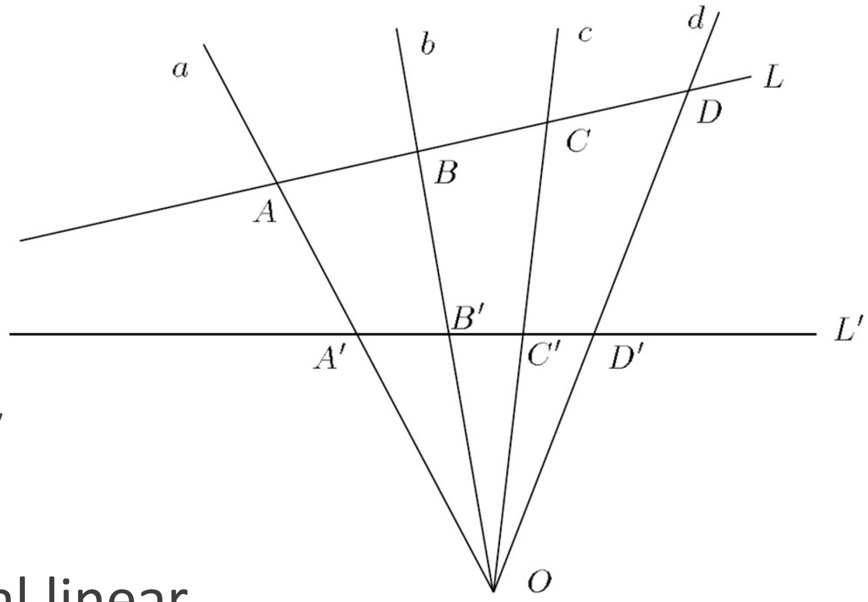
$$\boldsymbol{x} \mapsto \lambda\boldsymbol{x}$$

Cross ratio

$$(ABCD) = \frac{AC}{BC}\frac{BD}{AD} = \frac{\boldsymbol{a}\wedge\boldsymbol{c}}{\boldsymbol{b}\wedge\boldsymbol{c}}\frac{\boldsymbol{b}\wedge\boldsymbol{d}}{\boldsymbol{a}\wedge\boldsymbol{d}}$$

Distance between points

Invariant quantity



Can see that the RHS is invariant under a general linear transformation of the 4 points

$$\boldsymbol{a}\wedge\boldsymbol{b} \mapsto f(\boldsymbol{a})\wedge f(\boldsymbol{b}) = \det(f)\,\boldsymbol{a}\wedge\boldsymbol{b}$$

Ratio is invariant under rotations, translations and scaling

Projective plane

Points on a plane represented by vectors in a 3D GA. Typically align the 3 axis perpendicular to the plane, but this is arbitrary

a	$a \wedge b$	$a \wedge b \wedge c$
Point	Line	Plane

Interchange points and lines by duality. Denoted *

Intersection (meet) defined by

$$(A \vee B)^* = A^* \wedge B^*$$

For 2 lines

$$A \vee B = -I(IA) \wedge (IB)$$
$$= I\langle AB \rangle_2 = \langle IAB \rangle_1$$

For 3 lines to meet at a point

$$(A \vee B) \wedge C = 0$$
$$\implies \langle IABC \rangle_3 = 0$$

Reduces to simple statement

$$\langle ABC \rangle = 0$$

Example

$$A = b \wedge c, \quad B = c \wedge a, \quad C = a \wedge b$$

$$P = a \wedge a', \quad Q = b \wedge b', \quad R = c \wedge c'$$

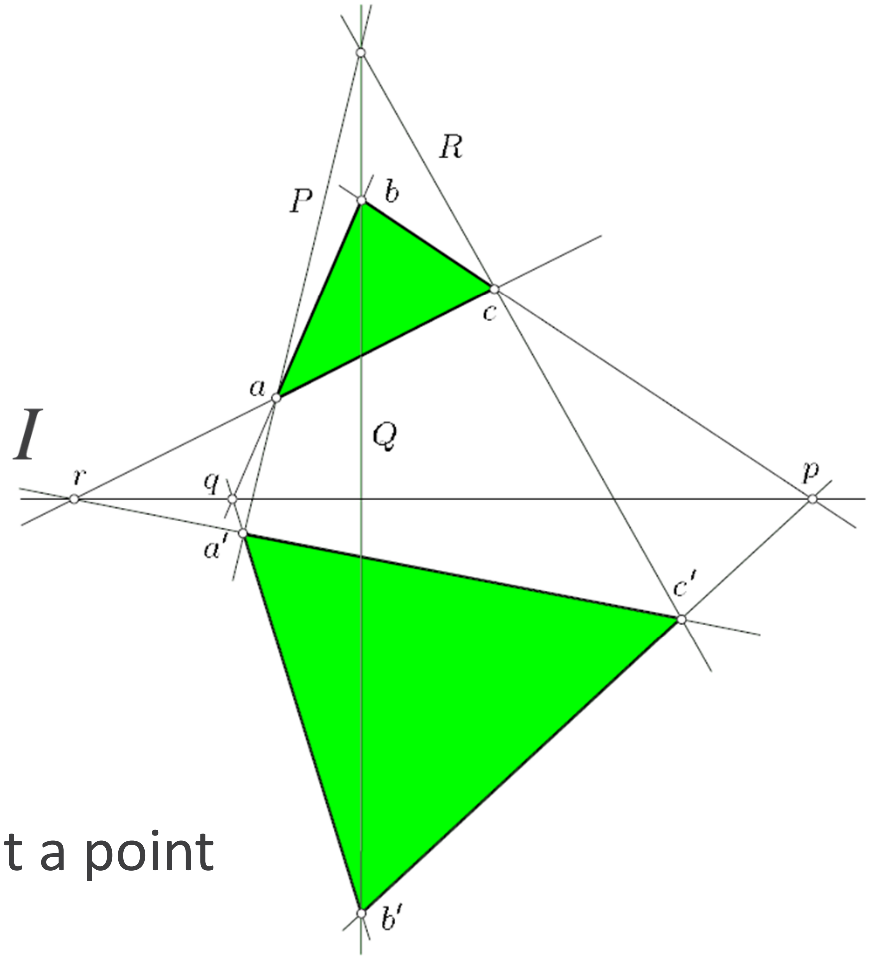
$$p = A \times A' I, \quad q = B \times B' I, \quad r = C \times C' I$$

Can prove the algebraic identity

$$I p \wedge q \wedge r = \langle a \wedge b \wedge c \, a' \wedge b' \wedge c' \rangle \langle PQR \rangle$$

These 3 points are collinear iff these 3 lines meet at a point

This is Desargues theorem. A complex geometric identity from manipulating GA elements.



Projective geometry of 3D space

a	$a \wedge b$	$a \wedge b \wedge c$	$a \wedge b \wedge c \wedge d$
Point	Line	Plane	Volume

Interchange points and planes by duality. Lines transform to other lines

In 4D we can define the object

$$B = \alpha e_1 \wedge e_2 + \beta e_3 \wedge e_4$$

This is homogenous, but NOT a blade. Also satisfies

$$B \wedge B = 2\alpha\beta I \neq 0$$

Bivectors form a 6 dimensional space
Blades represent lines
Test of intersection is

$$B \wedge B' = 0$$

2 Bivectors with non-vanishing outer product are 2 lines missing each other

Plucker coordinates and intersection

Condition that a bivector *B* represents a line is $B \wedge B = 0$

Write $B = (a_1 e_1 + a_2 e_2 + a_3 e_3) e_4 + b_1 e_2 e_3 + b_2 e_3 e_1 + b_3 e_1 e_2$

$$B \wedge B = \boldsymbol{a} \cdot \boldsymbol{b} e_1 e_2 e_3 e_4 = 0$$ Plucker's condition

A linear representation of a line, with a non-linear constraint

Suppose we want to intersect the line *L* with the plane *P*

$$x = L \vee P = (L^* \wedge P^*)^*$$
$$x = \langle IPL \rangle_1$$

Resources

geometry.mrao.cam.ac.uk
chris.doran@arm.com
cjld1@cam.ac.uk
@chrisjldoran
#geometricalgebra
github.com/ga